THE COHOMOLOGY OF A SPACE ON WHICH AN H-SPACE OPERATES

Edward Halpern

INTRODUCTION

An H-*space* consists of a topological space X with base point $e \in X$ and a (continuous) map $\Delta: X \times X \to X$ such that

$$\Delta i \simeq I, \qquad \Delta j \simeq I, \tag{1.1}$$

where I is the identity map, i and j are defined by $i(x) = (x, e)$ and $j(x) = (e, x)$ $(x \in X)$, and $\simeq$ means "is homotopic relative to e." The multiplication Δ is *homotopy-associative* if

$$\Delta(\Delta \times I) \simeq \Delta(I \times \Delta); \tag{1.2}$$

it is *homotopy-commutative* if

$$\Delta\theta \simeq \Delta, \tag{1.3}$$

where θ is defined by $\theta(x, y) = (y, x)$ $(x, y \in X)$.

An H-space X *operates* (on the right, and up to homotopy) on a topological space T if there is a (continuous) map $\overline{\Delta}: T \times X \to T$ such that

$$\overline{\Delta} i \simeq I, \tag{1.4}$$

$$\overline{\Delta}(I \times \Delta) \simeq \overline{\Delta}(\overline{\Delta} \times I). \tag{1.5}$$

In particular, if X is homotopy-associative we may regard it as operating on itself by right translations. A map $f: X \to T$ is said to *commute* with the operations of X on T and X if

$$\overline{\Delta}(f \times I) \cong f\Delta. \tag{1.6}$$

The main theorem is as follows:

THEOREM 1. *Let* K *be a field of characteristic zero. Let* X *be an* H-*space with homotopy-associative and homotopy-commutative multiplication which operates on a topological space* T, *with* X *and* T *arcwise connected and* $H^i(X, K)$ *and* $H^i(T, K)$ *finitely generated* (*all* i). *If* $f: X \to T$ *commutes with the operations of* X *on* T *and itself, then* $f^*H^*(T, K)$ *is a Hopf subalgebra. Moreover,*

$$H^*(T, K) \cong B \otimes C, \tag{1.7}$$

where B *and* C *are subalgebras of* $H^*(T, K)$ *such that* f^* *annihilates the elements of positive degree in* B *and is injective on* C.

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If in the hypothesis of the theorem we replace homotopy-commutativity of X by the assumption that $H^*(X, K)$ is an exterior algebra, and allow K to have any characteristic other than 2, we obtain a theorem of A. Borel [2, Théorèmes 3.6, 3.7]. Actually, we shall prove an algebraic theorem somewhat more general than the algebraic formulation of Theorem 1, from which it and the Borel theorem follow. Theorem 1 may be applied to Lie groups in two situations:

(1.8) T *a connected Lie group,* X *a closed connected commutative subgroup, and* f *the inclusion map.*

(1.9) X *a connected commutative Lie group,* T *the right coset space of* X *modulo a closed subgroup, and* f *the canonical projection.*

We remark that (1.8) and (1.9) are analogues of results of H. Samelson [7, Satz II] and J. Leray [5].

2. HYPERALGEBRAS

Throughout, Λ will denote a commutative ring with unit 1, and all Λ-modules will be unitary. As usual, $H \otimes H'$ will denote the tensor product of Λ-modules H and H'; if the latter are Λ-algebras, then the former is a Λ-algebra with multiplication defined by

$$(x \otimes x')(y \otimes y') = xy \otimes x'y'. \tag{2.1}$$

A Λ-*hyperalgebra* consists of a Λ-algebra H and an algebraic homomorphism $\Delta: H \to H \otimes H$. We call Δ the *coproduct* in H; it is *associative* if

$$(\Delta \otimes I)\Delta = (I \otimes \Delta)\Delta, \tag{2.2}$$

where I is the identity map of H. A *unit* for the hyperalgebra is simply an algebraic unit. A *homomorphism* of Λ-hyperalgebras $f: H \to H'$ is an algebraic homomorphism such that

$$\Delta' f = (f \otimes f)\Delta.$$

A *subhyperalgebra* $G \subset H$ is a subalgebra such that $\Delta(G) \subset G \otimes G$.

We recall that an *augmentation* of a Λ-module consists of homomorphisms $\alpha: H \to \Lambda$ and $\beta: \Lambda \to H$ such that $\alpha\beta = I$. If H^+ denotes the kernel of α, then H is a direct sum

$$H = \beta(\Lambda) + H^+. \tag{2.3}$$

A *homomorphism* of augmented Λ-modules $f: H \to H'$ is required to satisfy $\alpha' f = \alpha$ and $f\beta = \beta'$, where (α, β) and (α', β') are the corresponding augmentations. An *augmentation* of a Λ-*algebra* is an augmentation of the module, with α and β multiplicative.

