BINARY OPERATIONS ON SETS OF MAPPING CLASSES

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1. INTRODUCTION

1.1. SUMMARY OF RESULTS

Suppose that X and Y are topological spaces, with points $x_0 \in X$ and $y_0 \in Y$. The symbol $[X, x_0; Y, y_0]$ will be used to denote the set of homotopy classes of maps (continuous functions) from (X, x_0) to (Y, y_0). The chief purpose of this paper is to investigate two ways in which $[X, x_0; Y, y_0]$ may be equipped with a binary operation. The first way (homotopy theory) imitates the Hurewicz homotopy theory; (X, x_0) is held fixed, and the binary operations are so chosen that the class of the constant map is an identity element, and every map from a pair (Y, y_0) to a pair (Y', y_0') induces a homomorphism. It is shown that the binary operations exist if and only if (X, x_0) has (modified) Lusternik-Schnirelmann category at most two. The second way (cohomotopy theory) imitates the Borsuk-Spanier cohomotopy theory, but without dimensional restrictions. In this case, the existence of binary operations is related to the fact that Y is an H-space. For each of these theories, exact sequences of a natural sort are developed, together with certain lesser results.

The last section contains applications of the earlier parts of the paper. A result of Spanier and J. H. C. Whitehead (roughly, that a fibre contractible in its fibre space is an H-space) is presented in a somewhat strengthened form. Several results in the direction of describing $[X, x_0; Y, y_0]$ when X and Y are "simple" spaces are obtained. Finally, the problem of determining the structure maps on an H-space is solved in case the space has only two nontrivial homotopy groups.

1.2. NOTATION AND CONVENTIONS

All of the spaces considered in this paper are Hausdorff spaces. The unit interval $[0, 1]$ is denoted by I; the n-cube $I \times \cdots \times I$ (n factors), by I^n. The boundary of I^n is designated by $\dot{I}^n$.

If X and Y are spaces, or topological pairs or triples, then $[X; Y]$ is the set of homotopy classes of maps from X into Y. If $f: X \to Y$ and $g: Y \to Z$ are maps, then $g \circ f: X \to Z$ is the map given by $g \circ f(x) = g(f(x))$ for $x \in X$. The symbol $\{f\}$ denotes the homotopy class of the map f. A map $f: X \to Y$ induces functions

$$f_{\#}: [Z; X] \to [Z; Y] \qquad \text{and} \qquad f^{\#}: [Y; Z] \to [X; Z];$$

these functions are defined by $f_{\#}\{g\} = \{f \circ g\}$ when $\{g\} \in [Z; X]$, and by $f^{\#}\{g\} = \{g \circ f\}$ when $\{g\} \in [Y; Z]$.

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If $f_i: (X_i, A_i, B_i) \to (Y_i, C_i, D_i)$ $(i = 1, 2)$ are maps, then

$$f_1 \times f_2: (X_1, A_1, B_1) \times (X_2, A_2, B_2) \to (Y_1, C_1, D_1) \times (Y_2, C_2, D_2)$$

is the map given by $f_1 \times f_2(x_1, x_2) = (f_1(x_1), f_2(x_2))$ when $x_i \in X_i$. Some of the spaces A_i, B_i, C_i, D_i may be void, or they may consist of single points.

The terms CW-*complex* and *relative* CW-*complex* are used in the sense of [7, p. 73].

2. HOMOTOPY THEORY

2.1. DEFINITIONS

Let X be a topological space, let $x_0 \in X$, and let $\mathscr{C}$ be a collection of topological pairs of the form (Y, y_0) with $y_0 \in Y$. A *homotopy theory* on X, x_0, $\mathscr{C}$ is a system μ of binary operations, one defined on each of those sets $[X, x_0; Y, y_0]$ for which $(Y, y_0) \in \mathscr{C}$. The system is subject to the following two conditions.

H1. *For each* (Y, y_0) *in* $\mathscr{C}$, *the homotopy class* ε *of the constant map is a two-sided identity element.*

H2. *For all* (Y, y_0) *and* (Y', y_0') *in* $\mathscr{C}$ *and for each map* $f: (Y, y_0) \to (Y', y_0')$, *the induced function* $f_\#: [X, x_0; Y, y_0] \to [X, x_0; Y', y_0']$ *is a homomorphism.*

The symbol μ will be used to denote an individual member of the system of binary operations, as well as the whole system.

