HOMOTOPICALLY HOMOGENEOUS POLYHEDRA

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1. INTRODUCTION

If X is a space, let

$$\hat{X} = \{(x_1, x_2) \mid x_1 \neq x_2\} \subset X \times X,$$

and define $p: \hat{X} \to X$ by $p(x_1, x_2) = x_1$. We say that X is *homotopically homogeneous* (abbreviated h.h.) if $(\hat{X}, X, p)$ is a Hurewicz fiber space, that is, if $(\hat{X}, X, p)$ has the covering homotopy property for maps of any topological space. It follows immediately that if y, z are points of a path-wise connected h.h. space, then X - y and X - z have the same homotopy type.

A homogeneous polyhedron is clearly a manifold, and h.h. polyhedra turn out to be a kind of homotopy manifold. In particular, an h.h. polyhedron is a Kosiński r-polyhedron [5], and also a homotopy manifold as defined by Griffiths [4]. Thus h.h. polyhedra of dimensions 1, 2, 3 are manifolds, and 4-dimensional h.h. polyhedra are manifolds if the Poincaré Conjecture is true. No example is known of an h.h. polyhedron which is not a manifold.

Section 2 gives some examples. Manifolds, groups, and loop spaces are h.h., and closed cells are not. In Section 3 a class of *locally conical* spaces is considered, so that results are a little more general than for polyhedra. Some lemmas on covering homotopies for locally conical h.h. spaces are proved. The results of Section 3 are applied in Section 4 to show that locally conical h.h. spaces are Kosiński r-spaces, homotopy manifolds, and (hence) homology manifolds. Section 5 is devoted to the consideration of locally conical homology manifolds. A different proof is given of the theorem of Kwun and Raymond [6] that 3-dimensional locally conical generalized manifolds are locally euclidean. Combining these results with those of Section 4 shows that, modulo the Poincaré Conjecture, 4-dimensional h.h. polyhedra are manifolds.

2. EXAMPLES

Example 1. Any n-manifold (separable metric locally euclidean space) is an h.h. space.

We shall show that if M is an n-manifold, then $(\hat{M}, M, p)$ is a locally trivial fiber space. It will then follow from [2] that $(\hat{M}, M, p)$ is a Hurewicz fiber space. Let B be the open unit ball in E^n, and let C be the ball concentric with B and with radius 1/2. Given a point m in M, there exists a homeomorphism h of B into M sending the origin to m. Let h(B) = P and h(C) = Q. We need to define a homeomorphism

$$f: Q \times (M - m) \to p^{-1}(Q)$$

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such that $pf(x, y) = x$ for each x in $M - m$ and each y in Q.

If D is the diagonal of $Q \times Q$, then $p^{-1}(Q) = (Q \times M) - D$, and we define f to be the identity outside of $Q \times (P - m)$. For each x in Q, let f_x be the map of $P - m$ to $P - x$ induced by changing the representation of P from a cone over m to a cone over x. Then f is defined to be f_x on each $p^{-1}(x)$. The resulting f has the required properties, and M is h.h.

Example 2. Topological groups and loop spaces are h.h.

Let e be the identity of a topological group G. Consider the diagram

$$\begin{array}{ccc} G \times (G - e) & & \hat{G} \\ q\downarrow & & \downarrow p \\ G & \xrightarrow{\text{id.}} & G \end{array}$$

where $q(g, h) = g$. Define $\phi\colon G \times (G - e) \to \hat{G}$ by $\phi(g, h) = (g, gh)$. It is easy to verify that ϕ is a fiber-preserving homeomorphism. Hence $(\hat{G}, G, p)$ is a fiber space equivalent to the product $(G \times (G - e), G, q)$.

In the space Ω of loops from x_0 in a space X, let ω_0 be the null loop $\omega_0\colon I \to x_0$. We shall show that $(\hat{\Omega}, \Omega, p)$ is a fiber space with the same fiber homotopy type as the product fibering $(\Omega \times (\Omega - \omega_0), \Omega, q)$. First, to see that $(\hat{\Omega}, \Omega, p)$ is a fiber space, consider an arbitrary covering homotopy situation:

$$\begin{array}{ccc} Y & \xrightarrow{\phi} & \hat{\Omega} \\ & & \downarrow p \\ Y \times I & \xrightarrow{\Psi} & \Omega \end{array}$$

Let the loop $\Psi_t(y)$ be denoted by f_t, and let the path $f_t | [0, \tau]$ be denoted by $f_{t,\tau}$. Define $\Phi_\tau(y) = (f_t, f_\tau f_{t,\tau}^{-1} g)$, where $f_0 = f$ and $\phi(y) = (f, g)$. One verifies that Φ maps into $\hat{\Omega}$, and clearly $p\Phi = \Psi$.