An *augmentation* of a Λ-hyperalgebra H is an augmentation of the algebra with the additional properties

$$(2.4)\qquad \begin{cases} (I \otimes \beta\alpha)\,\Delta(x) = x \otimes \beta(1)\,, \\ (\beta\alpha \otimes I)\,\Delta(x) = \beta(1) \otimes x\,. \end{cases} \qquad (x \in H)$$

Let $\Delta'' = \Delta - \Delta'$, where Δ' is defined by

$$(2.5)\qquad \Delta'(x) = \begin{cases} x \otimes \beta(1) + \beta(1) \otimes x & (x \in H^+)\,, \\ \beta(1) \otimes \beta(1) & (x = \beta(1))\,. \end{cases}$$

A straightforward computation based on (2.4) and (2.5) shows that

$$(2.6)\qquad (I \otimes \beta\alpha)\Delta'' = 0, \qquad (\beta\alpha \otimes I)\Delta'' = 0\,,$$

which together with (2.3) proves

$$(2.7)\qquad \Delta''(x) \in H^+ \otimes H^+ \qquad (x \in H)\,.$$

An element x of an augmented Λ-hyperalgebra H is *primitive* if $\Delta''(x) = 0$. The subset of primitive elements form a submodule which we shall denote by π_0. The subalgebra generated by π_0 will be denoted by π. If $H = \pi$, then H is said to be a *primitive hyperalgebra.* If in addition $\pi_0 \cap D = 0$, where D is the submodule spanned by decomposable elements, then H is said to be *simply-primitive.*

We recall that a Λ-module H is *graded* if it is a (weak) direct sum of submodules H^i $(0 \leq i < \infty)$. The elements of H^i are *homogeneous* of *degree* i: H is of *finite type* if each H^i is finitely generated. A map $f\colon H \to H'$ is *homogeneous* of *degree* j if $f(H^i) \subset (H')^{i+j}$. A *homomorphism* of graded Λ-modules (and all subsequent graded structures) is required to be homogeneous of degree zero. The (graded) submodules $G \subset H$ are graded by $G^i = G \cap H^i$. The tensor product $H \otimes H'$ of graded Λ-modules is graded by

$$(H \otimes H')^i = \sum_{i=j+k} H^j \otimes H'^k.$$

If H is a graded Λ-module, we shall tacitly assume that it has a *standard* augmentation (α, β), namely, $\beta(\Lambda) = H^0$. Then a homomorphism $f\colon H \to H'$ of (graded) Λ-modules will be an isomorphism in dimension zero.

A Λ-algebra H is *graded* if it is a graded Λ-module and the multiplication satisfies $H^i \cdot H^j \subset H^{i+j}$. The product is *anticommutative* if

$$xy = (-1)^{ij}yx \qquad (x \in H^i,\ y \in H^j)\,.$$

If H and H' are graded Λ-algebras, the multiplication in $H \otimes H'$ is defined by

$$(2.1)' \qquad (x \otimes x')(y \otimes y') = (-1)^{ij}\, xy \otimes x'y' \qquad (x' \in H'^i,\ y \in H^j)\,.$$

A Λ-hyperalgebra H is *graded* if it is a graded algebra and Δ is homogenous of degree zero. The coproduct is *anticommutative* if $\theta\Delta = \Delta$, where θ is defined by

$$(2.8)\qquad \theta(x \otimes y) = (-1)^{ij}\, y \otimes x \qquad (x \in H^i,\ y \in H^j)\,.$$

Since H has a standard augmentation, it is clear that H^+ is precisely the submodule spanned by the elements of positive degree. Moreover, since Δ is homogeneous of degree zero, it follows from (2.7) that we may write

$$\Delta''(x) = \sum_{j=1}^{k} y_j \otimes z_j \qquad (x \in H^i), \tag{2.9}$$

where y_j and z_j have positive degrees with sum i.

By a *Hopf algebra* we shall mean a graded hyperalgebra with a unit and with an associative and anticommutative product. A *Hopf subalgebra* is then a subhyperalgebra that contains a unit.

