DIVERGENCE OF RANDOM POWER SERIES

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1. INTRODUCTION

Let $\phi_n(t)$ $(n = 0, 1, 2, \cdots)$ be the Rademacher functions, that is, let

$$\phi_n(t) = (-1)^j \text{ for } j/2^n \leq t < (j+1)/2^n \qquad (j = 0, 1, \cdots, 2^n - 1;\ n = 0, 1, 2, \cdots).$$

Given any sequence of complex numbers $\{a_n\} = \{a_0, a_1, a_2, \cdots\}$, we denote by $\mathscr{F}\{a_n\}$ the family of power series

$$P(z;\ t) = \sum_{n=0}^{\infty} \phi_n(t)\, a_n\, z^n \qquad (0 \leq t < 1). \tag{1}$$

It is well known that if

$$\sum_{n=0}^{\infty} |a_n|^2 = \infty, \tag{2}$$

then *almost all* series (1) diverge *almost everywhere* on $|z| = 1$. Here *almost all* refers to the set of t (in the usual Lebesgue sense), while *almost everywhere* refers to the set of z on the circumference of the unit circle (again in the usual sense).

Only recently [1] was it observed that some nontrivial interesting assertions similar to the above with *almost everywhere* replaced by *everywhere* can be made. Here a new result of this kind, going beyond that indicated in [1], will be established.

2. STATEMENT OF RESULTS

Our main result is the following

THEOREM. *Let $\{c_n\}_{n=0}^{\infty}$ be a monotone sequence of positive numbers tending to zero and satisfying the condition*

$$\limsup_{n \to \infty} \frac{\sum_{j=0}^{n} c_j^2}{\log 1/c_n} > 0. \tag{3}$$

If $\{a_n\}_0^{\infty}$ is a sequence of complex numbers satisfying the condition

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(4) $$|a_n| \geq c_n$$

for all n, *then almost all series of* $\mathscr{F}\{a_n\}$ *diverge everywhere on* $|z| = 1$.

As a specially important case, we have the

COROLLARY. *If* $\{a_n\}$ *satisfies the condition*

(5) $$|a_n| \geq c/\sqrt{n} \qquad (n > N)$$

for some $c > 0$, *then almost all series (1) diverge everywhere on* $|z| = 1$.

It will be seen from the proof that the statements above can be strengthened by substituting the words *have unbounded partial sums* for the word *diverge*.

Corresponding to any sequence $\{a_n\}$ of complex numbers differing from zero, the sequence

$$c_n = \min_{0 \leq k \leq n} |a_k| \qquad (n = 0, 1, 2, \cdots)$$

is monotone and satisfies (4). It is only necessary to check whether (3) holds in order to see whether the theorem can be applied.

3. PROOF OF THE THEOREM

The theorem being utterly trivial otherwise, we assume $a_n \to 0$. It may also be assumed, without loss of generality, that the limit superior in (3) is greater than 8. Then there exists, by (3) and (4), an increasing sequence of positive integers n_k $(k = 0, 1, \cdots)$ satisfying

(6) $$\sum_{j=n_{k-1}+1}^{n_k} |a_j|^2 > 8 \log 1/c_{n_k} \qquad (k = 1, 2, \cdots).$$

Since the sequence $\{c_n\}$ is monotone, (4) implies that

(7) $$|a_n| \geq c_{n_k} \text{ for } n \leq n_k \qquad (k = 1, 2, \cdots).$$

It follows from (6) that for all sufficiently large k there exist integers $n_{k,\ell}$ $(n_{k-1} = n_{k,0} < n_{k,1} < \cdots < n_{k,\gamma_k} = n_k)$ such that

(8) $$1 < \sum_{j=n_{k,\ell-1}+1}^{n_{k,\ell}} |a_j|^2 < 2 \qquad (\ell = 1, 2, \cdots, \gamma_k).$$

From (7) and (8) it follows that

(9) $$n_{k,\ell} - n_{k,\ell-1} < 2/c_{n_k}^2 \qquad (\ell = 1, 2, \cdots, \gamma_k),$$

while from (6) and (8) we have

$$\gamma_k > 4 \log 1/c_{n_k}. \tag{10}$$

For any z_0 on the circumference of the unit circle, we have either

$$\sum_{j=n_{k,\ell-1}+1}^{n_{k,\ell}} (\Re\{a_j z_0^j\})^2 > 1/2, \tag{11}$$

or the same inequality with the real part replaced by the imaginary part; since the treatment of the two cases is exactly the same, we assume that (11) holds.

