RELATIONS AMONG THE LOTOTSKY, BOREL AND OTHER METHODS FOR EVALUATION OF SERIES

Ralph Palmer Agnew

1. INTRODUCTION

It is our first purpose to correct and extend the development in [1] of relations between the Lototsky and Borel methods for evaluation of series. These results and known facts about evaluability of Fourier series are used in Section 6 to obtain further information about the Lototsky method. Section 7 gives results on consistency which imply that the Lototsky and Abel methods are consistent.

As in Sections 9 and 11 of [1], we shall be concerned with analytic extensions. When $f(z)$ is defined, by a convergent power series or otherwise, and analytic over some neighborhood of the origin, we use the symbol $\{f(z)\}^*$ to denote the analytic extension of $f(z)$ along radial lines from the origin. Thus, for example, $\Sigma_{n=0}^{\infty} z^n$ converges to $1/(1 - z)$ only when $|z| < 1$, but $\{\Sigma_{n=0}^{\infty} z^n\}^* = 1/(1 - z)$ for each complex $z = x + iy$ which is not on the real half-line $x \geq 1$.

2. METHODS FOR EVALUATION OF SERIES

As usual, a series $u_0 + u_1 + \cdots$ is said to be evaluable to S by the Abel method A, and we write $A(u_0 + u_1 + \cdots) = S$, if the series $\Sigma_{k=0}^{\infty} u_k r^k$ converges when $|r| < 1$ and

$$\lim_{r \to 1-} \sum_{k=0}^{\infty} u_k r^k = S \tag{1}$$

holds. In accordance with an idea of Silverman and Tamarkin [6], the series will be said to be evaluable to S by the generalized Abel method A^* if

$$\lim_{r \to 1-} \left\{ \sum_{k=0}^{\infty} u_k r^k \right\}^* = S. \tag{2}$$

As usual, the series is said to be evaluable to S by the Borel integral method BI if the series $\Sigma_{k=0}^{\infty} (u_k t^k)/k!$ converges for each t and

$$\int_0^{\infty} e^{-t} \sum_{k=0}^{\infty} \frac{u_k}{k!} t^k \, dt = S \tag{3}$$

holds. As in [1], the series is evaluable to S by the generalized Borel integral method BI^* if

Received April 20, 1959.

$$\int_0^\infty e^{-t}\left\{\sum_{k=0}^{\infty}\frac{u_k}{k!}t^k\right\}^* dt = S. \tag{4}$$

Here and elsewhere, integrals over finite intervals can be Riemann or Lebesgue integrals, and $\int_0^\infty = \lim_{h\to\infty}\int_0^h$. The series Σu_n is evaluable to S by the Borel exponential method B if

$$\lim_{t\to\infty} e^{-t}\sum_{k=0}^{\infty}\frac{s_k}{k!}t^k = S \tag{5}$$

when $s_0, s_1, \cdots$ is the sequence of partial sums of Σu_n.

As in [1], the constants p_{nk} denote the Stirling numbers defined by the polynomial identities

$$p_n(x) = x(x+1)(x+2)\cdots(x+n-1) = \sum_{k=1}^{n} p_{nk}x^k \qquad (n = 1, 2, \cdots) \tag{6}$$

with $p_{nk} = 0$ when $k < 1$ and when $k > n$. The principal property of these numbers which we need is the recursion formula

$$p_{n+1,k} = np_{n,k} + p_{n,k-1}\,. \tag{7}$$

The Lototsky method for evaluation of series was initially defined in terms of a sequence-to-sequence transformation involving the numbers p_{nk}. For our present purposes, it is convenient to use the series-to-series version given in Section 6 of [1]. After making adjustments of subscripts so that the series being evaluated has the form $u_0 + u_1 + u_2 \cdots$, we see that the Lototsky method involves the transformation $U_0 = u_0$ and

$$U_n = \sum_{k=1}^{n}\frac{p_{n,k}}{(n+1)!}u_k \qquad (n = 1, 2, \cdots). \tag{8}$$

The series is evaluable L to S if the series $U_0 + U_1 + U_2 + \cdots$ converges to S.

