A CHARACTERIZATION OF THE ANALYTIC OPERATOR AMONG THE LOEWNER-BENSON OPERATORS

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1. INTRODUCTION

C. Loewner [1] considered integral operators of the type

(1) $$y(t) = -\int_0^{2\pi} K(s)\,x(t-s)\,ds = -\int_0^{2\pi} K(t-s)\,x(s)\,ds,$$

where $K(t)$ is L-integrable on the interval $[0, 2\pi]$ and $x(t)$ ranges over the continuous 2π-periodic functions. He gave necessary and sufficient conditions that such operators generate only curves $\{x(t), y(t)\}$ of non-negative circulation, that is, curves whose index relative to any point not on them is non-negative. His conditions are

(a) $K(t)$ is (possibly after a change in its values on a set of measure zero) analytic in the open interval $(0, 2\pi)$ and

(b) $K'(t)$ can be represented, in the interval $(0, 2\pi)$, by a Laplace-Stieltjes integral

$$K'(t) = \int_{-\infty}^{\infty} e^{-rt}\,d\mu(r),$$

where $\mu(t)$ is a non-decreasing function.

D. C. Benson [2, 3] extended Loewner's result to include the case where $K(t)$ is not necessarily L-integrable on the closed interval $[0, 2\pi]$ but is such that the Cauchy Principal Value $P\int_0^{2\pi} K(t)\,dt$ exists. For a certain class of continuous periodic functions, he showed that, in order that the operator (1) (with the integral understood as a Cauchy Principal Value) generate only curves of non-negative circulation, it is again necessary and sufficient that conditions (a) and (b) hold.

Among the kernels which fall into Benson's class is the kernel $K(t) = -\cot t/2$. This kernel corresponds to what we have called the analytic operator—the operator that relates, on the boundary of the unit disk, the real and imaginary parts of a function continuous on the closed disk and analytic on the interior. The analytic operator

$$y(t) = P\int_0^{2\pi} \cot\frac{s}{2}\; x(t-s)\,ds$$

has the property that if $m(t)$ is a continuous mapping of the line onto itself induced by a one-to-one conformal map of the closed disk onto itself, then

Received June 1, 1961.

$$P\int_0^{2\pi} \cot\frac{s}{2}\; x[m(t) - s]\,ds - P\int_0^{2\pi} \cot\frac{s}{2}\; x[m(t - s)]\,ds = \text{const.}$$

This is a consequence of the fact that the pairs $\{x(m(t)),\ y(m(t))\}$ and

$$\left\{ x(m(t)),\ P\int_0^{2\pi} \cot\frac{s}{2}\; x[m(t - s)]\,ds \right\}$$

are the real and imaginary parts of analytic functions whose real parts coincide and whose imaginary parts, as is well known, can therefore differ by at most a constant.

C. J. Titus (private communication) posed the problem of determining whether the property of the analytic operator's "commuting up to a constant" with all such $m(t)$ characterizes it among the Loewner-Benson operators. It does, and in fact a much stronger characterization is possible.

2. THE CHARACTERIZATION

Let $y(t) = -P\int_0^{2\pi} K(s)\,x(t - s)\,ds$ be a Loewner-Benson operator, that is, let $K(t)$ satisfy the following three conditions.

(i) There exist an L-integrable function $\phi(t)$ on the closed interval $[0, 2\pi]$, and an $\alpha > 0$ such that $K(t) = \dfrac{\phi(t)}{t^{\alpha}(2\pi - t)^{\alpha}}$.

(ii) $P\int_0^{2\pi} K(t)\,dt$ exists.

(iii) The operator generates only curves of non-negative circulation, when operating on the class $X[0, 2\pi]$ of 2π-periodic functions that satisfy a Hölder condition of order α on $[0, 2\pi]$.

Let H denote the set of all real-valued continuous functions $h(t)$ defined for all t and satisfying the following four conditions.

