ON HYPERBOLIC CAPACITY AND HYPERBOLIC LENGTH

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1. INTRODUCTION

Let G be a simply connected region with at least two boundary points, and let $\zeta = \psi(z)$ be a function that maps G conformally onto $\{|\zeta| < 1\}$. Let E be a compact subset of G. Tsuji [8] has introduced the hyperbolic capacity of E with respect to G, which can be defined by

$$\operatorname{caph} E = \lim_{n\to\infty} \max_{z_1,\cdots,z_n \in E} \prod\prod_{\mu\neq\nu} \left| \frac{\psi(z_\mu) - \psi(z_\nu)}{1 - \overline{\psi(z_\mu)}\,\psi(z_\nu)} \right|^{1/n(n-1)}.$$

It does not depend on the choice of $\psi(z)$. Also, it is invariant under conformal mapping of G. For many purposes it is thus sufficient to choose $G = \{|z| < 1\}$ and $\psi(z) \equiv z$. Then

$$(1) \qquad \operatorname{caph} E = \lim_{n\to\infty} \max_{z_1,\cdots,z_n \in E} \prod\prod_{\mu\neq\nu} \left| \frac{z_\mu - z_\nu}{1 - \bar{z}_\mu z_\nu} \right|^{1/n(n-1)}.$$

It is always true that $0 \leq \operatorname{caph} E < 1$. The circle $\{|z| = \rho\}$ has hyperbolic capacity ρ.

Let E be a compact set in $\{|z| < 1\}$. Together with caph E we can consider the logarithmic capacity, cap E. From (1) and the corresponding definition of cap E we immediately obtain the following lemma.

LEMMA 1. *If* $E \subset \{|z| \leq \delta\}$ $(0 \leq \delta < 1)$, *then*

$$\frac{\operatorname{cap} E}{1+\delta^2} \leq \operatorname{caph} E \leq \frac{\operatorname{cap} E}{1-\delta^2}.$$

Hence $\operatorname{caph} E = 0$ if and only if $\operatorname{cap} E = 0$. If F is any compact plane set and if R is so large that $F \subset \{|z| < R\}$, we can consider the hyperbolic capacity $\operatorname{caph}_R F$ of F with respect to $\{|z| < R\}$. Since

$$\operatorname{caph}_R F = \operatorname{caph}_1(R^{-1} F) \text{ and } \operatorname{cap}(R^{-1} F) = R^{-1} \operatorname{cap} F,$$

Lemma 1 implies that $R \operatorname{caph}_R F \to \operatorname{cap} F$ as $R \to \infty$.

Let E be a connected compact set in G, and let S be the doubly connected region between E and the boundary ∂G of G. If $z = f(s)$ maps $\{\rho < |s| < 1\}$ conformally onto S, then [8]

$$(2) \qquad \rho = \operatorname{caph} E.$$

Hence the modulus of S is $\log(1/\operatorname{caph} E)$.

The hyperbolic metric in G is defined by the length element

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$$(1 - |\psi(z)|^2)^{-1} |\psi'(z)| |dz|.$$

For $G = \{|z| < 1\}$ the length element becomes $(1 - |z|^2)^{-1} |dz|$. In this case the geodesic through two given points z_1 and z_2 is the circle through z_1 and z_2 that is perpendicular to $\{|z| = 1\}$. The arc of the circle between z_1 and z_2 will be called the geodesic segment between z_1 and z_2. The hyperbolic distance between z_1 and z_2,

$$d(z_1, z_2) = \frac{1}{2}\log\left(1 + \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|\right) / \left(1 - \left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right|\right), \tag{3}$$

is a monotone function of $|z_1 - z_2|/|1 - \bar{z}_1 z_2|$.

We denote by $\rho(\lambda)$ the hyperbolic capacity of a geodesic segment of hyperbolic length λ. It is given by

$$\rho(\lambda) = \exp\left[-\frac{\pi}{2} K(\kappa')/K(\kappa)\right],$$

where

$$\kappa = \tanh \lambda = (e^{\lambda} - e^{-\lambda})/(e^{\lambda} + e^{-\lambda}).$$

