HOLOMORPHIC FUNCTIONS, OF ARBITRARILY SLOW GROWTH, WITHOUT RADIAL LIMITS

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By the well-known theorem of Fatou, if f(z) is holomorphic and bounded in $|z| < 1$ then f(z) possesses radial limits almost everywhere. This result was extended by Nevanlinna to meromorphic functions of bounded characteristic T(r) [4, p. 189]. A natural question raised by Lohwater and Piranian [2, p. 16] is this: if the condition of boundedness of T(r) be relaxed to the requirement that $T(r) < q(r)$, where $q(r) \to \infty$ slowly enough, can one still conclude that *some* radial limits must exist? Bagemihl, Erdös and Seidel [1, Theorem 7] have given an example of a *holomorphic* function without a radial limit for which $T(r) = O((1 - r)^{-8})$. Lohwater and Piranian [2] gave an example of a *meromorphic* function without radial limit for which $T(r) = O(-\log(1 - r))$. See also Noshiro [5, p. 90]. Mac Lane [3] gave an example of a *meromorphic* function, of arbitrarily slow growth, without asymptotic value (and hence without radial limit). The purpose of the present note is to derive a similar result for *holomorphic* functions. The method of proof and the precise statement of the result are different in the holomorphic case, since a holomorphic function must possess at least one asymptotic value (along some curve, not necessarily along some radius). For that reason the construction used in our example for meromorphic functions is completely inapplicable.

Let C_{-1} and C_1 be two fixed disjoint compact simple arcs in $|\zeta| < 1$, neither of which contains the origin, and such that each radius of $|\zeta| < 1$ intersects both C_{-1} and C_1. For example, we may use the two arcs $2\pi \leq \arg \zeta \leq 4\pi$ and $6\pi \leq \arg \zeta \leq 8\pi$ of the spiral $|\zeta| = 1 - (\arg \zeta)^{-1}$.

LEMMA 1. *There exists a function* $\phi(\zeta)$, *holomorphic in* $|\zeta| < 1$, *and a constant* $M > 1$ *such that*

$$|\phi(\zeta)| \leq M|\zeta| \qquad (|\zeta| < 1) \tag{1}$$

and

$$\begin{cases} \Re\, \phi(\zeta) \leq -1 & (\zeta \in C_{-1}), \\ \Re\, \phi(\zeta) \geq 1 & (\zeta \in C_1). \end{cases} \tag{2}$$

Proof. The three sets $\{0\}$, C_{-1} and C_1 may be enclosed in simply-connected neighborhoods, D_0, D_{-1}, D_1, whose closures are disjoint. Define the function $\phi_0(\zeta)$, holomorphic in $D_0 \cup D_{-1} \cup D_1$, by

$$\phi_0(\zeta) = 0 \ (\zeta \in D_0), \qquad \phi_0(\zeta) = -3 \ (\zeta \in D_{-1}), \qquad \phi_0(\zeta) = 3 \ (\zeta \in D_1).$$

Then, by Runge's theorem (see for example [6, p. 15]), there exists a polynomial $P(\zeta)$ approximating $\phi_0(\zeta)$ well enough so that

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$$|P(0)| < 1,$$

$$|P(\zeta) + 3| < 1 \qquad (\zeta \in C_{-1}),$$

$$|P(\zeta) - 3| < 1 \qquad (\zeta \in C_1).$$

Set $\phi(\zeta) = P(\zeta) - P(0)$ and let M denote the maximum of $|\phi(\zeta)|$ on $|\zeta| = 1$. Then (1) follows from Schwarz' lemma. Also, since $|P(0)| < 1$,

$$|\phi(\zeta) + 3| = |P(\zeta) + 3 - P(0)| \leq |P(\zeta) + 3| + |P(0)|$$
$$< |P(\zeta) + 3| + 1 < 1 + 1 = 2 \qquad (\zeta \in C_{-1}).$$

Thus $\Re\phi(\zeta) < -1$ on C_{-1}, and the first equation of (2) follows. The second is proved in a similar fashion.