Now consider the diagram

$$\begin{array}{ccc} \Omega \times (\Omega - \omega_0) & \overset{\alpha}{\underset{\beta}{\rightleftarrows}} & \hat{\Omega} \\ q\downarrow & & \downarrow p \\ \Omega & \xrightarrow{\text{id.}} & \Omega \end{array}$$

where $\alpha(\omega_1, \omega_2) = (\omega_1, \omega_1\omega_2)$ and $\beta(\omega_1, \omega_2) = (\omega_1, \omega_1^{-1}\omega_2)$. Then

$$\beta\alpha(\omega_1, \omega_2) = \beta(\omega_1, \omega_1\omega_2) = (\omega_1, \omega_1^{-1}\omega_1\omega_2)$$

and

$$\alpha\beta(\omega_1, \omega_2) = \alpha(\omega_1, \omega_1^{-1}\omega_2) = (\omega_1, \omega_1^{-1}\omega_2).$$

Now $\omega_1^{-1}\omega_1\omega_2 \neq \omega_0$ and $\omega_1\omega_1^{-1}\omega_2 \neq \omega_0$. Hence α and β map into appropriate subsets of $\Omega \times \Omega$, and it is easy to see that they constitute a fiber homotopy equivalence.

Example 3. A closed n-cell is not h.h.

We shall show that any contractible space X which is h.h. cannot have the fixed-point property. Choose x_0 in X, let $\Psi': X \times I \to X$ be a contraction of X to x_0, and let $\Psi_t = \Psi'_{1-t}$. Let $\phi(X) = (x_0, x_1)$ with $x_1 \neq x_0$. Then we have a covering homotopy situation

$$\begin{array}{rcl} \phi: X & \to & \hat{X} \\ & & \downarrow p \\ \Psi: X \times I & \to & X \end{array} .$$

We are assuming that X is h.h., so that there exists a covering $\Phi: X \times I \to \hat{X}$ of Ψ. Define f to satisfy $\Phi_1(x) = (x, f(x))$. Then f is continuous, and if $x = f(x)$ for some x, we contradict the fact that Φ_1 maps X into $\hat{X}$. Hence f is a map of X with no fixed point, and our assertion is proved.

Note that if X is the 0-cell, then $\hat{X}$ is empty, so that X is not h.h.

3. LOCALLY CONICAL SPACES

Definition 1. Let $\dot{N}$ denote the boundary of a closed neighborhood N of a point x in a space X. We say that N is a *conical neighborhood* of x if there exists a homeomorphism of N onto the cone $C\dot{N}$ which is the identity on $\dot{N}$ and sends x to the cone point. The space X is *locally conical* if each point of X has a conical neighborhood.

A conical neighborhood can be shrunk uniformly along rays toward the vertex of the cone, and when one conical neighborhood can be so obtained from another, we shall refer to them as concentric neighborhoods.

LEMMA 1. *If* N *and* M *are conical neighborhoods of* x, *then their boundaries* $\dot{N}$ *and* $\dot{M}$ *have the same homotopy type.*

Proof. Represent N and M as homeomorphs of $C\dot{N}$ and $C\dot{M}$. Let $N \supset M_1 \supset N_1$, where M_1 is concentric with M and N_1 is concentric with N. A deformation of $\dot{N}_1$, first outward along rays of M to $\dot{M}_1$, and then along rays of N to $\dot{N}$, gives a deformation of $\dot{N}_1$ to $\dot{N}$, and the homotopy paths all lie in N - x. Let α be the homotopy obtained by following the shrinking of $\dot{N}$ to $\dot{N}_1$ by the homotopy just described. Then the homotopy paths of α all lie in N - x and so can be pushed out to a deformation of $\dot{N}$. This shows that $\dot{M}$ dominates $\dot{N}$. Dually, $\dot{N}$ dominates $\dot{M}$, and the lemma follows.

Notation. If $U \subset X$, we let $U_y = \{(y, z) \mid z \in U, z \neq y\}$. That is, U_y is a copy of U - y in the fiber of $\hat{X}$ above y. Of course, if $y \notin U$, then U_y is homeomorphic with U.