Here $K(\kappa)$ denotes the complete elliptic integral of first kind and $\kappa' = (1 - \kappa^2)^{1/2}$. Let $\lambda = \lambda(\rho)$ be the inverse function. Then

$$\begin{aligned} \lambda(\rho) &= 4\rho + O(\rho^2) \quad \text{as } \rho \to 0, \\ \lambda(\rho) &= \frac{1}{4}\pi^2(1 - \rho)^{-1} + O(1) \quad \text{as } \rho \to 1. \end{aligned} \tag{4}$$

In Section 2, it will be proved that the hyperbolic capacity is not increased by certain projections. As a consequence, we obtain a new proof and a generalization of the following theorem due to Grötzsch [4]: *If* E *is a continuum in the unit disk which contains the points* 0 *and* a, *then the modulus of the ring-region between* E *and* $\{|z| = 1\}$ *becomes smallest if* E *is the segment* $[0, a]$.

In Sections 2 and 3, the hyperbolic length $\Lambda_0(E)$ of the shortest curve enclosing the given continuum $E \subset G$ is estimated from below and from above in terms of $\rho = \text{caph}\, E$. We shall find, for instance, that

$$\Lambda_0(E) \Big/ \frac{2\pi\rho}{1 - \rho^2}$$

lies between two positive absolute constants.

2. PROJECTIONS

A subset H of G is hyperbolically convex if the geodesic segment between any two points of H lies in H. Let K be a compact hyperbolically convex subset of G. For a point $z \in G$ we define its hyperbolic projection z^* on K as the point in K that is nearest to z in the hyperbolic metric. The projection E^* of E on K is defined as the set of the projections z^* of the points $z \in E$. The concept is invariant under conformal mapping of G.

THEOREM 1. *Let* E *be a compact set in* G, *and let* E^* *be its projection on the hyperbolically convex set* K. *Then*

$$\operatorname{caph} E^* \leq \operatorname{caph} E.$$

This is a generalization of a result about the Euclidean projection of the logarithmic capacity [6, Th. 1.]. We shall first prove three lemmas.

LEMMA 2. *If* $z_j = x_j + iy_j$, $|z_j| < 1$, *then*

$$\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| \geq \left|\frac{x_1 - x_2}{1 - x_1 x_2}\right|.$$

Proof. Computation shows that

$$\begin{aligned} &|z_1 - z_2|^2(1 - x_1x_2)^2 - |1 - \bar{z}_1 z_2|^2(x_1 - x_2)^2 \\ &\quad = (1 - |z_1|^2)(x_1 - x_2)^2 y_2^2 + (1 - |z_2|^2)(x_1 - x_2)^2 y_1^2 + (1 - x_1^2)(1 - x_2^2)(y_1 - y_2)^2 \\ &\quad + (x_1 - x_2)^2 y_1^2 y_2^2 \geq 0. \end{aligned}$$

LEMMA 3. *Let* K *be a hyperbolically convex set in* $\{|z| < 1\}$, *and let* z_j^* *be the projection of* z_j $(|z_j| < 1)$ *on* K. *Then*

$$\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| \geq \left|\frac{z_1^* - z_2^*}{1 - \bar{z}_1^* z_2^*}\right|. \tag{5}$$

Proof. Since both sides of (5) are invariant under linear transformations of $\{|z| < 1\}$, we may assume that $z_1^* = -a$, $z_2^* = a$. The geodesic segment between $-a$ and a is then the real segment $[-a, a]$. Because K is hyperbolically convex, it follows that $[-a, a] \subset K$. Since $-a = z_1^*$ is the point of K nearest to z_1, we easily see that $x_1 = \Re z_1 \leq -a$. Similarly, $x_2 = \Re z_2 \geq a$. Therefore

$$d(x_1, x_2) \geq d(-a, a) = d(z_1^*, z_2^*).$$

By Lemma 2, $d(z_1, z_2) \geq d(x_1, x_2)$, which implies inequality (5).

LEMMA 4. *Let* E *and* F *be two compact subsets of* $\{|z| < 1\}$. *If there is a mapping* $w = \phi(z)$ *from* E *onto* F *such that*

$$\left|\frac{\phi(z) - \phi(z')}{1 - \overline{\phi(z)}\,\phi(z')}\right| \leq \left|\frac{z - z'}{1 - \bar{z} z'}\right| \tag{6}$$

for all $z \in E$, $z' \in E$, *then* $\operatorname{caph} F \leq \operatorname{caph} E$.