3. A USEFUL FORMULA

Let $u_0 + u_1 + \cdots$ be a given series, and let

$$f(t) = \sum_{k=0}^{\infty}\frac{u_k}{k!}t^k \tag{9}$$

for those complex values of t for which the series converges. Supposing always that $|z| < 1$, let $\log[1/(1-z)]$ or $-\log(1-z)$ be unambiguously defined by the elementary formula

$$-\log(1-z) = z + z^2/2 + z^3/3 + \cdots,$$

and let

$$F(z) = f(-\log(1-z)) = \sum_{k=0}^{\infty} \frac{u_k}{k!}[-\log(1-z)]^k \tag{10}$$

for those complex values of z for which the series converges. We assume that there is a positive number R_1 such that the series in (9) converges when $|t| < R_1$ or, equivalently, that there is a positive number R such that $R \leq 1$ and the series in (10) converges when $|z| < R$. Then (10) defines a function $F(z)$ which is analytic when $|z| < R$; and accordingly,

$$F(z) = \sum_{k=0}^{\infty} \frac{F^{(n)}(0)}{n!} z^n \qquad (|z| < R). \tag{11}$$

Differentiation of (10) shows that the formula

$$F^{(n)}(z) = \frac{1}{(1-z)^n} \sum_{j=1}^{n} p_{nj} \sum_{k=j}^{\infty} \frac{u_k}{(k-j)!}[-\log(1-z)]^{k-j} \tag{12}$$

is valid when $n = 1$. Suppose that (12) holds for a given n. Considering the right side of (11) to be the product of $(1-z)^{-n}$ and another function, we differentiate this product and use the recursion formula (7) to obtain the result of replacing n by $n+1$. Thus it is proved that (12) holds when $n = 1, 2, 3, \cdots$. The use of (11), (10), and (12) gives

$$F(z) = u_0 + \sum_{n=1}^{\infty} \left[\sum_{j=1}^{n} \frac{p_{nj}}{n!} u_j \right] z^n \qquad (|z| < R). \tag{13}$$

This result and (8) yield the formula

$$F(z) = \sum_{n=0}^{\infty} (n+1) U_n z^n \qquad (|z| < R), \tag{14}$$

which involves the terms of the Lototsky transform of the series $u_0 + u_1 + \cdots$. Starting with (14), replacing $F(z)$ by the last member of (10), replacing the functions by their analytic extensions, and integrating the result gives the formula

$$\int_0^r \left\{ \sum_{k=0}^{\infty} \frac{u_k}{k!}[-\log(1-z)]^k \right\}^* dz = r \left\{ \sum_{n=0}^{\infty} U_n r^n \right\}^*, \tag{15}$$

which is valid when $|r|$ is sufficiently small. Changing the variable of integration in (15) by setting $t = -\log(1-z)$ and $z = 1 - e^{-t}$, we find

$$\int_0^{-\log(1-r)} e^{-t}\left\{\sum_{k=0}^{\infty}\frac{u_k}{k!}t^k\right\}^* dt = r\left\{\sum_{n=0}^{\infty} U_n r^n\right\}^*, \tag{16}$$

a relation which is valid when r is sufficiently small. This is the fundamental formula relating BI and L transforms of series. To make effective use of it, we need the following lemma.

LEMMA. *If one of the two series*

$$\sum_{k=0}^{\infty}\frac{u_k}{k!}t^k, \qquad \sum_{n=0}^{\infty} U_n r^n \tag{17}$$

has a positive radius of convergence, then so also does the other, and (16) *is valid when* $|r|$ *is sufficiently small.*

We have obtained the conclusion of the lemma under the hypothesis that the first series in (17) has a positive radius of convergence. We complete the proof by showing that if the second series in (17) has a positive radius of convergence, then so also does the first. Let Σu_n be a series whose Lototsky transform ΣU_n is such that $\Sigma |U_n| r^n < \infty$ for some fixed r $(0 < r < 1)$. To fit the notation of [1], let $s_n = u_0 + u_1 + \cdots + u_{n-1}$ and $\sigma_n = U_0 + U_1 + \cdots + U_{n-1}$ for each $n = 1, 2, \cdots$. Then $\Sigma |\sigma_n| r^n < \infty$. Choose a constant H such that $|\sigma_n| r^n < H$ for each n. Then, with the notation of [1, Section 5],

$$|s_n| \leq \sum_{k=1}^{n} k!\,|q_{nk}|\,|\sigma_k| \leq \sum_{k=1}^{n} k!|q_{nk}| H r^{-k}$$

$$\leq H r^{-n}\sum_{k=1}^{n} k|q_{nk}| = H r^{-n} Q_n \leq H' \frac{n!}{(r \log 2)^n}$$

for some constant H'. This implies that, for some constant H",

$$|u_n| \leq H'' n!/(r \log 2)^n.$$

Therefore the first series in (17) converges when $|t| < r \log 2$, and the lemma is proved. It follows from the lemma and from uniqueness of radial analytic extensions that, for each r in $0 < r < 1$ for which one of the two members of (16) exists, the two members both exist and are equal.

