# ON SOME PROBLEMS BY ERDÖS, HERZOG AND PIRANIAN

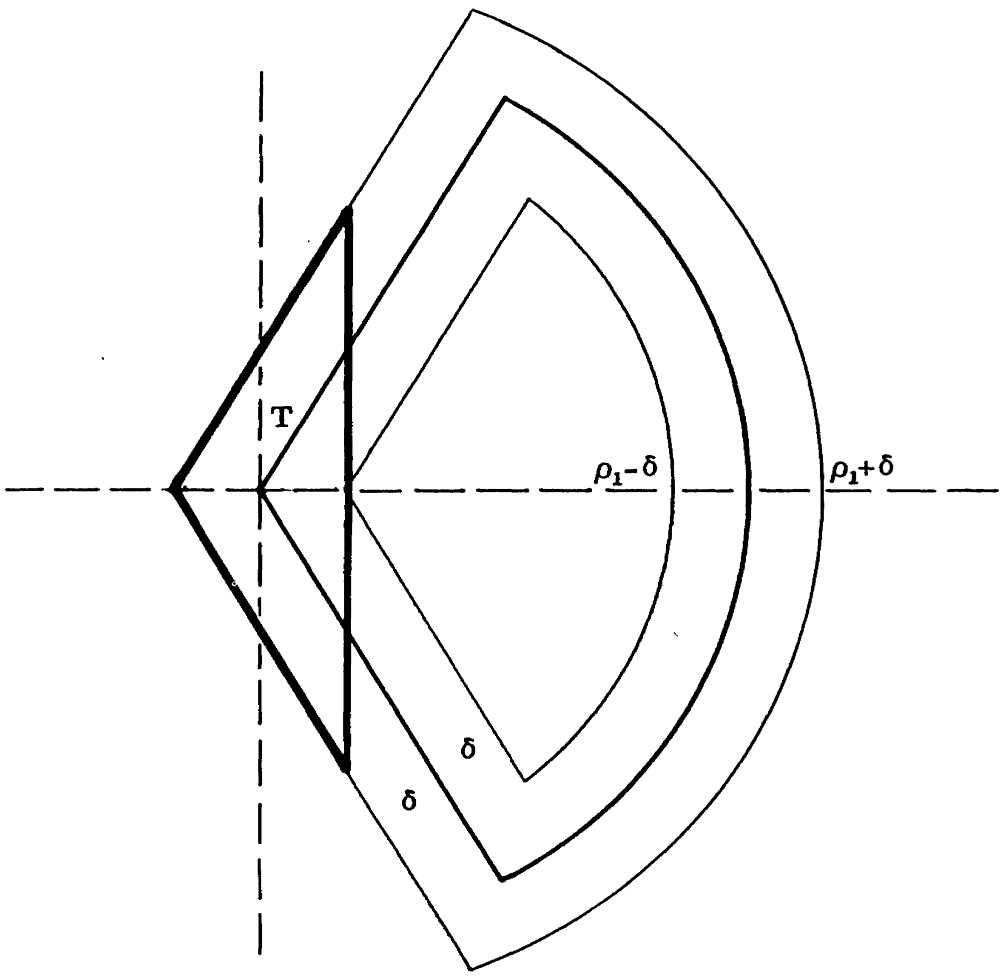
Chr. Pommerenke

Let $f(z)$ be a polynomial with highest coefficient 1. The lemniscate $|f(z)| = 1$ is denoted by $C$, and its interior $|f(z)| < 1$ by $E$. A paper by P. Erdös, F. Herzog and G. Piranian [1] raises some questions about the geometric properties of $C$ and $E$. I shall give an answer to a few of these questions.

The answer to the second part of Problem 10 in [1] is negative:

THEOREM 1. *There exists a polynomial* $f(z)$ *such that, for some point* $z_0$ *lying on the lemniscate* $C$ *of* $f(z)$ *and on a line of support of* $C$, $|z - z_0| < 2$ *for every* $z \in C$.

To show this, let $S(\rho)$ denote the closed sector $0 \leq |z| \leq \rho$, $|\arg z| \leq \pi/3$. Since the area of $S(2)$ is $4\pi/3 > \pi$, the transfinite diameter of $S(2)$ is greater than 1 (Pólya [3, p. 280]). Hence there exists a $\rho_1 < 2$ such that $S(\rho_1)$ has transfinite diameter 1. Now $S(\rho_1)$ can be approximated arbitrarily closely by the lemniscates $C$ of certain polynomials $f(z)$ (Faber [2, p. 100]). Let $0 < \delta < (2 - \rho_1)/6$. There




Received January 15, 1959.

exists a lemniscate C whose points lie within a distance $\delta$ of the boundary of $S(\rho_1)$, and which has at least one point of support $z_0$ lying in the heavily drawn triangle T in the figure. Since $|z_0| \leq 5\delta$,

$$|z - z_0| \leq \rho_1 + \delta + 5\delta < 2$$

for every $z \in C$.