Proof. Choose $w_1, \cdots, w_n$ in F so that

$$\prod_{\mu \neq \nu}\prod \left|\frac{w_\mu - w_\nu}{1 - \bar{w}_\mu w_\nu}\right|$$

has its maximum possible value. Because $\phi(z)$ maps E onto F, there exist points $z_\nu \in E$ such that $w_\nu = \phi(z_\nu)$. Now (6) and (1) immediately imply that $\operatorname{caph} F \leq \operatorname{caph} E$.

Proof of Theorem 1. We consider the mapping of E onto E^* that assigns to each $z \in E$ its projection $z^* \in E^*$. Because of Lemma 3 we may apply Lemma 4. Doing so, we obtain the inequality $\operatorname{caph} E^* \leq \operatorname{caph} E$.

THEOREM 2. *Let* E *be a compact subset of* G *with* $\rho = \operatorname{caph} E$. *Let* b *be the hyperbolic (linear) measure of the hyperbolic projection of* E *on a given geodesic. Then* $b \leq \lambda(\rho)$, *and equality holds if* E *is an arc of the geodesic.*

By the remark made after Lemma 1 and by (4), this is a generalization of a theorem of Pólya [5], who proved that the Euclidean projection of a compact plane set F on a line has measure no greater than $4 \operatorname{cap} E$.

Theorem 2 also generalizes a result of Grötzsch [4], which is (by (2)) equivalent to the following theorem: *Of all connected compact sets of given hyperbolic capacity, the segment has greatest hyperbolic diameter.*

There is a corresponding theorem for Euclidean (orthogonal) projections: *If* E *is a compact set in* $\{|z| < 1\}$ *and* E' *its Euclidean projection on the real axis, then* $\operatorname{caph} E' \leq \operatorname{caph} E$, *and* E' *has hyperbolic measure no greater than* $\lambda(\rho)$. This is another generalization of Pólya's theorem. The proof uses Lemma 2 and 4 and is otherwise analogous to the following proof.

Proof of Theorem 2. We may assume that $G = \{|z| < 1\}$ and that E is projected on the real axis. Let P be the projection, let $L(x)$ be the hyperbolic measure of $[-1, x] \cap P$, and let $\phi(x) = \tanh L(x)$. Then

$$L(x) = \frac{1}{2}\log(1 + \phi(x))/(1 - \phi(x)).$$

For $x_1 < x_2$ (x_1 and $x_2 \in P$) the hyperbolic measure of $[x_1, x_2] \cap P$ is

$$L(x_2) - L(x_1) = \frac{1}{2}\log\left(1 + \frac{\phi(x_2) - \phi(x_1)}{1 - \phi(x_2)\phi(x_1)}\right) \Big/ \left(1 - \frac{\phi(x_2) - \phi(x_1)}{1 - \phi(x_2)\phi(x_1)}\right).$$

On the other hand, this measure is no greater than $d(x_1, x_2)$. It follows that

$$\frac{\phi(x_2) - \phi(x_1)}{1 - \phi(x_2)\phi(x_1)} \leq \frac{x_2 - x_1}{1 - x_2 x_1} \tag{7}$$

The function $\phi(x)$ maps P onto the segment $[0, \tanh b]$. In view of (7), we can apply Lemma 4. Using Theorem 1, we obtain the inequalities

$$\operatorname{caph}[0, \tanh b] \leq \operatorname{caph} P \leq \operatorname{caph} E = \rho.$$

Since $[0, \tanh b]$ has hyperbolic length b, this completes the proof of Theorem 2.

3. LOWER ESTIMATES OF LENGTH BY HYPERBOLIC CAPACITY

THEOREM 3. *If* C *is a curve in* G *of hyperbolic length* Λ *and* $\rho = \operatorname{caph} C$, *then* $\Lambda \geq \lambda(\rho)$. *Equality holds if* C *is a geodesic segment.*

This is a generalization of the result [6, Th. 3] that every plane curve C has length at least $4 \operatorname{cap} C$.