(H1) $$h(t + 2\pi) = h(t) + 2\pi,$$

(H2) $$h'(t) > 0,$$

(H3) $$h''(t) \text{ exists},$$

(H4) $$P\int_0^{2\pi} K(s)\,x[h(t) - s]\,ds - P\int_0^{2\pi} K(s)\,x[h(t - s)]\,ds = \text{const.} \quad (\text{in } t).$$

(In [2] it is shown that for a $K(t)$ of the form assumed here, the integral

$$y(t) = P\int_0^{2\pi} K(s)\,x(t - s)\,ds$$

exists for each t whenever x(t) is in $X[0, 2\pi]$. That $X[0, 2\pi]$ contains $x[h(t)]$ and $x[h^{-1}(t)]$ whenever it contains x(t) is a consequence of (H2) and (H3).) It can be readily verified that H is a group under function composition, and that H always contains the translations $h_b(t) = t + b$.

THEOREM. *If* K(t) *is a Loewner-Benson kernel and there exists an* h(t) *in* H *which is not a translation, then* $K(t) = A \cot t/2 + B$ (a.e.), *for some* $A < 0$ *and some* B.

Suppose that h(t) is in H and that h(t) is not a translation. Without loss of generality, we may suppose that $h(0) = 0$ and $h'(0) \neq 1$; for under the assumption that h(t) is not a translation, there must exist a t_0 such that $h'(t_0) \neq 1$. If we let $\hat{h}(t) = h(t + t_0) - h(t_0)$, we find, in the light of the remark made above concerning the translations, that $\hat{h}(t)$ is in H and has the additional properties $\hat{h}(0) = 0$ and $\hat{h}'(0) \neq 1$. It can easily be verified that the set of all h(t) in H having the additional property $h(0) = 0$ is a subgroup of H. We shall denote it by H_0.

Conditions (a) and (b) allow us to assume that, on the open interval $(0, 2\pi)$, K(t) is continuous and K'(t) is continuous and positive. We may also take K(t) to be extended by the formula $K(t) = K(t + 2\pi)$. Finally, for convenience, we absorb the minus sign prefacing the operator into the kernel and regard K'(t) as negative. These preliminaries over, we prove the following:

LEMMA. *If* h(t) *is in* H_0, *then there exists a continuous,* 2π*-periodic function* G(s) *such that* $h'(s)K[h(t) - h(s)] - K(t - s) = G(s)$ *for all* s *and* t $(-\infty < s < +\infty, -\infty < t < +\infty, s \neq t \pmod{2\pi})$.

Proof. Let $F(t, s) = h^{-1\prime}(s)K[h^{-1}(t) - h^{-1}(s)] - K(t - s)$ for $0 \leq t \leq 2\pi$ and $0 \leq s \leq 2\pi$, $0 < |s - t| < 2\pi$. For fixed t $(0 \leq t \leq 2\pi)$, F(t, s) is continuous in s for all s $(0 < s < 2\pi)$ except possibly for $s = t$ or $s - t = 2\pi$, because the only possible discontinuities of K(t) occur at points of the form $2n\pi$. We show first that for any two points t_1 and t_2 in the open interval $(0, 2\pi)$, $F(t_1, s) = F(t_2, s)$ for all s such that $0 < s < 2\pi$ and $t_1 \neq s \neq t_2$. If we suppose the contrary: that there exist an s $(0 < s < 2\pi)$ and points t_1 and t_2 in the interval satisfying $t_1 \neq s \neq t_2$, for which $F(t_1, s) - F(t_2, s) > 0$, we are assured that there exists an interval [a, b], containing s and contained in the open interval $(0, 2\pi)$, such that $F(t_1, s) - F(t_2, s) > 0$ for all s in [a, b]. The interval may be taken to exclude t_1 and t_2.

Let X[a, b] be the subclass of continuous, 2π-periodic functions that satisfy a Hölder condition of order α on the interval [a, b] and which vanish on the complement of (a, b) in $[0, 2\pi]$. It is readily proved that these functions satisfy a Hölder condition of the same order on $[0, 2\pi]$ and thus form a sub-class of $X[0, 2\pi]$. Now, for any x(t) in X[a, b] and for t in the formulas below equal to t_1 or t_2, we may write

$$\int_a^b F(t, s)x(s)\,ds = \int_a^b h^{-1\prime}(s)K[h^{-1}(t) - h^{-1}(s)]x(s)\,ds - \int_a^b K(t - s)x(s)\,ds.$$

A change of variable $s = h(r)$ in the first of the integrals on the right-hand side above produces

$$\int_a^b F(t, s)x(s)\,ds = \int_{h^{-1}(a)}^{h^{-1}(b)} K[h^{-1}(t) - s]x[h(s)]\,ds - \int_a^b K(t - s)x(s)\,ds.$$