If X is h.h., then $(\hat{X}, X, p)$ is a Hurewicz fiber space, so that it is possible to choose at once covering homotopies for all covering homotopy situations [2]. This construction has a continuity property in that nearby covering homotopy situations yield nearby covering homotopies.

Let $y \in N$. Then there exists a unique path α from x to y and a chosen homotopy of X_x into $\hat{X}$ covering α. We denote this homotopy by $\Phi_y: X_x \times I \to \hat{X}$, and its final stage by ϕ_y. We also have a chosen homotopy $\Psi_y: X_y \times I \to X$ covering the inverse path α^{-1}, and we denote its final stage by ψ_y. Let $\gamma_y: X_x \times I \to X_x$ denote the homotopy between $\psi_y \phi_y$ and the identity obtained by shrinking the path α to the null path at x. This notation will be used in the remainder of this section and in Section 4.

Since X is a cartesian product, away from the diagonal, we should be able to choose covering homotopies (when X is h.h.) so that, beyond some distance from the diagonal, the second coordinate of a covering homotopy path will be constant. The next lemma gives such a result in the special case which we need later.

LEMMA 2. *Let* N *be a conical neighborhood of* x *in an h.h. space* X. *Then there exists a concentric* N'' *such that, for each* $y \in N''$, *there is a covering homotopy*

$$F: X_x \times I \to \hat{X}$$

covering the path α *from* x *to* y *in such a way that* $F \mid N''_x = \Phi_y$ *and, for* $z \notin N$, $F_t(x, z) = (\alpha(t), z)$ *for all* t.

Proof. Shrink N to concentric neighborhoods $N'' \subset N' \subset N$ such that for each t

$$\Phi_y(N''_x, t) \subset N'_{\alpha(t)} \, .$$

Now F is already defined on $(X_x - N) \times I$ and on $N''_x \times I$, by the conditions of the lemma. The remaining part of X_x, namely $N_x - N''_x$, consists of segments λ_{ab} of rays of N with a in $\dot{N}$ and b in $\dot{N}''$. Let $\beta: I \to \lambda_{ab}$ be linear, with $\beta(0) = b$, and let $\beta(1)$ be the midpoint of λ_{ab}. Let $\rho: \dot{N}''_x \times I \to X$ be such that

$$\Phi_y(w, t) = (\alpha(t), \rho(w, t))$$

for each w in $\dot{N}''_x$. Finally, we can define F_t on λ_{ab} as follows:

F_t stretches the part $[a, \beta(t)]$ of λ_{ab} linearly onto λ_{ab}, and it maps the part $[\beta(t), b]$ by $\rho \mid [0, t]$.

4. r-POLYHEDRA

Definition 2. A point x of a space X is an r*-point* if x has arbitrarily small (closed) neighborhoods U such that for each $y \in U$, there exists a (strong) deformation retraction of U - y onto $\dot{U}$ (see [5]). Such neighborhoods are called *canonical* neighborhoods. An r*-space* is a finite-dimensional compact metric space in which each point is an r-point.

LEMMA 3. *Let* $M \subset N$ *be concentric conical neighborhoods of a point* x *in a space* X. *Suppose that for each* y *in* M *there exists a deformation retraction of* N - y *onto* $\dot{N}$. *Then* M *is a canonical neighborhood.*

Proof. Let $y \in M$, and let W be the concentric conical neighborhood of x with y in its boundary $\dot{W}$. For each ray λ_b from x to a point b in $\dot{N}$, we define an isotopy which stretches the part of λ_b between W and $\dot{M}$ to the part of λ_b between W and $\dot{N}$. These isotopies on rays combine to give an isotopy ρ which stretches M - W onto N - W.

By hypothesis there exists a deformation retraction μ of N - y onto $\dot{N}$. Then $\rho^{-1}\mu\rho$ is a deformation retraction of M - y onto $\dot{M}$, and the lemma is proved.

Of course, if some conical neighborhood is canonical, then so is every concentric neighborhood.

LEMMA 4. *If* X *is a locally conical* r*-space, then conical neighborhoods are canonical.*

Proof. Given a conical neighborhood N of x, choose a concentric neighborhood M and a canonical neighborhood W such that $M \subset W \subset N$. If y is in M, then there exists a deformation retraction of W - y onto $\dot{W}$, and this can be pushed out to give a deformation retraction of N - y onto $\dot{N}$. By Lemma 3, M is canonical, and hence so is N.