Proof. Let $z = \chi(L)$ be a parametrization of C in terms of the hyperbolic length L. The function

$$\phi(\xi) = \chi\left(\frac{1}{2}\log(1+\xi)/(1-\xi)\right)$$

maps $[0, \tanh \Lambda]$ onto C. If $\xi_1 < \xi_2$, $z_1 = \phi(\xi_1)$, and $z_2 = \phi(\xi_2)$, then

$$d(z_1, z_2) \leq L_2 - L_1 = \frac{1}{2}\log\frac{1+\xi_2}{1-\xi_1}\cdot\frac{1-\xi_2}{1+\xi_1} = \frac{1}{2}\log\left(1+\frac{\xi_2-\xi_1}{1-\xi_2\xi_1}\right)\Big/\left(1-\frac{\xi_2-\xi_1}{1-\xi_2\xi_1}\right),$$

which by (3) implies that

$$\left|\frac{z_2-z_1}{1-\bar{z}_1 z_2}\right| \leq \frac{\xi_2-\xi_1}{1-\xi_2\xi_1}.$$

Therefore, an application of Lemma 4 yields the inequality

$$\rho = \operatorname{caph} C \leq \operatorname{caph}[0, \tanh\Lambda] = \rho(\Lambda).$$

We shall consider now only the case where E is connected. Let $\Lambda_0(E)$ denote the infimum of the hyperbolic lengths of the closed curves that separate E from ∂G. Thus $\Lambda_0(E)$ is the hyperbolic perimeter of the hyperbolically convex hull of E. Let $K(\kappa)$ again be the complete elliptic integral of the first kind, and let $\kappa' = (1-\kappa^2)^{1/2}$.

THEOREM 4. *Suppose* E *is connected. If* κ *is determined from the relation*

(8) $$\rho = \operatorname{caph} E = \exp[-\pi K(\kappa)/K(\kappa')],$$

then

(9) $$\Lambda_0(E) \geq \frac{1}{2}(1-\kappa)K(\kappa').$$

The quantity $\Lambda_0(E)$ is invariant under conformal mapping of G (not under conformal mapping of the doubly connected region between E and ∂G). Therefore we may assume that $G = \{|z| < 1\}$. We shall compare $\Lambda_0(E)$ with the hyperbolic length $2\pi\rho/(1-\rho^2)$ of the circle $\{|z| = \rho\}$, which has hyperbolic capacity ρ. For $0 < \rho < 1$, let

$$q(\rho) = \inf \Lambda_0(E)\Big/\frac{2\pi\rho}{1-\rho^2},$$

where the infimum is taken over all connected compact sets E with $\operatorname{caph} E = \rho$, and where

$$q(0) = \liminf_{r\to 0} q(r), \quad q(1) = \limsup_{r \to 1} q(r).$$

Then $q(\rho) \leq 1$ always. It is possible that actually $q(\rho) \equiv 1$. Theorem 4 only gives the following result.

COROLLARY 1.

$$q(0) = 1, \quad 1 \geq q(1) \geq \pi/4.$$

Computation shows that, for instance,

$$q(0.25) > 0.94, \quad q(0.50) > 0.84, \quad q(0.75) > 0.79.$$

LEMMA 5. *Let* $z = f(s)$ *be the univalent function that maps* $\rho < |s| < 1$ *onto the doubly connected region between* E *and* $\{|z| = 1\}$, *and for real* t *let*

$$g(s, t) = \frac{i}{2} e^{it} \frac{f'(e^{it})}{f(e^{it})} \cdot \frac{f(e^{it}) + f(s)}{f(e^{it}) - f(s)}. \tag{10}$$

If t *is fixed,* $g(s, t)$ *is univalent and analytic if* $\rho < |s| < 1/\rho$ *except for a simple pole at* $s = e^{it}$ *where*

$$g(s, t) = \frac{i}{2} \frac{e^{it} + s}{e^{it} - s} + b_0 + \cdots. \tag{11}$$

Also, $g(s, t)$ *is real for* $|s| = 1$. *If* $L(r, t)$ *is the Euclidean length of the set* $\{g(s, t): |s| = r\}$ *and if* $\Lambda(r)$ *is the hyperbolic length of* $\{f(s): |s| = r\}$, *then for* $\rho < r < 1$,

$$\Lambda(r) = \frac{1}{2\pi} \int_0^{2\pi} L(r, t)\, dt.$$

Proof. By the Schwarz reflection principle, $g(s, t)$ is univalent and meromorphic in $\{\rho < |s| < 1/\rho\}$. Hence $g(s, t)$ has only a pole at $s = e^{it}$, and (11) follows by computation. By (10), the image of $\{|s| = 1\}$ under $g(s, t)$ is a straight line that has to be the real axis by (11). From (10), we obtain the identity