Since $x[h(s)]$ vanishes outside of $(h^{-1}(a), h^{-1}(b))$, we have

$$P\int_0^{2\pi} K[h^{-1}(t) - s]x[h(s)]\,ds = \int_{h^{-1}(a)}^{h^{-1}(b)} K[h^{-1}(t) - s]x[h(s)]\,ds.$$

Similarly,

$$P\int_0^{2\pi} K(t - s)\,x(s)\,ds = \int_a^b K(t - s)\,x(s)\,ds.$$

Hence,

$$\int_a^b F(t, s)\,x(s)\,ds = P\int_0^{2\pi} K[h^{-1}(t) - s]x[h(s)]\,ds - P\int_0^{2\pi} K(t - s)\,x(s)\,ds.$$

If we replace the point t in the right-hand side of the above equation by $h(t)$, we do not change the value of the difference of the integrals, and we may also conclude that

$$\int_a^b F(t_1, s)\,x(s)\,ds = \int_a^b F(t_2, s)\,x(s)\,ds \qquad \text{for all } x(t) \text{ in } X[a, b].$$

But, if we choose an $x(t)$ in $X[a, b]$ which is positive on (a, b) and which vanishes on the complement of (a, b) in $[0, 2\pi]$, we find that

$$\int_a^b [F(t_1, s) - F(t_2, s)]\,x(s)\,ds > 0,$$

which contradicts the equation above. This establishes that $F(t_1, s) = F(t_2, s)$ for all s $(0 < s < 2\pi)$ and all t_1 and t_2 in the open interval $(0, 2\pi)$, provided that $t_1 \neq s \neq t_2$. Now, for t such that $0 < t < 2\pi$, since $F(t, s)$ is continuous in s at $s = 0$ and $s = 2\pi$, we have

$$F(t_1, 0) = \lim_{s \to 0} F(t_1, s) = \lim_{s \to 0} F(t_2, s) = F(t_2, 0).$$

Similarly, $F(t_1, 2\pi) = F(t_2, 2\pi)$. This, together with the equality $F(0, s) = F(2\pi, s)$ for $0 < s < 2\pi$, shows that if t_1 and t_2 are any two points in the closed interval $[0, 2\pi]$, then $F(t_1, s) = F(t_2, s)$ for all s $(0 \le s \le 2\pi)$ provided that $0 < |s - t_1| < 2\pi$ and $0 < |s - t_2| < 2\pi$.

We now show that there exists a function $G(s)$, defined and continuous for $0 \le s \le 2\pi$, such that $G(0) = G(2\pi)$ and such that, for each t in the closed interval $[0, 2\pi]$, $G(s) = F(t, s)$ provided that $0 < |s - t| < 2\pi$. We define

$$G(s) = \lim_{s' \to s} F(s, s').$$

That this limit exists can be seen from the following: If we choose $0 < |s'' - s| < 2\pi$, then, because $F(s'', s')$ is continuous in s' at $s' = s$, we have

$$\lim_{s' \to s} F(s, s') = \lim_{s' \to s} F(s'', s') = F(s'', s).$$

Suppose now that t and s are two points in the closed interval $[0, 2\pi]$ with $0 < |s - t| < 2\pi$; then $F(t, s') = F(s, s')$ for $s \neq s' \neq t$, and we see that

$$F(t, s) = \lim_{s' \to s} F(t, s') = \lim_{s' \to s} F(s, s') = G(s).$$

Further, since $\lim_{s \to t} G(s) = \lim_{s \to t} F(t, s) = G(t)$, it is proved that $G(t)$ is continuous. That $G(0) = G(2\pi)$ follows from the property $F(t, s) = G(s)$, condition (H1), and the periodicity of $K(t)$. Because H_0 is a group, we may drop the sign of the inverse in $F(t, s)$, and although we realize that $G(s)$ is determined by the group element for which the equation $F(t, s) = G(s)$ holds, we do not emphasize this and write simply

$$h'(s) K[h(t) - h(s)] - K(t - s) = G(s). \tag{2}$$

This completes the proof of the lemma. From now on we deal only with the functional equation (2).