If H is a primitive Hopf algebra, it is readily proved that its coproduct is associative and anticommutative. As a partial converse we have the following theorem [3, Theorem 2.10]:

(2.10) *If* H *is a Hopf algebra with an associative and anticommutative coproduct over a field of characteristic zero, then it is primitive.*

We also note the following theorem due to H. Samelson [7] and J. Leray [5]:

(2.11) *Let* H *be a Hopf algebra with associative coproduct over a field. If as an algebra* H *is an exterior algebra generated by elements of odd degrees, then it is primitive.*

If H is a graded Λ-module, its *dual* is the Λ-module $H_* = \Sigma H_i$, where $H_i = \text{Hom}(H^i, \Lambda)$. (We use lower rather than upper stars, since topologically H will correspond to the homology module.) If H and H' are torsion-free graded Λ-modules of finite type, we may canonically identify $(H \otimes H')_* = H_* \otimes H'_*$. If H is a torsion-free, graded Λ-hyperalgebra of finite type with coproduct Δ, its dual H_* is evidently a Λ-algebra with product

$$uv = \Delta_*(u \otimes v) \qquad (u, v \in H_*).$$

(Actually, H_* is a Λ-hyperalgebra with coproduct ϕ_*, where ϕ_* is the transpose of the product in H; but we shall not use this.) By standard duality arguments we see that Δ_* is associative or anticommutative if Δ is associative or anticommutative, respectively, and that H_* has a unit if H has a unit. In particular, if H is a Hopf algebra, its dual is called a *Pontrjagin algebra.*

Let H be a Λ-algebra. If $x \in H$, its *height* h(x) is the least positive integer such that $x^{h(x)} = 0$; if no such integer exists, we define $h(x) = \infty$. If H is graded and anticommutative and x is a homogeneous element not in the center of H, then $h(x) = 2$.

Let H be an algebra over a field K. If $X \subset H$ is a subset such that the inclusion map induces an algebraic isomorphism

$$i: \bigotimes_{x \in X} K[x]/(x^{h(x)}) \cong H \qquad \text{(weak tensor product)},$$

we say that H is a *truncated polynomial algebra in the elements of* X. We shall write $H = K[X, h]$, where h is the *height function* on X. If H is graded, then X is required to consist of homogeneous elements (of positive degree), and i to be homogeneous of degree zero.

A field K of characteristic p is *perfect* if $p = 0$ or if each element of K has a pth root in K. We cite the following theorems, where H is a Hopf algebra over a perfect field:

(2.12) *We may represent* $H = K[X, h]$; *moreover, if* $x \in X$ *is in the center of* H, *then*

$$h(x) = \begin{cases} \infty & \text{if } p = 0, \\ p^i \ (1 \leq i \leq \infty) & \text{if } p \neq 0. \end{cases}$$

(2.13) *If* H *is primitive, then we may choose the elements of* X *primitive, in which case the primitive subspace* π_0 *has as basis the set*

$$\begin{cases} X \cup \{1\} & \text{if } p = 0, \\ \{x^{p^i};\ x \in X,\ 0 \leq p^i < h(x)\} & \text{if } p \neq 0. \end{cases}$$

The former is due to A. Borel [1, Théorème 6.1]. In the proof given in [1], H is assumed to be of finite type, but this is easily dispensed with. When $p \neq 0$, the condition that K be perfect may be replaced by $\gamma(H) = 0$, where $\gamma\colon H \to H$ is defined by $\gamma(x) = x^p$. In the proof we may then choose for X any minimal system of generators for H, and the argument reduces essentially to the proof in the case $p = 0$. For the proof of (2.13), see [3, Theorem 2.7, Corollaries 2.5, 2.6].

PROPOSITION 2.1. *If* H *is primitive, it is simply-primitive if and only if* $p = 0$ *or* $\gamma(H) = 0$.

COROLLARY 2.2. *If* H *is simply-primitive, then we may write* $H = K[X, h]$, *where the elements of* X *are primitive and together with* 1 *form a basis for* π_0; *moreover, if* $x \in X$ *is in the center of* H, *then*

$$h(x) = \begin{cases} \infty & \text{if } p = 0, \\ p & \text{if } p \neq 0. \end{cases}$$

COROLLARY 2.3. *If* H *is simply-primitive and of finite type, then we may write*

$$H = K[\{x_i\}, h], \qquad H_* = K[\{u_i\}, h_*],$$

where the x_i *are primitive,* u_i *and* x_i *correspond under duality, and* $h(x_i) = h_*(u_i)$.

Proposition 2.1 and Corollary 2.2 follow readily from (2.12) and (2.13); Corollary 2.3 follows from the preceding corollary and a duality theorem [4, Theorems 6.4, 6.5].

LEMMA 2.4. *Let* H *be a simply-primitive Hopf algebra over a field* K. *If* $G \subset H$ *is a subalgebra, then we may write* $H = K[X, h]$, *so that* X *consists of primitive elements and* $(X \cup \{1\}) \cap G$ *is a basis for* $\pi_0 \cap G$.

The proof is easy. Represent H as in Corollary 2.2. We may replace X by an X' such that $(X' \cup \{1\}) \cap G$ is a basis for $\pi_0 \cap G$. It remains to show that $H = K[X', h']$. But this follows readily from the fact that the transformation $X \to X'$ induces an automorphism of H.