The remainder of this note deals with the case where E is connected (in which case f(z) is called a K-polynomial in [1]). This special case seems much easier than the general one, since use can be made of the theory of univalent functions. The set E is connected if and only if all the zeros of the derivative f'(z) are contained in E ([1, p. 142]).

The following theorem answers part of Problem 12 in [1]:

THEOREM 2. *If* E *is connected, the length of* C *is at least* $2\pi$, *with equality only for* $f(z) = z^n$.

The region G outside of C is simply connected. Let

$$w = g(z) = (f(z))^{1/n} = (z^n + \cdots)^{1/n} = z + \cdots.$$

This function is regular and uniform in $G \cup C$ (except for the simple pole at $\infty$). Since all the zeros of $f'(z)$ belong to E, we have

$$g'(z) = \frac{1}{n} f'(z)\, (f(z))^{\frac{1}{n}-1} \neq 0,$$

for $z \in G \cup C$. On C we have $|g(z)| = 1$. Therefore g(z) is univalent in $G \cup C$ (Pólya and Szegö [4, Vol. 1, Section III, p. 122, Problem 193]). Its inverse function

$$z = \psi(w) = w + b_0 + \frac{b_1}{w} + \cdots$$

maps $|w| \geq 1$ conformally onto $G \cup C$. The length of C is

$$\int_0^{2\pi} |\psi'(e^{i\theta})|\, d\theta \geq \left|\int_0^{2\pi} \psi'(e^{i\theta})\, d\theta\right| = 2\pi,$$

with equality only for $\psi(w) = w$, that is, for $f(z) = z^n$.

Next, I shall establish the conjecture in Problem 14:

THEOREM 3. *Let* $\zeta = (z_1 + \cdots + z_n)/n$, *where* $z_1, \cdots, z_n$ *are the zeros of* f(z). *If* E *is connected, then* C *is contained in the circle* $|z - \zeta| < 2$.

To prove this, let $z = \psi(w)$ be the function in the last proof. From the relation

$$w = g(z) = (f(z))^{1/n} = (z^n - n\zeta z^{n-1} + \cdots)^{1/n} = z - \zeta + \frac{d_1}{z} + \cdots,$$

we get

$$z = \psi(w) = w + \zeta + \cdots.$$

Since this function maps $|w| = 1$ onto C, it follows [4, Vol. 2, Section IV, p. 25, Problem 140] that for every $c \in C$

$$|c - \zeta| \leq 2,$$

with equality only for $\psi(w) = w + \zeta + \dfrac{e^{i\alpha}}{w}$. Since no polynomial $f(z)$ corresponds to the latter function $\psi(w)$, equality can not occur.

THEOREM 4. *If* E *is connected and has diameter* d *and width* b, *then*

$$2 \leq d < 4, \qquad 0 < b^2 \leq 32/3, \qquad b^2 + d^2 \leq 64/3.$$

*Proof.* A consideration of the function $\psi(w)$ used above shows that $2 \leq d < 4$, and that the bounds are sharp [4, Vol. 2, Section IV, p. 24, Problem 141].

To prove the inequality on b, we note that, for each value c on C, the function

$$\begin{aligned}(\psi(w^2) - c)^{\frac{1}{2}} &= (w^2 - (b_0 - c) + b_1/w^2 + \cdots)^{\frac{1}{2}} \\ &= w + \frac{1}{2}(b_0 - c)/w + \left(\frac{1}{2}b_1 - \frac{1}{8}(b_0 - c)^2\right)/w^3 + \cdots\end{aligned}$$

is regular and univalent in $|w| > 1$. Hence

$$\left|\frac{1}{2}(b_0 - c)\right|^2 + 3\left|\frac{1}{2}b_1 - \frac{1}{8}(b_0 - c)^2\right|^2 \leq 1 \tag{1}$$

[4, Volume 2, Section IV, p. 24, Problem 136]. It will be convenient to write

$$(b_0 - c)\exp\left(-\frac{i}{2}\arg b_1\right) = 2(x + iy), \qquad |b_1| = \beta.$$

Then $0 \leq \beta < 1$, and condition (1) takes the form

$$\left|x + iy\right|^2 + \frac{3}{4}\left|\beta - x^2 + y^2 - 2ixy\right|^2 \leq 1,$$

that is,

$$y^2 + \frac{3}{4}(y^4 + 2\beta y^2 + \beta^2) + x^2\left\{1 + \frac{3}{4}[x^2 + 2(y^2 - \beta)]\right\} \leq 1. \tag{2}$$