4. RELATIONS AMONG METHODS

We now state and prove some consequences of (16).

THEOREM 1. *The methods* BI* *and* A*L *are equivalent methods for evaluation of series; that is,* BI* ~ A*L.

Suppose first that Σu_n is evaluable BI* to S. Then the left member of (16) exists when $0 < r < 1$ and converges to S as $r \to 1$. It then follows from (16) that

$$\lim_{r \to 1} \left\{ \sum_{k=0}^{\infty} U_n r^n \right\}^* = S.$$

But $\Sigma U_n r^n$ is the Abel transform of the Lototsky transform of Σu_n, and it follows that the series is evaluable A*L to S. Thus A*L ⊃ BI*. A similar argument shows that if Σu_n is evaluable A*L to S, then it is also evaluable BI* to S. Thus BI* ⊃ A*L. This completes the proof.

In [1], it was erroneously asserted that BI* ~ AL. The following example clarifies this matter. Let Σu_n be the series whose Lototsky series-transform ΣU_n has terms defined by the identity

$$U_0 + U_1 z + U_2 z^2 + \cdots = (1 + 2z)^{-1}.$$

Then Σu_n is evaluable A*L, and hence evaluable BI*; but the series is not evaluable AL because the radius of convergence of $\Sigma U_n z^n$ is less than 1.

It is a trivial consequence of Theorem 1 that A*L ⊃ BI and BI* ⊃ AL. The following theorem is less obvious.

THEOREM 2. AL ⊃ BI.

To prove this theorem, let Σu_n be evaluable BI to S. Then the series in (9) must converge for each t, and it follows that the series in (10) must converge for each z for which $|z| < 1$. Therefore the star superscripts can be removed from the series in (16). The result follows.

It is not possible to modify the above argument to prove that BI ⊃ AL. In fact, the weaker relation BI ⊃ L is false. Some explicit examples of series evaluable L but nonevaluable BI are given in [1].

THEOREM 3. *The methods* L *and* BI *are consistent.*

This theorem follows from Theorem 2 and a standard argument. Suppose that Σu_n is evaluable BI to S_1 and is evaluable L to S_2. Since A is regular, the series is evaluable AL to S_2. But AL ⊃ BI, and therefore $S_2 = S_1$. This proves the theorem. Since BI ⊃ B, it follows that L and B are consistent.

THEOREM 4. *The inclusion relations* BI ⊃ L *and* L ⊃ BI *are both false; that is,* L *and* BI *have overlapping evaluability fields. Likewise,* L *and* B *have overlapping evaluability fields.*

We have already noted that the relation BI ⊃ L is false. Since BI ⊃ B, the relation B ⊃ L must also be false. The proof is completed in the next section. In [1], the questions whether L ⊃ BI and L ⊃ B were left unanswered.

5. AN EXAMPLE OF A SERIES EVALUABLE BI AND B BUT NONEVALUABLE L

Let $u_0 + u_1 + \cdots$ be the series whose terms are defined by the identity

$$\sum_{n=0}^{\infty} \frac{u_n}{n!} t^n = f(t) = \exp[-ie^t]. \qquad (18)$$

This series is obviously evaluable BI. Moreover, with the aid of formulas given by Hardy [4, page 182], it is easy to see that the series is also evaluable by the Borel exponential method B. In accordance with (10) and (14), the Lototsky transform $U_0 + U_1 + \cdots$ is determined by the identity

$$\sum_{n=0}^{\infty} (n+1)U_n z^n = F(z) = f(-\log(1-z)) = \exp[-i/(1-z)], \tag{19}$$

in which the functions are analytic and the series converges when $|z| < 1$. Without estimating the terms U_n, we shall show that ΣU_n diverges, and hence that Σu_n is not evaluable L. If we suppose that ΣU_n converges, then there is a constant M such that $|U_n| \leq M$, and (19) implies that

$$|F(z)| \leq M \sum_{n=0}^{\infty} (n+1)|z|^n = M(1-|z|)^{-2} \tag{20}$$

when $|z| < 1$. But when $0 < \theta < \pi/2$ and $z = \frac{1}{2} + \frac{1}{2}e^{2i\theta}$, we find that $|z| = \cos\theta < 1$, $1 - |z| = 2\sin^2(\theta/2)$, and $(1-z)^{-1} = 1 + i\cot\theta$. These formulas and (19) imply that

$$(1-|z|)^2 |F(z)| = 4\sin^4\frac{\theta}{2}\, e^{\cot\theta} > 4\sin^4\frac{\theta}{2}\,\frac{\cot^5\theta}{5!}. \tag{21}$$