Let $H = K[X, h]$ be a graded algebra; we may assume X well-ordered in such a way that $x < y$ if x has lower degree than y. By a *normal monomial* we shall mean a product of the form

(2.14) $$M = x_1^{m_1} x_2^{m_2} \cdots x_t^{m_t} \qquad (0 \leq m_i < h(x_i)),$$

where the $x_i \in X$ and $x_i < x_{i+1}$. Its *length* is $m_1 + \cdots + m_t$; its *width* is the number of positive m_i; its *terminal factor* is x_j if $m_j > 0$ and $m_i = 0$ for $i > j$.

Now let $H = K[X, h]$ be a Hopf algebra with coproduct $\triangle$, and assume each x in X to be primitive. By induction on width, the following formula is readily proved:

(2.15) $$\triangle(M) = \sum \operatorname{sg}(RS)\,[R, S]\, R \otimes S,$$

where the summation extends over distinct pairs of normal monomials

$$R = x_1^{r_1} x_2^{r_2} \cdots x_t^{r_t}, \qquad S = x_1^{s_1} x_2^{s_2} \cdots x_t^{s_t} \qquad (r_i + s_i = m_i),$$

and where $\operatorname{sg}(RS)$ and $[R, S]$ are defined by

$$RS = \operatorname{sg}(RS)\, M, \qquad [R, S] = \prod_{i=1}^{t} m_i!/r_i!\, s_i!\,.$$

PROPOSITION 2.5. *Let* H *be a simply-primitive Hopf algebra over a field* K. *A subalgebra* $G \subset H$ *is a Hopf subalgebra if and only if it is primitively generated.*

Proof. By Lemma 2.4, we may represent $H = K[X, h]$, where each x in X is primitive and $(X \cup \{1\}) \cap G$ is a basis for $\pi_0 \cap G$. If G is primitively generated, it is clearly a Hopf subalgebra (in fact, $G = K[X \cap G, h'])$.

Let G be a Hopf subalgebra. We shall first show that if M is a normal monomial in the elements of X and $M \in G$, then M is a normal monomial in the elements of $X' = X \cap G$. Write $M = \pm N x_i$, where x_i is a nontrivial factor of M and N is normal. Applying (2.15), we may write

(2.16) $$\triangle(M) = \pm [N, x_j] N \otimes x_j \pm \sum_{S \neq x_i} \operatorname{sg}(RS)\,[R, S] R \otimes S\,.$$

Since G is a Hopf subalgebra, $\triangle(G) \subset G \otimes G$ and hence, from (2.16), we see that $N \otimes x_i \in G \otimes G$. Thus x_i is in G, and the assertion is proved.

A general element g in G may be written (uniquely)

$$g = \sum k_i M_i \qquad (k_i \neq 0,\ k_i \in K),$$

where the M_i are distinct normal monomials in the elements of X. Applying (2.15), we see that

$$\triangle(g) = \sum k_i \operatorname{sg}(R, S)\,[R, S] R \otimes S\,.$$

Since G is a Hopf subalgebra, $R \otimes S \in G \otimes G$. By the preceding result, R and S, and hence M, are normal monomials in the elements of X'. Thus G is primitively generated.

3. GENERALIZED HYPERALGEBRAS, CAP PRODUCTS

Let A be a Λ-algebra and H a Λ-hyperalgebra with coproduct Δ. We say that $\{A, H, \overline{\Delta}\}$ is a *generalized* Λ-*hyperalgebra* if $\overline{\Delta}: A \to A \otimes H$ is an algebraic homomorphism such that

(3.1) $$(\overline{\Delta} \otimes I)\overline{\Delta} = (I \otimes \Delta)\overline{\Delta},$$

where I is the identity map (of H or A). If H has augmentation (α, β), we further require

(3.2) $$(I \otimes \beta\alpha)\overline{\Delta}(a) = a \otimes \beta(1) \qquad (a \in A).$$

If H is a graded hyperalgebra, we require A to be a graded algebra and $\overline{\Delta}$ homogeneous of degree zero. If both A and H are of finite type, then $\{A, H, \overline{\Delta}\}$ is said to be of *finite type*. If a is homogeneous, it follows from (3.2) that we may write

(3.3) $$\overline{\Delta}(a) = a \otimes \beta(1) + \sum_{j=1} a_j \otimes x_j \qquad (a_j \in A,\ x_j \in H),$$

where each x_j has positive degree. A *map* f: $\{A, H, \overline{\Delta}\} \to \{A', H, \overline{\Delta}'\}$ of generalized Λ-hyperalgebras is an algebraic homomorphism $f: A \to A'$ such that

(3.4) $$(f \otimes I)\overline{\Delta} = \overline{\Delta}' f.$$

In particular, we may regard a hyperalgebra H with an associative coproduct Δ as a generalized hyperalgebra $\{H, H, \Delta\}$.