3. PROOF OF THE THEOREM

In what follows, we suppose that $G(0) = 0$ and $h'(0) < 1$. If this is not the case, we may reduce the general case to this by first setting

$$P = \frac{G(0)}{h'(0) - 1} \qquad \text{and} \qquad \hat{K}(t) = K(t) - P.$$

Then $\hat{K}'(t) < 0$, and the functional equation (2) is satisfied by $\hat{K}(t)$, the original $h(t)$, and $\hat{G}(s) = G(s) - P[h'(s) - 1]$. It is clear that $\hat{G}(s)$ is continuous for all s and that $\hat{G}(0) = G(2\pi) = 0$. We assume, then, that $G(0) = 0$ in (2). Now, if the original $h(t)$ in (2) is such that $h'(0) > 1$, we proceed as follows: we replace s in (2) by $h^{-1}(s)$, and t by $h^{-1}(t)$; and multiplying throughout by $h^{-1\prime}(s)$, we obtain

$$h'[h^{-1}(s)]\, h^{-1\prime}(s) K(t - s) - K[h^{-1}(t) - h^{-1}(s)] h^{-1\prime}(s) = G[h^{-1}(s)] h^{-1\prime}(s).$$

Since $h'[h^{-1}(s)] h^{-1\prime}(s) = 1$, the above equation may be rewritten as

$$h^{-1\prime}(s) K[h^{-1}(t) - h^{-1}(s)] - K(t - s) = - G[h^{-1}(s)] h^{-1\prime}(s).$$

This is a functional equation of the same form as (2) and, since $h^{-1}(0) = 0$, we see that $h'(0) h^{-1\prime}(0) = 1$ and consequently $h^{-1\prime}(0) < 1$. We also note that the right-hand side of the newest functional equation is continuous for all s and vanishes for $s = 0$ and $s = 2\pi$.

Next, we show that $\lim_{t \to 0} t K(t)$ exists and is positive, and that the functional equation

$$h'(t) K[h(t)] - K(t) = G(t) + M h''(t)/h'(t) \tag{3}$$

(where $M = \lim_{t \to 0} t K(t)$) holds for $0 < t < 2\pi$. For each fixed s $(0 \leq s \leq 2\pi)$, the left-hand side of (2) is differentiable (as a function of t) for each t such that $0 \leq t \leq 2\pi$ and $0 < |s - t| < 2\pi$ and

$$h'(t) h'(s) K'[h(t) - h(s)] - K'(t - s) = 0. \tag{4}$$

For each fixed t $(0 \leq t \leq 2\pi)$, the left-hand side of (2) is differentiable (as a function of s) for each s such that $0 \leq s \leq 2\pi$ and $0 < |s - t| < 2\pi$, and

$$h''(s)K[h(t) - h(s)] - [h'(s)]^2 K'[h(t) - h(s)] + K'(t - s) = G'(s). \tag{5}$$

Solving (2) for $K[h(t) - h(s)]$ and (4) for $K'[h(t) - h(s)]$, and substituting in (5), we get

$$\frac{h''(s)}{h'(s)}[G(s) + K(t - s)] - \frac{h'(s)}{h'(t)} K'(t - s) + K'(t - s) = G'(s). \tag{6}$$

After dividing (6) throughout by $h'(s)$, we may write the result as

$$-\frac{h''(s)}{[h'(s)]^2}[G(s) + K(t - s)] + \frac{1}{h'(s)}[G'(s) - K'(t - s)] + \frac{1}{h'(t)} K'(t - s) = 0, \tag{7}$$

from which we conclude that

$$\frac{\partial}{\partial s}\left[\frac{1}{h'(s)}[G(s) + K(t - s)] - \frac{1}{h'(t)} K(t - s)\right] = 0$$

for $0 \leq t \leq 2\pi$, $0 \leq s \leq 2\pi$, $0 < |s - t| < 2\pi$. Hence, for each fixed t $(0 < t < 2\pi)$, the function

$$\phi(s, t) = \frac{1}{h'(s)}[G(s) + K(t - s)] - \frac{1}{h'(t)} K(t - s) \tag{8}$$

is constant in s for $0 \leq s < t$ and for $t < s \leq 2\pi$. Since

$$\phi(2\pi, t) = \phi(0, t) = \frac{1}{h'(0)} K(t) - \frac{1}{h'(t)} K(t), \tag{9}$$

$\phi(s, t) = \phi(0, t)$ for all s $(0 \leq s \leq 2\pi,\ s \neq t)$.