We say that the *category* of (X, x_0) is at most n (written $\operatorname{cat}(X, x_0) \leq n$) if there are closed subsets $X_1, \cdots, X_n$ of X and homotopies $R_i: (X \times I, x_0 \times I) \to (X, x_0)$ $(i = 1, 2, \cdots, n)$ such that

1) $X = X_1 \cup \cdots \cup X_n$,

2) $R_i(x, 0) = x$ for each $x \in X$ $(i = 1, \cdots, n)$,

3) $R_i(x, 1) = x_0$ for each $x \in X_i$ $(i = 1, \cdots, n)$.

The subsets $X_1, \cdots, X_n$ are called *categorical subsets*.

Note that if $\operatorname{cat}(X, x_0) \leq n$, then X has Lusternik-Schnirelmann category at most n. Conversely, if X is smooth about some point $x_0 \in X$ (every map from $X \times I \times 0 \cup x_0 \times I \times I$ to X can be extended to a map from $X \times I \times I$) and has Lusternik-Schnirelmann category at most n, then $\operatorname{cat}(X, x_0) \leq n$. In particular, the two ideas are equivalent for locally finite CW-complexes.

2.2. THE FUNDAMENTAL THEOREM

Corresponding to each pair (X, x_0), we define the space $X \vee X$ to be the set $X \times x_0 \cup x_0 \times X \subset X \times X$, and x_0^2 to be the point $(x_0, x_0) \in X \vee X$.

THEOREM 2.2A. *Suppose that a pair* (X, x_0) *and a collection* $\mathscr{C}$ *of pairs are given, and that* $\mathscr{C}$ *contains* (X, x_0) *and* $(X \vee X, x_0^2)$. *There is a homotopy theory* μ *on* X, x_0, $\mathscr{C}$ *if and only if* $\operatorname{cat}(X, x_0) \leq 2$.

Proof. Suppose $\operatorname{cat}(X, x_0) \leq 2$, with categorical subsets X_1, X_2 and homotopies R_1, R_2. Let $(Y, y_0) \in \mathscr{C}$ and $\alpha, \beta \in [X, x_0; Y, y_0]$ be given. Select representative maps $f \in \alpha$ and $g \in \beta$, and in terms of these define a function $h: (X, x_0) \to (Y, y_0)$ as follows:

$$h(x) = \begin{cases} f(R_2(x, 1)) & \text{for } x \in X_1, \\ g(R_1(x, 1)) & \text{for } x \in X_2. \end{cases}$$

This function is defined and continuous, since X_1 and X_2 are closed and

$$f(R_1(x, 1)) = f(R_2(x, 1)) = y_0$$

for x in $X_1 \cap X_2$. The homotopy class of h is independent of the representatives f and g. Set $\mu(\alpha, \cdot\beta) = \{h\}$. The verifications of H1 and H2 are routine. The binary operation obtained in this fashion is said to be *induced by the categorical decomposition.*

Now suppose that μ is a homotopy theory on X, x_0, $\mathscr{C}$ and that (X, x_0) and $(X \vee X, x_0^2)$ are in $\mathscr{C}$. We define four maps

$$f_1, f_2: (X, x_0) \to (X \vee X, x_0^2), \qquad g_1, g_2: (X \vee X, x_0^2) \to (X, x_0)$$

as follows. If $x \in X$, then $f_1(x) = (x, x_0)$ and $f_2(x) = (x_0, x)$. The map g_1 is given by $g_1(x, x_0) = x$ and $g_1(x_0, x) = x_0$, while $g_2(x, x_0) = x_0$ and $g_2(x_0, x) = x$. Let h be a representative of $\mu(\{f_1\}, \{f_2\}) \in [X, x_0; X \vee X, x_0^2]$, and set $X_1 = h^{-1}(X \times x_0)$, $X_2 = h^{-1}(x_0 \times X)$. Since $g_1 \circ f_1$ is the identity on X, and $g_1 \circ f_2$ is constant, $g_1 \circ h$ is homotopic to the identity, for

$$\{g_1 \circ h\} = g_{1\#}\mu(\{f_1\}, \{f_2\}) = \mu(\{g_1 \circ f_2\}, \{g_1 \circ f_2\}) = \{g_1 \circ f_1\}.$$

Let $R_2: (X \times I, x_0 \times I) \to (X, x_0)$ be this homotopy (that is, let $R_2(x, 0) = x$ and $R_2(x, 1) = g_1 \circ h(x)$, for $x \in X$). From the definitions of R_2 and X_2 it follows that $R_2(x, 1) = x_0$ for $x \in X_2$. The homotopy R_1 is similarly constructed. Thus $\operatorname{cat}(X, x_0) \leq 2$.