THEOREM 1. *A locally conical point of an h.h. space is an* r*-point.*

Proof. We shall show that a conical neighborhood is canonical. By Lemma 3 it suffices to show that if N is a conical neighborhood of x, then there exists a concentric neighborhood M such that to each $y \in M$, there corresponds a deformation retraction of N - y onto $\dot{N}$.

We choose concentric neighborhoods $M \subset N'' \subset N' \subset N$ such that

(i) for $y \in M$, $\psi_y(M_y) \subset N''_x$,

(ii) $\phi_y(\dot{N}''_x) \subset N'_y - M_y$,

(iii) homotopies in N' can be patched with the identity outside of N, as in Lemma 3.

Now we map M_y into $\dot{N}''_x$ by ψ_y and follow it by the standard deformation retraction of N''_x - x onto $\dot{N}''_x$. Next $\dot{N}''_x$ is mapped back into X_y by ϕ_y. By shrinking the path α (from y to x), we obtain a homotopy in X_y which deforms $\dot{M}_y$ out to $\phi_y(\dot{N}''_x)$. By (i) the image does not contain x, and hence we can retract out to $\dot{N}'$ by a deformation. We patch this with the identity outside of N and can then get a deformation retraction of N_y - y onto $\dot{N}$ as required. Hence N is a canonical neighborhood, and the theorem is proved.

COROLLARY 1. *If* X *is a compact, metric, finite-dimensional locally conical h.h. space, then* X *is an* r*-space.*

Kosiński [5] has proved that for an r-polyhedron, the boundaries of star neighborhoods of points have the homotopy type of spheres. It follows that such a space is a homotopy manifold [4], and hence also a homology generalized manifold. So we have the following proposition.

COROLLARY 2. *An h.h. polyhedron is an* r*-polyhedron, and a homotopy manifold, and hence a homology generalized manifold.*

5. POLYHEDRAL GENERALIZED MANIFOLDS

We shall use gcm to mean "generalized closed manifold defined over a field of coefficients" (see Wilder [7]). An n-gcm is *spherelike* if it has the homology of S^n.

THEOREM 2. *Let* X *be a locally contractible metric continuum. Then* X *is a spherelike* n-gcm *if and only if its suspension* SX *is an orientable* (n + 1)-gcm.

Proof. Suppose SX is a spherelike n-gcm. Now SX is a product of X with a line, except at suspension points, and the product of gcm's is a gcm. The local homology at the suspension points can be based on neighborhoods homeomorphic with a cone over X. Since X is spherical, the suspension points have the right local homology, and SX is an (n + 1)-gcm. Clearly, SX is orientable.

Now suppose SX is an (n + 1)-gcm. SX is easily seen to be locally contractible and simply connected. Also, X is deformation-free in SX, and therefore we may apply Theorem 1.1 of [1], which implies that under these circumstances X is an n-gcm. Since the homology works as in the first half of the proof, it follows that X must have the homology of S^n.

COROLLARY 3 (Kwun and Raymond). *A locally conical* 3-gcm *is a manifold.*

Proof. Let N be a conical neighborhood of a point s in a 3-gcm X. Take another copy N' of N, and identify N and N' along the boundaries $\dot{N}$ and $\dot{N}'$. The result M is the suspension of $\dot{N}$, and it is an orientable 3-gcm, since X is a gcm. By Theorem 2, $\dot{N}$ is a spherelike 2-gcm, and hence $\dot{N} = S^2$ (see [7]). It follows that N is a 3-cell and X is a manifold.

Exactly the same type of argument shows the following.

COROLLARY 4. *If* X *is a polyhedral* n-gcm, *and* β *is the boundary of a star neighborhood of a point in* X, *then* β *is a spherelike* (n - 1)-gcm. *Similarly, the boundary of a star neighborhood in* β *is a spherelike* (n - 2)-gcm, *etc.*

Combining the results above, we see that if X is a connected n-dimensional h.h. polyhedron, and β is the boundary of a star neighborhood of a point x in X, then β is an (n - 1)-gcm with the homotopy type of S^{n-1}.

THEOREM 3. *If the Poincaré Conjecture is true, then each* 4-*dimensional connected h.h. polyhedron is a manifold.*

Proof. If β is the boundary of a star neighborhood, then β is a polyhedral 3-gcm with the homotopy type of S^3. By Corollary 3, β is a manifold, and by the Poincaré Conjecture it is S^3. Hence the space is locally euclidean.

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