Let $\{A, H, \overline{\Delta}\}$ be of finite type. Assume that A and H are torsion-free, and let A_* and H_* be their dual module and algebra, respectively. Assume further that H has a unit (also to be denoted by 1). We may pair H_* and A to A as follows: For $u \in H_i$ and $a \in A^j$ we define the *cap product* $u \frown a \in A^{j-i}$ by

(3.5) $$\langle u \frown a, e\rangle = \langle\overline{\Delta}(a), e \times u\rangle \qquad (e \in A_{j-i}),$$

where $\langle\, ,\rangle$ refers to the duality pairing $A \times A_* \to \Lambda$. We shall prove the following properties:

(3.6) $$1 \frown a = a,$$

(3.7) $$uv \frown a = u \frown (v \frown a),$$

(3.8) $$f(u \frown a) = u \frown f(a),$$

where f: $\{A, H, \overline{\Delta}\} \to \{A', H, \overline{\Delta}'\}$ is a map.

(3.9) $$u \frown ab = a(u \frown b) + (-1)^{ik}(u \frown a)b \qquad (u \in H_i,\ b \in A^k),$$

if u is orthogonal to the decomposable elements of H.

If we take u = 1 in (3.5) and apply (3.3), we get (3.6). To prove (3.7), let Δ_* and $\overline{\Delta}_*$ denote the transposes of Δ and $\overline{\Delta}$ respectively. Then

$$<uv \frown a, e> = <\bar{\Delta}(a), e \otimes uv>$$
$$= <\bar{\Delta}(a), (I \otimes \Delta_*)(e \otimes u \otimes v)>$$
$$= <(I \otimes \Delta)\bar{\Delta}(a), e \otimes u \otimes v>;$$
$$<u \frown (v \frown a), e> = <\bar{\Delta}(v \frown a), e \otimes u>$$
$$= <v \frown a, \bar{\Delta}_*(e \otimes u)>$$
$$= <\bar{\Delta}(a), \bar{\Delta}_*(e \otimes u) \otimes v>$$
$$= <\bar{\Delta}(a), (\bar{\Delta}_* \otimes I)(e \otimes u \otimes v)>$$
$$= <(\bar{\Delta} \otimes I)\bar{\Delta}(a), e \otimes u \otimes v>.$$

Thus (3.7) follows from (3.1). Let f_* be the transpose of f; then

$$<u \frown f(a), e'> = <\bar{\Delta}' f(a), e' \otimes u> \qquad (e' \in A_*),$$
$$= <(f \otimes I)\bar{\Delta}(a), e' \otimes u>$$
$$= <\bar{\Delta}(a), (f_* \otimes I)(e' \otimes u)>$$
$$= <\bar{\Delta}(a), f_*(e') \otimes u>$$
$$= <u \frown a, f_*(e')>$$
$$= <f(u \frown a), e'>,$$

and this proves (3.8). Finally, to prove (3.9), choose an additive basis $\{x_1, x_2, \cdots\}$ for H such that

$$<x_1, u> = \lambda \neq 0 \quad (\lambda \in \Lambda), \qquad <x_i, u> = 0 \quad (i > 1).$$

Relative to this basis, we may write

(3.10) $$\bar{\Delta}(a) = a \otimes 1 + \sum a_i \otimes x_i,$$

(3.11) $$\bar{\Delta}(b) = b \otimes 1 + \sum b_i \otimes x_i,$$

which applied in (3.5) gives

(3.12) $$u \frown a = \lambda a_1, \qquad u \frown b = \lambda b_1.$$

Multiplying (3.10) and (3.11), we get

$$\bar{\Delta}(ab) = ab \otimes 1 + (ab_1 + (-1)^{ik} a_1 b) \otimes x_1 + \sum_{i>1} c_i \otimes x_i + \sum d_j \otimes z_j \qquad (c_i, d_j \in A),$$

where the z_j are products of the x_i. Applying this in (3.5), we obtain

(3.13) $$u \frown ab = \lambda(ab_1 + (-1)^{ik} a_1 b).$$

Thus (3.9) follows from (3.12) and (3.13).