Returning to (2) and setting $s = 0$, we obtain

$$h'(0)K[h(t)] - K(t) = 0. \tag{10}$$

Solving (10) for $K(t)$ and substituting the result for the first member of the right-hand side of (9), we get

$$\phi(0, t) = K[h(t)] - \frac{1}{h'(t)} K(t). \tag{11}$$

Rewriting (8), where it is now known that $\phi(s, t) = \phi(0, t)$, as

$$h'(t)G(s) + [h'(t) - h'(s)]K(t - s) = h'(t)h'(s)\phi(0, t), \tag{12}$$

and substituting (11) in (12), we obtain

$$h'(t)G(s) + [h'(t) - h'(s)]K(t - s) = h'(s)\{h'(t)K[h(t)] - K(t)\}. \tag{13}$$

Now let λ $(0 < \lambda < 2\pi)$ be such that $h''(\lambda) \neq 0$. Should no such λ exist, then $h'(t)$ would be constant on the interval $0 < t < 2\pi$ and therefore constant on the closed interval $0 \leq t \leq 2\pi$. Since $h(0) = 0$ and $h(2\pi) = 2\pi$, it follows that $h(t) = t$ on the closed interval. This has been disallowed. From (13), we have

$$(\lambda - s)K(\lambda - s) = \frac{h'(s)\{h'(\lambda)K[h(\lambda)] - K(\lambda)\} - h'(\lambda)G(s)}{[h'(\lambda) - h'(s)]/(\lambda - s)},$$

(where the denominator of the right-hand side is non-zero for s sufficiently close to λ). Letting s approach λ, we conclude that, since the right-hand side has a limit, the same is true of the left-hand side. But this is equivalent to the assertion that $\lim_{t \to 0} t\,K(t)$ exists. We may further conclude that $\lim_{s \to t}(t - s)K(t - s)$ exists for all t $(0 \leq t \leq 2\pi)$. We denote this limit by M, and remark that since it is a two-sided limit, and since $K(t)$ is periodic,

$$\lim_{t \to 0+} tK(t) = \lim_{t \to 2\pi - 0}(t - 2\pi)K(t - 2\pi) = \lim_{t \to 2\pi - 0}(t - 2\pi)K(t) = M.$$

Writing (13) as

$$h'(t)G(s) + \frac{[h'(t) - h'(s)]}{t - s}(t - s)K(t - s) = h'(s)\{h'(t)K[h(t)] - K(t)\}$$

and letting s approach t, we obtain

$$h'(t)G(t) + h''(t)M = h'(t)\{h'(t)K[h(t)] - K(t)\},$$

which we rewrite as

$$h'(t)K[h(t)] - K(t) = G(t) + Mh''(t)/h'(t). \tag{14}$$

Let $K_0(t) = [K(t) - K(-t)]/2$. Interchanging s and t in (2) and setting $s = 0$, we obtain $h'(t)K[-h(t)] - K(-t) = G(t)$. Subtracting this equation from (14) and multiplying by $1/2$, we get

$$h'(t)K_0[h(t)] - K_0(t) = \frac{M}{2}\frac{h''(t)}{h'(t)}. \tag{15a}$$

The functions $h'(0)K[-h(t)]$ and $K(-t)$ can differ by at most a constant on the interval $0 < t < 2\pi$, because their derivatives are equal — as can be verified by setting $s = 0$ in (4) after interchanging s and t. Thus, $h'(0)K[-h(t)] - K(-t) = -2C$. Subtracting this from (10) and multiplying by $1/2$, we get

$$h'(0)K_0[h(t)] - K_0(t) = C. \tag{15b}$$

We show now that $M > 0$. Suppose $M = 0$. The mean-value theorem applied to $h(t)$ on the interval $0 \leq t \leq 2\pi$ assures us that, since $h(0) = 0$ and $h(2\pi) = 2\pi$, while $h'(0) \neq 1$, we can find a λ $(0 < \lambda < 2\pi)$ such that $h'(\lambda) = 1$. From (15a) we conclude that $K_0[h(\lambda)] = K_0(\lambda)$. But $K_0'(t) = [K'(t) + K'(-t)]/2 < 0$ for $0 < t < 2\pi$. Hence $K_0(t)$ is strictly monotonic on the interval, and this implies that $h(\lambda) = \lambda$. Setting $t = \lambda$ and $s = 0$ in (4), we conclude that $h'(\lambda)h'(0) = 1$. Since $h'(\lambda) = 1$, it follows that $h'(0) = 1$, which is a contradiction. Since $M \neq 0$, neither of $K(0+)$ and $K(2\pi - 0) = K(0-)$ is finite. Since $K'(t) < 0$, we conclude that