If the quantity in braces is negative, then

$$3[x^2 + 2(y^2 - \beta)] < -4,$$

and this implies that $y^2 < \beta - 2/3 \leq 1/3$. If the quantity in braces is nonnegative, then (2) implies that

$$y^2 + \frac{3}{4}(y^4 + 2\beta y^2 + \beta^2) - 1 \leq 0,$$

and therefore that

(3) $$y^2 \leq -\frac{2}{3} - \beta + \frac{4}{3}\sqrt{1 + \frac{3}{4}\beta}\,.$$

The right member of this inequality is a decreasing function of $\beta$ ($\beta > 0$). Therefore its maximum occurs at $\beta = 0$, and it follows that $y^2 \leq 2/3$. Since b is at most four times the greatest possible value of $|y|$, $b^2 \leq 32/3$.

To obtain a useful relation between the quantities $x^2 + y^2 = r^2$ and $\beta$, we write (2) in the form

$$r^2 + \frac{3}{4}(r^4 - 2\beta r^2 + \beta^2) \leq 1 - 3\beta y^2 \leq 1\,.$$

This inequality implies that

(4) $$r^2 \leq -\frac{2}{3} + \beta + \frac{4}{3}\sqrt{1 - \frac{3}{4}\beta}\,.$$

Inequalities (3) and (4) now lead to the result

$$y^2 + r^2 \leq \frac{4}{3}\left(\sqrt{1 + \frac{3}{4}\beta} + \sqrt{1 - \frac{3}{4}\beta} - 1\right).$$

Since the function $\sqrt{1 + t}$ is concave, it follows that $y^2 + r^2 \leq 4/3$ for all c on C, and therefore that $b^2 + d^2 \leq 64/3$.

*Remarks.* Since $(b + d)^2 \leq 2(b^2 + d^2)$ and $bd \leq (b^2 + d^2)/2$, the theorem implies that

$$b + d < \sqrt{128/3} \approx 6.53, \qquad bd < 32/3\,.$$

It is clear that our upper bounds for b, b + d and bd can be slightly improved. To obtain lower bounds on the suprema of these quantities, we consider first the function

(5) $$z = (w^3 + w^{-3})^{\frac{1}{3}} = w + \cdots,$$

which maps the domain $|w| > 1$ onto the plane from which the three rectilinear segments

$$\left[-2^{\frac{1}{3}} e^{\frac{2\pi i}{3}k},\ 2^{\frac{1}{3}} e^{\frac{2\pi i}{3}k}\right] \qquad (k = 1, 2, 3)$$

have been deleted. Since the development of (5) begins with the term w, the configuration L of the three segments has transfinite diameter 1, and therefore it can be approximated by certain lemniscates C. It is easy to see that L has

$$b = \sqrt{3}\, 2^{\frac{1}{3}} > 2.18\,,$$

whereas Erdös, Herzog and Piranian [1, Problem 15] conjectured that $b \leq 2$ in all cases.

The function

$$z = (w^2 + \alpha + w^{-2})^{\frac{1}{2}} = w + \cdots \qquad (-2 < \alpha < 2)$$

maps $|w| > 1$ onto the plane from which the two rectilinear segments

$$[-(2+\alpha)^{\frac{1}{2}}, (2+\alpha)^{\frac{1}{2}}] \quad \text{and} \quad [-i(2-\alpha)^{\frac{1}{2}}, i(2-\alpha)^{\frac{1}{2}}]$$

have been deleted. If we take $\alpha = 1$, the configuration of the two segments has transfinite diameter 1 and

$$b = \sqrt{3}, \qquad d = 2\sqrt{3}, \qquad b + d = 3\sqrt{3} > 5.19.$$

If we take $\alpha = 2/3$, we get

$$b = 4\sqrt{2}/3, \qquad d = 4\sqrt{6}/3, \qquad bd = 32\sqrt{3}/9 > 6.15.$$

From this we deduce that

$$\sup b \geq \sqrt{3}\, 2^{\frac{1}{3}}, \qquad \sup (b + d) \geq 3\sqrt{3}, \qquad \sup bd \geq 32\sqrt{3}/9,$$

for the class of $f(z)$ for which $E$ is connected.

## REFERENCES

1. P. Erdös, F. Herzog and G. Piranian, *Metric properties of polynomials*, J. Analyse Math. 6 (1958), 123-148.

2. G. Faber, *Über Tschebyscheffsche Polynome*, J. Reine Angew. Math. 150 (1920), 79-106.

3. G. Pólya, *Beitrag zur Verallgemeinerung des Verzerrungssatzes auf mehrfach zusammenhängende Gebiete, II*, S.-B. Preuss. Akad. Wiss. Berlin. Kl. Math. Phys. Tech. (1928), 280-282.

4. G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Berlin (1925).

University Göttingen