Note that the binary operation induced by this categorical decomposition is precisely μ.

2.3. RELATIONS BETWEEN HOMOTOPY THEORIES

Suppose that $\operatorname{cat}(X, x_0) \leq 2$ and $\operatorname{cat}(X', x_0') \leq 2$ with categorical subsets $X_1, X_2 \subset X$ and $X_1', X_2' \subset X'$, and with homotopies $R_i: (X \times I, x_0 \times I) \to (X, x_0)$, $R_i': (X' \times I, x_0' \times I) \to (X', x_0')$ for $i = 1, 2$. Suppose that $F: (X, x_0) \to (X', x_0')$ is a map.

THEOREM 2.3A. *If* $F^{-1}(X_i') \subset X_i$ $(i = 1, 2)$, *then* $F^\#$ *is a homomorphism with respect to the binary operations induced by these categorical decompositions.*

The proof of this result is routine. By choosing F to be the identity map on (X, x_0), we obtain

COROLLARY 2.3B. *The binary operation is independent of the contractions of the categorical subsets.*

An example presented in Section 4 shows that the binary operation does, in general, depend on the choice of categorical subsets.

2.4. SUSPENDED SPACES

Important examples of spaces of category two are the suspended spaces. The following definition of suspended space will be used. Let X_0 be a topological space, and let $x_0 \in X_0$. The suspension SX_0 of X_0 is formed from $X_0 \times I$ by identifying $K = (X_0 \times 0) \cup (X_0 \times 1) \cup (x_0 \times I)$ with a point $\bar{x}_0 \in SX_0$. That is, the points of SX_0 are the points of $(X_0 \times I) - K$, together with a point $\bar{x}_0$; and a basis for the open sets of SX_0 consists of the open sets in $(X_0 \times I) - K$, together with the sets $(U - K) \cup \bar{x}_0$ such that $U \supset K$ and U is open in $X_0 \times I$. It follows immediately from this definition that the restriction to $(X_0 \times I) - K$ of the natural map θ: $(X_0 \times I, K) \rightarrow (SX_0, \bar{x}_0)$ is a homeomorphism. Two *cones* over X_0 are defined by $\hat{X}_0 = \theta(X_0 \times [\frac{1}{2}, 1])$ and $\check{X}_0 = \theta(X_0 \times [0, \frac{1}{2}])$.

THEOREM 2.4A. $\operatorname{Cat}(SX_0, \bar{x}_0) \leq 2$.

Proof. Let $\hat{X}_0$ and $\check{X}_0$ be the categorical subsets. The homotopies are easily constructed.

To simplify the notation, X_0 will be considered the same as $\theta(X_0 \times \frac{1}{2})$; then $\bar{x}_0 = x_0$. When X is a suspended space, $[X, x_0; Y, y_0]$ is a group; associativity is easily verified, and the inverse of an element $\{f\}$ is the class of the map sending $\theta(x, t)$ into $f(\theta(x, 1 - t))$.

Suppose π_1, π_2, π_3 are sets of homotopy classes of maps, and ξ_1: $\pi_1 \rightarrow \pi_2$, ξ_2: $\pi_2 \rightarrow \pi_3$ are functions. The sequence

$$\pi_1 \rightarrow \pi_2 \rightarrow \pi_3$$

is said to be *exact* when $\xi_1(\pi_1)$ is the set of elements of π_2 which ξ_2 carries into the class of the constant map in π_3.

Let X_0 be a topological space, and p: $E \rightarrow B$ a fibre map satisfying the homotopy lifting condition for maps from $X_0 \times I^n$, for each positive integer n. Let $x_0 \in X_0$, $b_0 \in B$, $F = p^{-1}(b_0)$, $y_0 \in F$, i: $F \subset E$, and $X = SX_0$ (formed relative to x_0). If X_0 is smooth about x_0, then a function ∂: $[X, x_0; B, b_0] \rightarrow [X_0, x_0; F, y_0]$ may be defined as follows. Let α be an element of $[X, x_0; B, b_0]$, and let f: $(X, x_0) \rightarrow (B, b_0)$ be a representative of α. The map f induces a map $\bar{f}$: $X_0 \times I \rightarrow B$ such that $\bar{f}(X_0 \times I \cup x_0 \times I) = b_0$. By the homotopy lifting condition, there is a map G: $X_0 \times I \rightarrow E$ such that $G(X_0 \times 0) = y_0$ and $p \circ G = f$. Note that $p \circ G(x, 1) = f(x, 1) = b_0$, so that $G(x, 1) \in F$. In view of the smoothness condition, we may assume that $G(x_0, t) = y_0$ for $t \in I$. Thus a map ∂f: $(X, x_0) \rightarrow (F, y_0)$ is defined by $\partial f(x) = G(x, 1)$. A homotopy of f induces homotopies of $\bar{f}$, G and hence of ∂f, and therefore the class of ∂f is independent of the representative f of α. Let $\partial\alpha = \{\partial f\}$.