THEOREM 1. *Let* $\mu(r)$ *be a function on* $[0, 1)$ *satisfying*

(3) $$0 < \mu(r) \uparrow \infty \qquad (r \uparrow 1).$$

Then there exists a function f(z), *holomorphic in* $|z| < 1$, *satisfying*

(4) $$|f(z)| \leq \mu(r) \qquad (0 \leq r = |z| < 1),$$

and, for all θ,

(5) $$\limsup_{r \to 1} \Re f(re^{i\theta}) = +\infty, \qquad \liminf_{r \to 1} \Re f(re^{i\theta}) = -\infty.$$

Proof. We shall construct f(z) as a series

(6) $$f(z) = \sum_{n=1}^{\infty} A_n \phi(z^{\lambda_n}),$$

where $\phi(\zeta)$ denotes the function of Lemma 1. Here the A_n will be positive constants and the λ_n positive integers, which we determine inductively. Once λ_n is determined, we shall denote by K_{-n} and K_n, respectively, the preimages of C_{-1} and C_1 under the map $z \to z^{\lambda_n}$. Also, by ρ_n we shall denote the minimum ρ such that K_{-n} and K_n are contained in $|z| \leq \rho < 1$. Note that if $\{\lambda_n\}$ is an increasing sequence, then $\{\rho_n\}$ increases.

Let $A_1 = 1$, and choose λ_1 so that

(7) $$A_1 |\phi(z^{\lambda_1})| < 2^{-1} \mu(|z|) \qquad (|z| < 1),$$

which is possible by (1) and (3). Once $A_1, \lambda_1, A_2, \lambda_2, \cdots, A_{n-1}, \lambda_{n-1}$ have been determined, choose A_n so that

(8) $$\Re \sum_{\nu=1}^{n} A_\nu \phi(z^{\lambda_\nu}) \begin{cases} < -n & (z \in K_{-n}) \\ > n & (z \in K_n). \end{cases}$$

That such a choice of A_n is possible, and doesn't depend on λ_n which is still to be chosen, is easily seen: (8) follows from (1) and (2) provided

$$A_n > n + M \sum_{\nu=1}^{n-1} A_\nu .$$

Then choose $\lambda_n > \lambda_{n-1}$ large enough so that

$$A_n |\phi(z^{\lambda_n})| < 2^{-n} \qquad (|z| \leq \rho_{n-1}) \tag{9}$$

and

$$A_n |\phi(z^{\lambda_n})| < 2^{-n} \mu(|z|) \qquad (|z| < 1), \tag{10}$$

which is possible by (1) and (3).

Clearly it follows from (10) that $f(z)$ is holomorphic in $|z| < 1$ and satisfies (4). Recalling the definition of ρ_n and the fact that $\rho_n \uparrow$ (since $\lambda_n \uparrow$), we see from (8) and (9) that, for $z \in K_n$,

$$\Re f(z) > n - \sum_{\nu=n+1}^{\infty} 2^{-\nu} > n - 1 .$$

Considering K_{-n} in a similar fashion and noting that each radius of $|z| < 1$ meets every $K_{\pm n}$, we see that (5) follows and Theorem 1 is proved.

THEOREM 2. *Let* $p(r)$ *be a function on* $[0, 1)$ *satisfying* $0 < p(r) \uparrow \infty$. *Then there exists a function* $F(z)$, *holomorphic in* $|z| < 1$, *such that*

$$|F(z)| \leq p(|z|) \qquad (0 \leq |z| < 1), \tag{11}$$

and, for every θ,

$$\limsup_{r \to 1} |F(re^{i\theta})| = +\infty, \qquad \liminf_{r \to 1} |F(re^{i\theta})| = 0 . \tag{12}$$

Also, $F(z)$ *has no zeros in* $|z| < 1$.

Remark. That F has no radial limits is a trivial consequence of (12).

Proof. Assume for the moment that $p(0) > 1$. Take $\mu(r) = \log p(r)$, and let $f(z)$ be the corresponding function of Theorem 1. Set $F(z) = e^{f(z)}$. Then (11) and (12) are simple consequences of (4) and (5). If $p(0) \leq 1$, use $p^*(r) = \max(2, p(r)) \leq 2p(r)/p(0)$ to obtain a function $F^*(z)$, and set $F(z) = p(0) F^*(z)/2$.

Remark. We have constructed $F(z)$ so that its maximum modulus, $M(r)$, satisfies the condition $M(r) \leq p(r)$. As is well known [4, p. 220], $T(r) \leq \log M(r)$, hence we can clearly replace (11) by a condition of the form $T(r) \leq q(r)$.

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