Let H be a simply-primitive Hopf algebra of finite type over a field K, and represent H and H_* as in Corollary 2.3. (According to the adopted convention, the corresponding sequence of degrees of the x_i is monotanically nondecreasing.) The normal monomial (2.14) may be written

$$M = x_1^{m_1} x_2^{m_2} \cdots x_i^{m_i} \cdots \qquad (0 \leq m_i < h(x_i)), \tag{3.14}$$

with finitely many $m_i > 0$. Let $\mu(M)$ be the corresponding normal monomial

$$\mu(M) = u_1^{m_1} u_2^{m_2} \cdots u_i^{m_i} \cdots . \tag{3.15}$$

Since H has an associative coproduct, we may regard H as a generalized hyperalgebra $\{H, H, \Delta\}$. We assert that if x_j is the terminal factor of M, then

$$u_j \frown M = \frac{\partial M}{\partial x_j}. \tag{3.16}$$

The proof is easy. Write $M = Nx_j$, where N is normal; then, applying (3.5) and (2.15), we get

$$\langle u_j \frown M, e\rangle = sg(Nx_j)\,[N, x_j]\langle N, e\rangle \qquad (e \in A_*).$$

Since $sg(Nx_j) = 1$ and $[N, x_j] = m_j$, (3.6) follows.

PROPOSITION 3.1. *Let* H *be a simply-primitive Hopf algebra of finite type over a field* K. *A subalgebra* $G \subset H$ *is a Hopf subalgebra if and only if it is stable under cap-products.*

Proof. In view of Proposition 2.5, it suffices to prove that G is stable if and only if it is primitively generated. We represent H and H_* as in Corollary 2.3; moreover, by Lemma 2.4, we may assume that $(\{x_i\} \cup \{1\}) \cap G$ is a basis for $\pi_0 \cap G$.

Let $g \in G$ be a homogeneous element. We may express g uniquely in the form

$$g = k_1 M_1 + k_2 M_2 + \cdots + k_s M_s \qquad (k_i \neq 0,\ k_i \in K), \tag{3.17}$$

where the M_i are (nonzero) normal monomials in the elements x_i.

(i) Suppose that G is primitively generated. Then each M_i is a normal monomial in the elements of $X' = \{x_i\} \cap G$. From (3.16) we see that for all $j \geq 1$, $u_j \frown g \in G$. In view of (3.7), G is therefore stable under cap-products.

(ii) Suppose that G is stable under cap-products. Let $x_{i_1}, x_{i_2}, \cdots, x_{i_t}$ $(i_1 < i_2 < \cdots < i_t)$ be the elements of X that have positive multiplicity in some M_i in (3.17). Suppose x_j is the first of these elements which is not in X'. Let x_j occur with maximum multiplicity in M_1; then we may write $M_1 = \pm x_j N_1$, where N_1 is a normal monomial. By stability of G, the element $g_j = \mu(N_1) \frown g$ is in G. We may write

$$g_j = kx_j + L_j + P_j \qquad (k \neq 0,\ k \in K),$$

where L_j is a linear polynomial in those $x_{i_1}, \cdots, \hat{x}_j, \cdots, x_{i_t}$ ($\hat{x}_j$ means that x_j is omitted) that have the same degree as x_j, and where P_j is a polynomial in those $x_{i_1}, \cdots, x_{i_t}$ that have lower degrees. By the assumption on x_j, clearly $P_j \in G$; hence $kx_j + L_j \in G$. Moreover, since L_j is linear, it follows that

$$kx_j + L_j \in \pi_0 \cap G.$$

Since $X' \cup \{1\}$ is a basis for $\pi_0 \cap G$ we must have $x_j \in X'$. This contradicts the assumption on x_j, and hence $x_{i_1}, x_{i_2}, \cdots, x_{i_t} \in X'$. Thus G is primitively generated.

THEOREM 2. *Let* $\{A, H, \overline{\Delta}\}$ *be a generalized simply-primitive Hopf algebra of finite type over a field* K *with characteristic* p. *If* $f: \{A, H, \overline{\Delta}\} \to \{H, H, \Delta\}$ *is a map, then* f(A) *is a Hopf subalgebra. Moreover, if one of the three conditions*

(a) $p = 0$,

(b) $p \neq 0$, $\gamma(A) = 0$,

(c) $p \neq 2$, H an exterior algebra $\Lambda_K(X)$,

holds, then

$$A \cong B \otimes C, \tag{3.18}$$

where B *and* C *are subalgebras of* A *such that* f *annihilates the positive degree elements of* B *and is injective on* C.