$$g'(s, t) = \frac{\partial}{\partial s} g(s, t) = \frac{ie^{it} f'(e^{it}) f'(s)}{(f(e^{it}) - f(s))^2};$$

hence

$$\begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} L(r, t)\, dt &= \frac{1}{2\pi} \int_0^{2\pi} r \int_0^{2\pi} |g'(re^{i\theta}, t)|\, d\theta\, dt \\ &= r \int_0^{2\pi} \left(\frac{1}{2\pi} \int_0^{2\pi} \frac{|f'(e^{it})|}{|f(e^{it}) - f(re^{i\theta})|^2}\, dt \right) |f'(re^{i\theta})|\, d\theta. \end{aligned} \tag{12}$$

Since $|f(e^{it})| = 1$ and since $f(e^{it})$ is univalent, we may substitute $e^{i\tau}$ for $f(e^{it})$. Because $|f(re^{i\theta})| < 1$, we find that

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{|f'(e^{it})|}{|f(e^{it}) - f(re^{i\theta})|^2}\, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{|e^{i\tau} - f(re^{i\theta})|^2}\, d\tau = \frac{1}{1 - |f(re^{i\theta})|^2}.$$

Hence (12) implies that

$$\frac{1}{2\pi} \int_0^{2\pi} L(r, t)\, dt = r \int_0^{2\pi} \frac{|f'(re^{i\theta})|}{1 - |f(re^{i\theta})|^2}\, d\theta = \Lambda(r).$$

Let A be a plane compact set. Its analytic capacity (see, for example, [1] or [7]) is defined to be

$$\alpha = \alpha(A) = \sup_g |a_1|,$$

where the supremum is taken over all functions $g(\zeta) = a_0 + a_1\zeta^{-1} + \cdots$ analytic in the exterior of A that satisfy the inequality $|g(\zeta)| \leq 1$.

LEMMA 6. *If* $0 \leq \rho < 1$,

$$A = \left\{ \left| \zeta - i\frac{1+\rho^2}{1-\rho^2} \right| \leq \frac{\rho}{1-\rho^2} \right\} \cup \left\{ \left| \zeta + i\frac{1+\rho^2}{1-\rho^2} \right| \leq \frac{\rho}{1-\rho^2} \right\}$$

has analytic capacity $\alpha = (2\pi)^{-1}(1-\kappa)K(\kappa')$, *where* κ *is defined by* (8).

Proof. The function

$$\text{(13)} \qquad w = \frac{\pi}{K(\kappa')}\int_1^{1/\omega} [(1-u^2)(1-\kappa^2u^2)]^{-1/2}\,du = \frac{\pi}{K(\kappa')}\frac{1}{\omega} + \cdots$$

maps the w-plane, cut along $[-1, +1]$, onto the rectangle

$$\{ |\Re w| < \pi K(\kappa)/K(\kappa'),\ |\Im w| < \pi \},$$

where $[-1, -\kappa]$ is mapped onto the left side of the rectangle, $[-\kappa, -\kappa]$ onto the lower and upper sides, and $[\kappa, 1]$ onto the right side; see for instance [2, p. 17]. Therefore

$$\zeta = \frac{1}{2}i(1+e^w)/(1-e^{-w})$$

maps the ω-plane, cut along $[-1, -\kappa]$ and $[\kappa, 1]$, onto the exterior region H of A. Near $\omega = \infty$ (13) implies that

$$\zeta = \frac{1}{2}i(2+\cdots)/(\pi K(\kappa')^{-1}\omega^{-1}+\cdots) = i\pi^{-1}K(\kappa')\omega + \cdots.$$

Hence the function

$$\text{(14)} \qquad \omega^* = i\pi^{-1}K(\kappa')\omega = \zeta + \cdots$$

maps H onto the ω^*-plane, cut along two segments that lie on the imaginary axis and that have total length $2\pi^{-1}(1-\kappa)K(\kappa')$. Since the analytic capacity of a linear set is one quarter of its measure [7] and since the analytic capacity is invariant under a conformal mapping of the form (14), it follows that A has analytic capacity $\alpha = (2\pi)^{-1}(1-\kappa)K(\kappa')$.