$$K(0+) = +\infty, \qquad K(2\pi - 0) = -\infty, \qquad M > 0.$$

We solve (10) for $h(t)$, obtaining

$$h(t) = K^{-1}\left(\frac{1}{h'(0)} K(t)\right) \qquad (0 \leq t \leq 2\pi), \tag{16}$$

where K^{-1} is the inverse of "K restricted to $[0, 2\pi]$." Consider $t_0 = K^{-1}(0)$. We see that

$$h(t_0) = K^{-1}\left(\frac{1}{h'(0)} K(t_0)\right) = t_0.$$

If t_1 is any fixed point for $h(t)$ in the open interval $0 < t < 2\pi$, setting $t = t_1$ in (10), we conclude that $K(t_1) = 0$ and hence $t_0 = t_1$. We have established

Property 1. $h(t)$ has one and only one fixed point in the open interval $0 < t < 2\pi$. If t_0 is this fixed point, then $K(t_0) = 0$.

We proceed to establish a few other properties of $h(t)$.

Property 2. $h'(t_0)\, h'(0) = 1$.

This follows immediately upon setting $t = t_0$ and $s = 0$ in (4).

Property 3. If $0 < t < t_0$, then $0 < h(t) < t < t_0$. If $t_0 < t < 2\pi$, then $t_0 < t < h(t) < 2\pi$.

Suppose that $0 < t < t_0$. In this case, $K(t) > 0$ and

$$\frac{1}{h'(0)} K(t) > K(t) > 0.$$

Since K^{-1} is a strictly decreasing function, we find that

$$h(t) = K^{-1}\left(\frac{1}{h'(0)} K(t)\right) < K^{-1}[K(t)] = t < K^{-1}(0) = t_0.$$

If we suppose that $t_0 < t < 2\pi$, then $K(t) < 0$ and

$$\frac{1}{h'(0)} K(t) < K(t) < 0.$$

Hence

$$h(t) = K^{-1}\left(\frac{1}{h'(0)} K(t)\right) > K^{-1}[K(t)] = t > K^{-1}(0) = t_0.$$

We set $Y(t) = \frac{2}{M} K_0(t)$, for convenience, and observe that $Y(t)$ satisfies the system

$$h'(t)\, Y[h(t)] - Y(t) = \frac{h''(t)}{h'(t)}, \tag{16a}$$

$$h'(0)\, Y[h(t)] - Y(t) = \frac{2C}{M}. \tag{16b}$$

Integrating (16a) between the limits t_0 and $0 < t < 2\pi$, we have

(17) $$\int_{t_0}^{h(t)} Y(s)\,ds - \int_{t_0}^{t} Y(s)\,ds = \log\frac{h'(t)}{h'(t_0)}.$$

Let $F(t) = e^{Q(t)}$, where $Q(t) = \int_{t_0}^{t} Y(s)\,ds$. From this and (17), we have

$$\frac{F[h(t)]}{F(t)} = \frac{h'(t)}{h'(t_0)}.$$

With the aid of Property 2, we may write

(18) $$F[h(t)] = h'(0)\,h'(t)\,F(t).$$

Differentiating (16b), we have

(19) $$h'(0)\,h'(t)\,Y'(h(t)) = Y'(t).$$

Combining (18) and (19), we obtain

(20) $$F[h(t)]\,Y'[h(t)] = F(t)\,Y'(t),$$

which holds for all t $(0 < t < 2\pi)$.