Since the last member of (21) is not bounded, we have a contradiction of (20). Therefore Σu_n is not evaluable L.

6. FOURIER SERIES

The Lototsky sequence transform $\sigma_1, \sigma_2, \cdots$ of a sequence $s_0, s_1, \cdots$ is defined by

$$\sigma_n = \sum_{k=1}^{n} p_{nk} s_{k-1}. \tag{22}$$

Starting with the familiar Dirichlet formula

$$s_k(x) = \frac{2}{\pi}\int_0^{\pi/2} \frac{f(x+2t) + f(x-2t)}{2}\,\frac{\sin(2k+1)t}{\sin t}\,dt \tag{23}$$

for the partial sums of the ordinary Fourier series of a function f which has period 2π and is integrable over a period, we can replace $\sin(2k+1)t$ by the imaginary part of $e^{it}e^{2kit}$ and use (22) and (6) to find that the elements of the L transform of the Fourier series are

$$\sigma_n(x) = \int_0^{\pi/2} \frac{f(x+2t) + f(x-2t)}{2} L_n(t)\,dt, \tag{24}$$

where

$$L_n(t) = \frac{2}{n!\,\pi \sin t} \Im\, e^{it}(e^{2it} + 1)(e^{2it} + 2) \cdots (e^{2it} + n - 1). \tag{25}$$

In particular,

$$L_1(t) = \frac{2}{\pi}, \quad L_2(t) = \frac{4}{\pi}\cos^2 t, \quad L_3(t) = \frac{16}{3\pi}\cos^4 t,$$

$$L_4(t) = \frac{\cos^2 t}{3\pi}[-1 + 4\cos^2 t + 16\cos^4 t], \tag{26}$$

$$L_5(t) = \frac{\cos^2 t}{15\pi}[-5 + 48\cos^4 t + 64\cos^6 t],$$

$$L_6(t) = \frac{\cos^2 t}{45\pi}[-13 - 20\cos^2 t + 64\cos^4 t + 192\cos^6 t + 128\cos^8 t].$$

The first of the formulas

$$1 = \int_0^{\pi/2} L_n(t)\,dt, \qquad L_n = \int_0^{\pi/2} |L_n(t)|\,dt \tag{27}$$

is an obvious consequence of the fact that $\sigma_n(x) \equiv 1$ when $f(x) \equiv 1$, and the second de-fies the Lebesgue constants L_n for the Lototsky transformation.

With the aid of familiar facts about Fourier series that can be found in [4] or [7], we give an indirect proof that L is not Fourier-effective and hence that the sequence $L_1, L_2, \cdots$ is unbounded. Let $f(x)$ be a continuous function of period 2π having a Fourier series $u_0(x) + u_1(x) + \cdots$ which fails to be evaluable by the Borel integral method BI when x has a particular value x_0. Then $u_n(x_0) \to 0$ as $n \to \infty$, and hence the series in the integrand in the left member of (16) is convergent for each complex t. Therefore $\Sigma u_n(x_0)$ is nonevaluable BI*. Theorem 1 then implies that $\Sigma u_n(x_0)$ is nonevaluable A*L and hence is nonevaluable L. Thus the methods A*L, AL, and L are not Fourier-effective, and our result is established.

This implies that the method A*L cannot include any of those methods for evaluation of series that are Fourier-effective. In particular, when $r > 0$ and C_r denotes the Cesàro method of order r, the relations

$$A^*L \supset C_r, \quad AL \supset C_r, \quad C_rL \supset C_r, \quad L \supset C_r, \quad A^*L \supset A, \quad AL \supset A, \quad L \supset A$$

are all false.

7. CONSISTENCY OF A*L AND A*

The results of the preceding section leave open the question whether the Lototsky method L is consistent with the Abel and Cesàro methods A and C_r. We now give affirmative answers to this and other questions by proving the following theorem.