Proof of Theorem 4. (a) Let us assume first that the function f(s) of Lemma 5 is analytic and univalent for $|s| = \rho$. Since g(s, t) is real for $|s| = 1$, the annulus $\{\rho < |s| < \rho^{-1}\}$ is mapped by g onto a region symmetric with respect to the real axis. Its complement consists of two compact components, F_t and its conjugate $\overline{F}_t$. The transformation

$$\zeta = \frac{1}{2} i(e^{it} + s)/(e^{it} - s)$$

maps $\{\rho < |s| < \rho^{-1}\}$ onto the exterior region H of the set A of Lemma 6. Let $s = s(\zeta)$ be the inverse of this transformation. Then $g(s(\zeta), t)$ is univalent in H and analytic, except for a pole at $\zeta = \infty$ where $g(s(\zeta), t) = \zeta + \cdots$ by (11). It follows that $F_t \cup \overline{F}_t$ has the same analytic capacity α as A. Therefore [3] the (Euclidean) perimeter of $F_t \cup \overline{F}_t$ is at least $2\pi\alpha$, and the perimeter $L(\rho, t)$ of F_t is at least $\pi\alpha$. From Lemma 5, it follows that

$$\Lambda(1) \geq \pi\alpha . \tag{15}$$

(b) Given $\varepsilon > 0$, let C be a closed analytic curve of hyperbolic length less than $\Lambda_0(E) + \varepsilon$ that separates E and $\partial G = \{|z| = 1\}$. By (15), the hyperbolic perimeter of C is at least $\pi\beta$, where β has the same relation to caph C as α has to caph E. Hence $\Lambda_0 + \varepsilon > \pi\beta$. Since E lies in the interior of C, we see that caph C $\geq$ caph E and therefore that $\beta \geq \alpha$ and $\Lambda_0 + \varepsilon > \pi\alpha$ for every $\varepsilon > 0$.

4. UPPER ESTIMATES OF LENGTH BY HYPERBOLIC CAPACITY

THEOREM 5. *If* E *is connected, then*

$$\Lambda_0(E) \leq \inf_{\rho < r < 1} \frac{2\pi}{(2 \log r/\rho)^{1/2}} \left[\sum_{n=1}^{\infty} \frac{nr^{6n}}{1 - r^{4n}} + \frac{r^2}{(1 - r^2)^2} \right]^{1/2}, \tag{16}$$

and also

$$\Lambda_0(E) \leq \frac{\pi^2 \rho^{1/2}}{\log 1/\rho}. \tag{17}$$

The first inequality is better than the second one for small ρ; the second one is better than the first for ρ near to 1. Inequality (17) follows almost immediately from known principles. We could have obtained a slightly better bound. Again, taking $G = \{|z| < 1\}$, we compare $\Lambda_0(E)$ with the hyperbolic length of $\{|z| = \rho\}$. For $0 < \rho < 1$, let

$$Q(\rho) = \sup \Lambda_0(E) / \frac{2\pi\rho}{1 - \rho^2},$$

where the supremum is taken over all connected compact E with caph E = ρ, and let

$$Q(0) = \limsup_{r \to 0} Q(r), \quad Q(1) = \limsup_{r \to 0} Q(r).$$

Then

$$1.31 < Q(0) < 1.46 .$$

This is only another way of writing some results on the logarithmic capacity of a plane set [6, Th. 5], as Lemma 1 shows. By computation, it follows from Theorem 5 that, for instance,

$$Q(0.25) < 1.85, \quad Q(0.50) < 2.50, \quad Q(0.75) < 3.19,$$

where inequality (16) is used for $\rho = 0.25$ and 0.50 and inequality (17) is used for $\rho = 0.75$. For $\rho = 1$, Theorem 5 implies bounds for $Q(1)$.

COROLLARY 2.

$$\frac{\pi}{2} \leq Q(1) \leq \pi.$$

The case where E is a geodesic segment gives the lower bound, as (4) shows. The upper bound follows immediately from (17).