We define a sequence of functions $h_n(t)$ $(0 \leq t \leq 2\pi,\ n = \cdots, -1, 0, 1, \cdots)$. Let $h_0(t) = t$, and for $k > 0$, let

$$h_k(t) = h[h_{k-1}(t)] \quad \text{and} \quad h_{-k}(t) = h^{-1}[h_{-k+1}(t)].$$

We shall show that for each fixed t $(0 < t < 2\pi)$,

$$\lim_{n\to+\infty} h_n(t) = \begin{cases} 0 & \text{if } 0 < t < t_0, \\ t_0 & \text{if } t = t_0, \\ 2\pi & \text{if } t_0 < t < 2\pi, \end{cases}$$

and

$$\lim_{n\to-\infty} h_n(t) = t_0 \qquad \text{if } 0 < t < 2\pi.$$

From Property 3, it is readily seen that for $0 < t < t_0$ and $k \geq 0$, we have

(21) $$0 < h_k(t) < h_{k-1}(t) < \cdots < h_1(t) < t < h_{-1}(t) < \cdots < h_{-k+1}(t) < h_{-k}(t) < t_0$$

and for $t_0 < t < 2\pi$, we have

(22) $$t_0 < h_{-k}(t) < h_{-k+1}(t) < \cdots < h_{-1}(t) < t < h_1(t) < \cdots < h_{k-1}(t) < h_k(t) < 2\pi.$$

It is clear that $h_n(0) = 0$, $h_n(t_0) = t_0$, and $h_n(2\pi) = 2\pi$ for all n $(-\infty < n < +\infty)$. From (21), it is evident that for $0 < t < t_0$,

$$\lim_{n\to+\infty} h_n(t) \geq 0 \quad \text{and} \quad \lim_{n\to-\infty} h_n(t) \leq t_0.$$

Similarly, from (22) it is evident that for $t_0 < t < 2\pi$,

$$\lim_{n\to+\infty} h_n(t) \leq 2\pi \qquad \text{and} \qquad \lim_{n\to-\infty} h_n(t) \geq t_0 .$$

In all of these relations, equality holds; for, if we suppose that $t_1 = \lim_n h_n(t)$, then, since $h(t)$ is continuous, we have $\lim_n h[h_n(t)] = h(t_1)$. But

$$\lim_n h[h_n(t)] = \lim_n h_n(t) = t_1 .$$

However, the only fixed points of $h(t)$ in the interval $0 \leq t \leq 2\pi$ are 0, t_0, and 2π.

Returning to (20), one readily verifies that for $-\infty < n < +\infty$ and each fixed t $(0 < t < 2\pi)$, we have

$$F[h_n(t)]\, Y'[h_n(t)] = F(t)\, Y'(t) .$$

Since $\lim_{n\to-\infty} h_n(t) = t_0$ for $0 < t < 2\pi$ and since $F(t)$ and $Y'(t)$ are continuous at $t = t_0$, we see, on letting $n \to -\infty$, that

$$F(t)\, Y'(t) = \lim_{n\to-\infty} F[h_n(t)]\, Y'[h_n(t)] = F(t_0)\, Y'(t_0) .$$

Since $F(t_0) = 1$, we have, for all t $(0 < t < 2\pi)$,

$$F(t)\, Y'(t) = Y'(t_0) . \tag{23}$$

From (23) we easily deduce the differential equation $Y''(t) = -Y'(t)\, Y(t)$, a first integral of which, when we use the boundary condition $Y(\pi) = 0$ ($Y(t)$ is odd and 2π-periodic), is

$$Y'(t) = -\frac{[Y(t)]^2}{2} + Y'(\pi).$$

Since $Y'(\pi) < 0$, we set $-\frac{a^2}{2} = Y'(\pi)$ and write

$$\frac{-\frac{1}{|a|}\, Y'(t)}{1 + \left(\frac{Y'(t)}{|a|}\right)^2} = \frac{|a|}{2} .$$

Integrating once more, we obtain

$$P \cot^{-1} \frac{Y(t)}{|a|} = \frac{|a|}{2}\, t + \text{const.}, \tag{24}$$

where $P \cot^{-1} u$ is that branch of the inverse cotangent for which $0 < P \cot^{-1} u < \pi$. Now, because

$$\lim_{t\to0+} Y(t) = +\infty \qquad \text{and} \qquad \lim_{t\to0+} P \cot^{-1} \frac{Y(t)}{|a|} = 0 ,$$

the constant in (24) is 0. Hence

$$Y(t) = |a| \cot \frac{|a|}{2} t.$$

$Y(\pi) = 0$ implies that a is an odd integer, and since $Y(t)$ is continuous in $0 < t < 2\pi$, it follows that $|a| = 1$. We have thus established that

$$K_0(t) = \frac{M}{2} \cot \frac{t}{2}. \tag{25}$$

From (25) and (15b) we now have

$$h'(0) \cot \frac{h(t)}{2} - \cot \frac{t}{2} = \frac{2}{M} C.$$

We evaluate C by setting $t = t_0$, obtaining

$$h'(0) \cot \frac{h(t)}{2} - \cot \frac{t}{2} = [h'(0) - 1] \cot \frac{t_0}{2}. \tag{26}$$