Proof. The first part follows at once from (3.8) and Proposition 3.1. To prove the second part, we may therefore assume that $f(A) = H$. Represent H and H_* as in Corollary 2.3. We assert that there is a sequence $\{y_i\}$ of elements of A with the following properties:

$$f(y_i) = x_i, \tag{3.19}$$

$$u_j \frown y_i = \delta_{ij} \quad \text{(the Kronecker delta)}. \tag{3.20}$$

We proceed by induction. Choose y_1 so that $f(y_1) = x_1$. Assume that $y_1, y_2, \cdots, y_k$ $(k \geq 1)$ have been chosen so that (3.19) and (3.20) hold for all i and j less than or equal to k. By an argument similar to the proof of (3.16) we may prove that if N is a normal monomial in $y_1, y_2, \cdots, y_k$ and y_j is its terminal factor, then

$$u_j \frown N = \frac{\partial N}{\partial x_j}. \tag{3.21}$$

For $1 \leq i \leq k$, we define

$$\sigma_i(a) = a - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!} (u_i^m \frown a) y_i^m \qquad (a \in A). \tag{3.22}$$

Note that $\sigma_i(a)$ is a finite sum, since $u_i^m \frown a = 0$ for m sufficiently large. In particular, if u_i has odd degree, then

$$\sigma_i(a) = a - (u_i \frown a) y_i. \tag{3.22'}$$

We assert that

$$(3.23) \qquad u_i \frown \sigma_i(a) = 0 .$$

If u_i has odd degree, then, by (3.9) and (3.22'),

$$u_i \frown \sigma_i(a) = u_i \frown a - [(u_i \frown a) - (u_i^2 \frown a)y_i] = 0 .$$

If u_i has even degree, we get

$$u_i \frown \sigma_i(a) = u_i \frown a - \sum_{m=1}^{\infty} \left[\frac{(-1)^{m-1}}{(m-1)!}(u_i^m \frown a)y_i^{m-1} + \frac{(-1)^{m-1}}{m!}(u_i^{m+1} \frown a)y_i^m\right].$$

A straightforward computation shows that the summation reduces to $u_i \frown a$, and hence (3.23) is proved.

Now choose $a_{k+1} \in A$ so that $f(a_{k+1}) = x_{k+1}$, and define

$$(3.24) \qquad a_{j,k} = \sigma_j \sigma_{j+1} \cdots \sigma_k(a_{k+1}) \qquad (1 \leq j \leq k) .$$

Finally, define $y_{k+1} = a_{1,k}$. We must prove that

$$(3.25) \qquad f(y_{k+1}) = x_{k+1} ,$$

$$(3.26) \qquad u_{k+1} \frown y_i = \delta_{i,k+1} \qquad (1 \leq i \leq k+1) ,$$

$$(3.27) \qquad u_i \frown y_{k+1} = 0 \qquad (1 \leq i \leq k) .$$

Using (3.8), we have

$$f(a_{k,k}) = f\sigma_k(a_{k+1}) = x_{k+1} - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!}(u_k^m \frown x_{k+1})x_k^m = x_{k+1} ,$$

since $u_k \frown x_{k+1} = 0$. Now let $1 < j \leq k$, and assume that $f(a_{j,k}) = x_{k+1}$. Then

$$f(a_{j-1,k}) = f\sigma_{j-1}(a_{j,k}) = x_{k+1} - \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!}(u_{j-1}^m \frown x_{k+1})x_{j-1}^m = x_{k+1} ,$$

since $u_{j-1} \frown x_{k+1} = 0$ for $1 < j \leq k$. Thus, by induction on j, we have $f(a_1, k) = x_{k+1}$, and (3.25) is proved.

We obtain (3.26) from (3.25) as follows. Using (3.8), we have

$$(3.28) \qquad f(u_{k+1} \frown y_i) = \delta_{i,k+1} \qquad (1 \leq i \leq k+1) .$$

If y_i has lower degree than u_{k+1}, then both sides of (3.28) are clearly zero. If y_i and u_{k+1} have the same degree then, since f is an isomorphism in degree zero, (3.26) follows from (3.28).

The proof of (3.27) is also by induction. Let $1 \leq i \leq k$. Then, by (3.23), we have $u_i \frown a_{i,k} = 0$. Now let $1 < j \leq i$, and assume that $u_i \frown a_{j,k} = 0$. Then

$$u_i \frown a_{j-1,k} = u_i \frown \sigma_{j-1}(a_{j,k}) = u_i \frown a_{j,k} \pm \sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m!}(u_{j-1}^m u_i \frown a_{j,k}) y_{j-1}^m = 0$$

by the inductive assumption. Thus by induction on j we have $u_i \frown a_{1,k} = 0$, and (3.27) is proved.

Let B be the subalgebra of A which is generated by 1 and all a such that $u_i \frown a = 0$ for all i. Let $b \in B$ have positive degree; then

$$u_i \frown f(b) = f(u_i \frown b) = f(0) = 0 \qquad (\text{all } i).$$

Thus f(b) must be zero.