Proof of Theorem 5. (a) We shall first prove (16). By Lemma 5, the function

$$g(s, t) - \frac{i}{2}(e^{it} + s)/(e^{it} - s)$$

is analytic if $\rho < |s| < 1/\rho$, and is real for $|s| = 1$. Hence, its Laurent expansion has the form

$$g(s, t) - \frac{i}{2}\frac{e^{it} + s}{e^{it} - s} = \sum_{n=1}^{\infty} \bar{b}_n s^{-n} + b_0 + \sum_{n=1}^{\infty} b_n s^n.$$

For $\rho < |s| < 1$ this implies that

$$g(s, t) = \sum_{n=1}^{\infty} \bar{b}_n s^{-n} + \left(\frac{1}{2}i + b_0\right) + \sum_{n=1}^{\infty} (b_n + ie^{-int})s^n.$$

Therefore the region enclosed by the curve $\{g(s, t): |s| = r\}$ has (Euclidean) area

$$\begin{aligned} A(r, t) &= -\pi \sum_1^{\infty} n|b_n|^2 r^{-2n} + \pi \sum_1^{\infty} n|b_n + ie^{-int}|^2 r^{2n} \\ &\leq -\pi \sum_1^{\infty} n|b_n|^2 r^{-2n} + \pi \sum_1^{\infty} n|b_n|^2 r^{2n} + 2\pi \sum_1^{\infty} n|b_n|r^{2n} + \frac{\pi r^2}{(1 - r^2)^2}. \end{aligned} \tag{18}$$

Schwarz's inequality implies that

$$\sum_1^{\infty} n|b_n|r^{2n} \leq B(r)\left(\sum_1^{\infty} \frac{nr^{4n}}{r^{-2n} - r^{2n}}\right)^{1/2}, \quad \text{where } B(r)^2 = \sum_1^{\infty} n|b_n|^2 (r^{-2n} - r^{2n}).$$

Therefore (18) yields the result

$$A(r, t) \leq \pi \sum_1^{\infty} \frac{nr^{6n}}{1 - r^{4n}} + \frac{\pi r^2}{(1 - r^2)^2}. \tag{19}$$

Let $L(r, t)$ again be the Euclidean length of $\{g(s, t): |s| = r\}$. Applying Schwarz's inequality, we find that

$$\int_\rho^r L(u, t)^2 u^{-1}\, du = \int_\rho^r \left(\int_0^{2\pi} |g'(ue^{i\theta})|\, d\theta\right)^2 u\, du$$

$$\leq 2\pi \int_\rho^r \int_0^{2\pi} |g'(ue^{i\theta}, t)|^2 u\, d\theta\, du$$

$$= 2\pi(A(r, t) - A(\rho, t)) \leq 2\pi A(r, t)$$

because $g(s, t)$ is univalent. By Lemma 5 and Schwarz's inequality, we thus obtain the result

$$\int_\rho^r \Lambda(r)^2 u^{-1}\, du = \int_\rho^r \left(\frac{1}{2\pi}\int_0^{2\pi} L(u, t) dt\right)^2 u^{-1}\, du$$

$$\leq \frac{1}{2\pi}\int_\rho^r \int_0^{2\pi} L(u, t)^2 u^{-1}\, dt\, du$$

$$\leq \int_0^{2\pi} A(r, t)\, dt.$$

Since $\Lambda_0(E) \leq \inf_{\rho<u<1} \Lambda(u)$, it follows that

$$\Lambda_0^2 \log r/\rho \leq \int_0^{2\pi} A(r, t)\, dt,$$

and (16) may now be obtained from (19).

(b) To establish (17) we shall prove that for $|s| = \rho^{1/2}$,

$$\frac{|f'(s)|}{1 - |f(s)|^2} \leq \frac{\pi}{2\rho^{1/2} \log 1/\rho}.$$

We may assume $s = \rho^{1/2}$. The function

$$\phi(\zeta) = \exp\left[\left(\frac{i}{\pi}\log\frac{1+\zeta}{1-\zeta} - \frac{1}{2}\right)\log 1/\rho\right] = \rho^{1/2}\left(1 + \frac{2i}{\pi}\log 1/\rho \cdot \zeta + \cdots\right)$$

is analytic for $|\zeta| < 1$ and maps $\{|\zeta| < 1\}$ onto the universal covering surface of $\{\rho < |z| < 1\}$. Hence, the function h,

$$h(\zeta) = \frac{f(\phi(\zeta)) - f(\rho^{1/2})}{1 - \overline{f(\rho^{1/2})}\, f(\phi(\zeta))} = \frac{f'(\rho^{1/2})}{1 - |f(\rho^{1/2})|^2}\cdot\frac{2i\rho^{1/2}}{\pi}\log 1/\rho \cdot \zeta + \cdots,$$

is analytic for $|\zeta| < 1$ and satisfies the inequality $|h(\zeta)| < 1$. By Schwarz's lemma the coefficient of ζ has absolute value no greater than 1.

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