If we denote by $\mu\{t: \cdots\}$ the Lebesgue measure of the set of numbers t $(0 \leq t < 1)$ satisfying the condition to the right of the colon in the braces, it follows from the well-known inequalities of Kolmogorov (see for example [2, p. 235]) that (11) implies

$$\mu\left\{t: \max_{n_{k,\ell-1} < m \leq n_{k,\ell}} \left| \sum_{j=n_{k,\ell-1}+1}^{m} \phi_j(t)\, a_j z_0^j \right| < \frac{1}{e}\right\} < \frac{1}{e}$$

for $\ell = 1, 2, \cdots, \gamma_k$ and all sufficiently large k. Therefore, by independence of the events corresponding to different ℓ,

$$\mu\left\{t: \max \left| \sum_{j=\alpha}^{\beta} \phi_j(t)\, a_j z_0^j \right| < \frac{1}{e}\right\} < e^{-\gamma_k}, \tag{12}$$

the maximum being taken over all integers α, β satisfying

$$n_{k-1} < \alpha < \beta \leq n_k, \qquad \beta - \alpha < 2/c_{n_k}^2. \tag{13}$$

Let us put

$$\lambda_k = [8\pi e c_{n_k}^{-3}] + 1, \tag{14}$$

the square brackets denoting the integral part. From (12) we have

$$\mu\left\{t: \min_{s=1,2,\cdots,\lambda_k} \max \left| \sum_{j=\alpha}^{\beta} \phi_j(t)\, a_j \exp(2\pi i s j/\lambda_k) \right| < \frac{1}{e}\right\} < \lambda_k e^{-\gamma_k}, \tag{15}$$

the maximum being again taken over all α, β satisfying (13). But

$$\left| \sum_{j=\alpha}^{\beta} \phi_j(t)\, a_j z^j \right| - \left| \sum_{j=\alpha}^{\beta} \phi_j(t)\, a_j z_0^j \right| \leq (\beta - \alpha) \left| z - z_0 \right| \sum_{j=\alpha}^{\beta} \left| a_j \right|$$

for z and z_0 on the circumference of the unit circle. Hence, if

$$\left| \sum_{j=\alpha}^{\beta} \phi_j(t)\, a_j z_0^j \right| \geq \frac{1}{e}$$

for some $|z_0| = 1$, then

$$\left| \sum_{j=\alpha}^{\beta} \phi_j(t)\, a_j z^j \right| \geq \frac{1}{e} - \frac{\pi}{\lambda_k}(\beta - \alpha) \sum_{j=\alpha}^{\beta} |a_j|, \tag{16}$$

for all z on $|z| = 1$ whose argument differs from that of z_0 by less than π/λ_k. From (7) we have

$$\sum_{j=\alpha}^{\beta} |a_j| \leq c_{n_k}^{-1} \sum_{j=\alpha}^{\beta} |a_j|^2,$$

whence we obtain, by (8), (13), and (14),

$$\frac{\pi}{\lambda_k}(\beta - \alpha) \sum_{j=\alpha}^{\beta} |a_j| < \frac{1}{2e}.$$

Thus we have from (15) and (16)

$$\mu\left\{ t: \min_{|z|=1} \max \left| \sum_{j=\alpha}^{\beta} \phi_j(t)\, a_j z^j \right| > \frac{1}{2e} \right\} > 1 - \lambda_k e^{-\gamma_k}. \tag{17}$$

Since the right-hand side of (17) tends to 1 by (10) and (14), it follows that for almost all t the event in the braces occurs for infinitely many k. This completes the proof.

4. REMARKS

4.1. We do not know whether the condition (3) is best possible; we can, however, show that (3) cannot be replaced by (2). Indeed, *there exists a monotone sequence satisfying* (2) *such that almost all series of* $\mathscr{F}(a_n)$ *have on every arc of* $|z| = 1$ *a set of points of convergence whose power is that of the continuum.* The main new tool in the construction is the following

LEMMA. *For every* $\alpha < \beta$ *and every* $\varepsilon > 0$ *there are only* $o(2^n)$ *choices of sign for which*

$$\min_{\alpha \leq t \leq \beta} \ \max_{1 \leq m \leq n} \left| \sum_{j=1}^{m} \pm e^{2\pi i j t} \right| > \varepsilon \sqrt{n}.$$

4.2. Results similar to those of the present paper but pointing, so to speak, in the opposite direction were given—in a different form—by Salem and Zygmund [3]. Thus, according to Theorem 5.1.5 of [3], almost all series of $\mathscr{F}\{a_n\}$ are uniformly convergent in $|z| \leq 1$ if $\Sigma_{n=0}^{\infty} |a_n|^2 < \infty$ and the remainders $R_n = \Sigma_{j=n}^{\infty} |a_j|^2$ are small enough so that

$$\sum \frac{R_n^{1/2}}{n(\log n)^{1/2}} < \infty. \tag{18}$$

Moreover, according to Theorem 5.5.1 of [3], if $|a_n|$ is decreasing and $R_n(\log n)^p$ is increasing for some $p > 1$, then (18) is necessary and sufficient for the almost everywhere uniform convergence of $\mathscr{F}(a_n)$.

4.3. We could obviously replace the Rademacher functions $\phi_n(t)$ by other independent functions, for example, by the Steinhaus functions.

REFERENCES

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