THEOREM 2.4B. *The sequence*

$$\cdots \rightarrow [X, x_0; E, y_0] \xrightarrow{p\#} [X, x_0; B, b_0] \xrightarrow{\partial} [X_0, x_0; F, y_0] \xrightarrow{i\#} [X_0, x_0; E, y_0] \xrightarrow{p\#} [X_0, x_0; B, b_0]$$

is exact.

The verification of exactness is routine. Note that not all of the functions involved need be homomorphisms.

THEOREM 2.4C. *If* $\operatorname{cat}(X_0, x_0) \leq 2$ *and* $\mathscr{C}$ *is arbitrary, then* X, x_0, $\mathscr{C}$ *has a homotopy theory such that* ∂ *is a homomorphism.*

Proof. Let A_1 and A_2 be the categorical subsets of X_0, and

$$T_1: (X_0 \times I, x_0 \times I) \rightarrow (X_0, x_0)$$

the contractions. That is, let $T_i(x, 0) = x$ for $x \in X_0$, and $T_i(x, 1) = x_0$ for $x \in A_i$ $(i = 1, 2)$. Set $X_i = \theta(A_i \times I)$ and $R_i(\theta(x, t), s) = \theta(T_i(x, s), t)$; then $X = X_1 \cup X_2$ is a categorical decomposition. Let $\bar{\mu}$ designate the binary operation induced by this decomposition. A straightforward computation shows that ∂ is a homomorphism with respect to this binary operation.

Let X' be the suspension of a space X_0', formed relative to some point $x_0' \in X_0'$. Let θ': $(X_0' \times I, X_0' \times \dot{I} \cup x_0' \times I) \to (X', x_0')$ be the analogue of θ. A map

$$\phi: (X_0, x_0) \to (X_0', x_0')$$

induces a map $S\phi$: $(X, x_0) \to (X', x_0')$ by $S\phi(\theta(x, t)) = \theta'(\phi(x), t)$.

THEOREM 2.4D. *The diagram*

$$\begin{array}{ccccccccc}
\cdots \to & [X, x_0; B, b_0] & \xrightarrow{\partial} & [X_0, x_0; F, y_0] & \xrightarrow{i_\#} & [X_0, x_0; E, y_0] & \xrightarrow{p_\#} & [X_0, x_0; B, b_0] \\
& \uparrow S\phi^\# & & \uparrow \phi^\# & & \uparrow \phi^\# & & \uparrow \phi^\# \\
\cdots \to & [X', x_0'; B, b_0] & \xrightarrow{\partial'} & [X_0', x_0'; F, y_0] & \xrightarrow{i_\#} & [X_0', x_0'; E, y_0] & \xrightarrow{p_\#} & [X_0', x_0'; B, b_0]
\end{array}$$

is commutative.

The proof follows immediately from the definitions of the various objects in the diagram.

Let (Y, Y') be a topological pair, let $y_0 \in Y'$, and let i: $(Y', y_0) \to (Y, y_0)$ and j: $(Y, y_0, y_0) \to (Y, Y', y_0)$ denote inclusion maps. A function

$$\partial: [\hat{X}_0, X_0, x_0; Y, Y', y_0] \to [X_0, x_0, Y', y_0]$$

is given by $\partial(\{f\}) = \{(f\,|(X_0, x_0)\}$ when f: $(\hat{X}_0, X_0, x_0) \to (Y, Y', y_0)$. The map j induces a function $j_\#$: $[SX_0, x_0; Y, y_0] \to [\hat{X}_0, X_0, x_0; Y, Y', y_0]$ as follows. If f: $(SX_0, x_0) \to (Y, y_0)$ is a given map and r_1: $(SX_0, \check{X}_0) \to (SX_0, x_0)$ is defined by $r_1(\theta(x, t)) = \theta(x, 2t)$, then $j_\#(\{f\}(= \{(f \circ r_1)\,|\,(X_0, X_0, x_0)\}$.