THEOREM 5. *The methods* A*L *and* A* *are consistent methods for evaluation of series.*

Because of Theorem 1, this is a consequence of the following theorem.

THEOREM 6. *The methods* BI* *and* A* *are consistent methods for evaluation of series.*

To prove this theorem, we suppose that Σu_n is a given series which is evaluable A* to V_1 and is evaluable BI* to V_2. Then, for some positive number R, the series in

$$(28) \qquad A(r) = \left\{ \sum_{k=0}^{\infty} r^k u_k \right\}^* = (1 - r)\left\{ \sum_{k=0}^{\infty} r^k s_k \right\}^* = V_1 + o(1)$$

converge when $|r| < R$ and (28) holds as $r \to 1^-$. This implies that

$$(29) \qquad \limsup_{n\to\infty} |u_n|^{1/n} \leq R^{-1}, \qquad \limsup_{n\to\infty} |s_n|^{1/n} \leq R^{-1}.$$

Moreover,

$$(30) \qquad F(t) = \int_0^t e^{-x} \left\{ \sum_{k=0}^{\infty} \frac{x^k}{k!} u_k \right\}^* dx = V_2 + o(1)$$

as $t \to \infty$. Since (29) implies that the series in (30) converges for all values of x, we can remove the star superscript. Moreover we can replace u_k by $s_k - s_{k-1}$, where $s_{-1} = 0$, to put (30) in the form

$$(31) \qquad F(t) = \int_0^t \frac{d}{dx}\left\{ e^{-x} \sum_{k=0}^{\infty} \frac{x^{k+1}}{(k+1)!} s_k \right\} dt = V_1 + o(1);$$

and hence

$$(32) \qquad F(t) = e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} s_{k-1} = V_1 + o(1).$$

This establishes, for the case in which we are interested, validity of familiar formulas involving the BI transform of a sequence and the B transform of the sequence obtained by prefixing a zero; see [4, page 182] and [5]. The remainder of the proof is a modification of the proof of Doetsch [2] that B and A are consistent; see also Gaier [3] and references given there. If we take Laplace transforms of the first and second members of (32), we find that

$$(33) \qquad \int_0^\infty e^{-st} F(t)\, dt = \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \int_0^\infty e^{-(s+1)t} t^k\, dt \right\} s_{k-1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+1}} s_{k-1} = \sum_{k=0}^{\infty} \frac{1}{(s+1)^{k+2}} s_k$$

when $(s + 1)^{-1} < R$, the termwise integration being justified by the fact that we obtain convergent integrals and series by replacing each function and number in (33) by its absolute value. With the aid of (32), we see that the first member of (33) exists and is analytic over the half-plane $s = \sigma + it$ $(\sigma > 0)$. Putting $s = (1 - r)/r$ in (33) therefore shows that the formula

$$\int_0^\infty e^{-[(1-r)/r]t} F(t)\,dt = r^2 \left\{ \sum_{k=0}^{\infty} r^k s_k \right\}^* \tag{34}$$

is valid when $0 < r < 1$. Consequently,

$$\frac{1-r}{r} \int_0^\infty e^{-[(1-r)/r]t} F(t)\,dt = r \left[(1-r) \left\{ \sum_{k=0}^{\infty} r^k s_k \right\} \right]^* \tag{35}$$

when $0 < r < 1$. With the aid of (28) and (32), we can let $r \to 1$ in (35) to obtain the equality $V_2 = V_1$ and complete the proof of Theorem 6.

REFERENCES

1. R. P. Agnew, *The Lototsky method for evaluation of series*, Michigan Math. J. 4 (1957), 105-128.
2. G. Doetsch, *Über den Zusammenhang zwischen Abelscher und Borelscher Summabilität*, Math. Ann. 104 (1931), 403-414.
3. D. Gaier, *Complex Tauberian theorems for power series*, Trans. Amer. Math. Soc. 75 (1953), 48-68.
4. G. H. Hardy, *Divergent series*, Oxford, 1949.
5. J. Karamata, *Über die Indexverschiebung beim Borelschen Limitierungsverfahren*, Math. Z. 45 (1939), 635-641.
6. L. L. Silverman and J. D. Tamarkin, *On the generalization of Abel's theorem for certain definitions of summability*, Math. Z. 29 (1929), 161-170.
7. A. Zygmund, *Trigonometrical series*, Warszawa-Lwow, 1935.

Cornell University