We now show that

$$h_n'(0) \cot \frac{h_n(t)}{2} - \cot \frac{t}{2} = [h_n'(0) - 1] \cot \frac{t_0}{2} \tag{27}$$

holds for each $n \geq 1$ and for each t $(0 < t < 2\pi)$. If $n = 1$, then (26) assures us that (27) holds. Suppose (27) to hold for $n \geq 1$. Replacing t by $h(t)$ in (27), multiplying by $h'(0)$, and using the relation $h_m'(0) = [h'(0)]^m$ (which holds for all $m \geq 1$), we obtain

$$h_{n+1}'(0) \cot \frac{h_{n+1}(t)}{2} - h'(0) \cot \frac{h(t)}{2} = [h_{n+1}'(0) - h'(0)] \cot \frac{t_0}{2}.$$

Adding this to (26), we see that (27) holds for $n + 1$.

Now, since $\lim_{n \to +\infty} h_n(t) = 0$ for $0 < t < t_0$ and $\lim_{n \to +\infty} h_n(t) = 2\pi$ for $t_0 < t < 2\pi$, we may (for n sufficiently large and $t \neq t_0$) rewrite (27) as

$$h_n'(0) = \frac{\cot \frac{t}{2} + [h_n'(0) - 1] \cot \frac{t_0}{2}}{\cot \frac{h_n(t)}{2}}. \tag{28}$$

If $0 < t < t_0$, then for n sufficiently large, we have

$$\frac{h_n'(0)}{h_n'(t)} = \frac{\cot \frac{t}{2} + [h_n'(0) - 1] \cot \frac{t_0}{2}}{h_n(t) \cot \frac{h_n(t)}{2}}. \tag{29}$$

Letting $n \to +\infty$ in (29), we find that

$$\lim_{n\to+\infty} \frac{h_n'(0)}{h_n(t)} = \left(\cot\frac{t}{2} - \cot\frac{t_0}{2}\right)/2 .$$

If $t_0 < t < 2\pi$, then for sufficiently large n, we write

$$\frac{h_n'(0)}{h_n(t) - 2\pi} = \frac{\cot\frac{t}{2} + [h_n'(0) - 1]\cot\frac{t_0}{2}}{[h_n(t) - 2\pi]\cot\frac{h_n(t)}{2}}. \tag{30}$$

Letting $n \to +\infty$, we find that

$$\lim_{n\to+\infty} \frac{h_n'(0)}{h_n(t) - 2\pi} = \left(\cot\frac{t}{2} - \cot\frac{t_0}{2}\right)/2 .$$

Now, from (10) and the definition of $h_n(t)$ $(n \geq 1)$, a simple induction establishes that

$$h_n'(0)\, K[h\ \ (t)] = K(t) \tag{31}$$

holds for $0 < t < 2\pi$. Writing (31) as

$$\frac{h_n'(0)}{h_n(t)}\, h_n(t)\, K[h_n(t)] = K(t)$$

or as

$$\frac{h_n'(0)}{h_n(t) - 2\pi}\, [h_n(t) - 2\pi]\, K[h_n(t)] = K(t) ,$$

depending on whether $0 < t < t_0$ or $t_0 < t < 2\pi$, and letting $n \to +\infty$, we conclude, in the light of our remarks (in the paragraph which follows (13)) concerning the limits

$$\lim_{t\to 0+} t\,K(t) \quad \text{and} \quad \lim_{t\to 2\pi-0} (t - 2\pi)\,K(t) ,$$

that for $0 < t < 2\pi$, we have

$$K(t) = \frac{M}{2}\left(\cot\frac{t}{2} - \cot\frac{t_0}{2}\right).$$

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