Let C be the subalgebra of A generated by $y_1, y_2, \cdots$. It is clear that y_i is in the center of A if and only if the corresponding x_i is in the center of H; moreover, from (3.19), $h(y_i) \geq h(x_i)$. In particular, if y_i is not central, then $h(y_i) = h(x_i) = 2$. Suppose y_i is central; then

(a) if $p = 0$, $h(y_i) = h(x_i) = \infty$,

(b) if $p \neq 0$ and $\gamma(A) = 0$, then $h(y_i) = h(x_i) = p$,

(c) if $p \neq 2$ and $H = \Lambda_K(\{x_i\})$, then y_i and x_i have the same odd degree, and hence $h(y_i) = h(x_i) = 2$.

Thus if (a), (b), or (c) holds, then $h(y_i) = h(x_i)$ for all i, and hence f is injective on C.

It remains to prove (3.18). First we shall show that A = BC, in other words, that if $a \in A$, then

$$a = b_1 c_1 + \cdots + b_s c_s \qquad (b_i \in B,\ c_i \in C).$$

If N is a normal monomial in the y_i, let $\mu(N)$ denote the normal monomial obtained by replacing each y_i in N by the corresponding u_i.

Clearly each $a \in A$ is annihilated by all $\mu(N)$ of length j, if j is sufficiently large. Thus it suffices to show that if $a \in A$ is annihilated by all $\mu(N)$ of a given length j, then $a \in BC$. The proof is by induction on j. For $j = 1$ this follows from the definition of B. Assume it for some $j \geq 1$; let a be annihilated by all $\mu(N)$ of length $j + 1$, and consider the element

$$a' = a - \sum (\mu(N) \frown a)N, \tag{3.29}$$

where the summation extends over all N of length j. Clearly $\mu(N) \frown a$ is in B, and hence the sum is in BC. Let N' be of length j; we have

$$\begin{aligned}\mu(N') \frown a' &= \mu(N') \frown a - \sum \mu(N') \frown [(\mu(N) \frown a)N] \\ &= \mu(N') \frown a - \mu(N') \frown a \pm \sum (\mu(N')\,\mu(N) \frown a)N = 0,\end{aligned}$$

the sum vanishing by the assumption on a. Thus by the inductive assumption, $a' \in BC$. From (3.29) it then follows that $a \in BC$, and the induction is complete. Thus A = BC.

It remains to show that B and C are linearly disjoint. Suppose

(3.30) $$b_1 N_1 + b_2 N_2 + \cdots + b_t N_t = 0,$$

where the $b_i \in B$ and the N_i are distinct normal monomials in the y_i. Assume that N_1 is the longest monomial; it suffices to show that $b_1 = 0$. This follows if we apply $\mu(N_1) \frown$ to both sides of (3.30).

4. PROOF OF THE MAIN THEOREM

Let K be a field of characteristic p, and X an arcwise connected H-space with $H^*(X, K)$ of finite type. It is well known that if Δ is the product in X, then $H^*(X, K)$ is a Hopf algebra with coproduct

(4.1) $$\Delta^*: H^*(X, K) \to H^*(X, K) \otimes H^*(X, K).$$

(In (4.1), and (4.2) below, we use implicitly the isomorphism given by the Künneth formula.) The arcwise connectedness of X implies that $H^*(X, K) \cong K \cdot 1$, and hence we have a standard augmentation. If we assume that X is homotopy-associative and homotopy-commutative, then it follows from (1.2) and (1.3) that Δ^* is associative and anticommutative.

Let T be an arcwise connected space, with $H^*(T, K)$ of finite type, on which X operates (on the right and up to homotopy). The map $\bar{\Delta}: T \times X \to T$ induces a homomorphism of graded algebras

(4.2) $$\bar{\Delta}^*: H^*(T, K) \to H^*(T, K) \otimes H^*(X, K).$$

Conditions (1.4) and (1.5) then imply (3.1) and (3.2) (the latter with $\bar{\Delta}^*$ in place of $\bar{\Delta}$). Thus $\{H^*(T, K), H^*(X, K), \bar{\Delta}^*\}$ is a generalized Hopf algebra of finite type over K.

Let $f: X \to T$ commute with the operations of X on T and on itself (by right translations). Then (1.6) implies that $f^*: H^*(T, K) \to H^*(X, K)$ is a map of generalized Hopf algebras. Thus by the first part of Theorem 2, $f^*H^*(T, K)$ is a Hopf subalgebra. If we assume $p = 0$, then by (2.10) and Proposition 2.1, $H^*(X, K)$ is simply-primitive. Thus Theorem 1 follows from Theorem 2.

If in place of the homotopy-commutativity of X we assume that $H^*(X, K)$ is an exterior algebra and $p \neq 2$, then by (2.11) it is primitive and, moreover, simply-primitive. Thus the theorem of Borel follows from Theorem 2.

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The University of Michigan