THEOREM 2.4E. *The sequence*

$$\cdots\cdot \to [SX_0, x_0, Y', y_0] \xrightarrow{i_\#} [SX_0, x_0; Y, y_0] \xrightarrow{j_\#} [\hat{X}_0, X_0, x_0; Y, Y', y_0]$$

$$\xrightarrow{\partial} [X_0, x_0; Y', y_0] \xrightarrow{i_\#} [X_0, x_0; Y, y_0]$$

is exact.

The proof is similar to the corresponding proof for Hurewicz homotopy groups. A weak form of Theorem 2.4B may be derived from this sequence.

3. COHOMOTOPY

3.1. DEFINITIONS

Let Y be a topological space, let $y_0 \in Y$, and let $\mathscr{C}$ be a collection of pairs of the form (X, x_0). A *cohomotopy* theory on $Y, y_0, \mathscr{C}$ is a collection μ of binary operations, one on each $[X, x_0; Y, y_0]$ with $(X, x_0) \in \mathscr{C}$, and satisfying the following two postulates.

C1. *For each* $(X, x_0) \in \mathscr{C}$, *the class of the constant map is the two-sided identity element of* $[X, x_0; Y, y_0]$.

C2. *For all* (X, x_0) *and* (X', x_0') *in* $\mathscr{C}$ *and for every map* $F: (X, x_0) \to (X', x_0')$, *the induced function* $F^{\#}: [X', x_0'; Y, y_0] \to [X, x_0; Y, y_0]$ *is a homomorphism.*

A topological space Y is said to be an H-space if there is a point $y_0 \in Y$ and a map $\lambda: (Y \times Y, y_0^2) \to (Y, y_0)$ such that the maps $y \to \lambda(y, y_0)$ and $y \to \lambda(y_0, y)$ are homotopic to the identity map on Y under homotopies which leave y_0 fixed. The point y_0 will be called the identity element, and the map λ, a structure map.

3.2. FUNDAMENTAL RESULTS

THEOREM 3.2A. *Suppose* (Y, y_0) *and* $(Y \times Y, y_0^2)$ *are in* $\mathscr{C}$. *Then* $Y, y_0, \mathscr{C}$ *has a cohomotopy theory if and only if* Y *is an* H-*space with identity element* y_0.

Proof. Suppose that Y is an H-space [7, p. 14], that $(X, x_0) \in \mathscr{C}$ and that $\alpha, \beta \in [X, x_0; Y, y_0]$. If $f \in \alpha$ and $g \in \beta$ are representative maps, define

$$\mu(\alpha, \beta) = \{\lambda \circ (f \times g) \cdot d\},$$

where $d: (X, x_0) \to (X \times X, x_0^2)$ is the diagonal map: $d(x) = (x, x)$ for $x \in X$. It is immediate that this definition does not depend on the choice of the representatives f and g.

Now suppose $f(X) = y_0$. Since $y \to \lambda(y_0, y)$ is homotopic to the identity, the map $x \to \lambda(f(x), g(x)) = \lambda(y_0, g(x))$ is homotopic to g. A similar statement holds in case $g(X) = y_0$. Thus Postulate C1 is satisfied.

If $(X, x_0), (X', x_0') \in \mathscr{C}$, if $F: (X', x_0') \to (X, x_0)$ is a map, and if

$$d': (X', x_0') \to (X' \times X', x_0'^2)$$

is given by $d'(x) = (x, x)$ for $x \in X'$, then

$$F^{\#}\mu(\alpha, \beta) = \{\lambda \circ (f \times g) \circ d \circ F\} = \{\lambda \circ (f \times g) \circ (F \times F) \circ d'\} = \mu(F^{\#}\alpha, F^{\#}\beta).$$

Thus Postulate C2 is satisfied.

Conversely, suppose that $(Y, y_0), (Y \times Y, y_0^2) \in \mathscr{C}$ and that μ is a cohomotopy theory on $Y, y_0, \mathscr{C}$. Let $j_1, j_2: (Y, y_0) \to (Y \times Y, y_0^2)$ be the two injections $j_1(y) = (y, y_0)$ and $j_2(y) = (y_0, y)$ for $y \in Y$, and let $p_1, p_2: (Y \times Y, y_0^2) \to (Y, y_0)$ be the projections $p_1(y, y') = p_2(y', y) = y$ for $y, y' \in Y$. The H-space structure map

$$\lambda: (Y \times Y, y_0^2) \to (Y, y_0)$$

is chosen to be any representative of $\mu(\{p_1\}, \{p_2\}) \in [Y \times Y, y_0^2; Y, y_0]$. Note that

$$\{\lambda \circ j_1\} = j_1^{\#}\mu(\{p_1\}, \{p_2\}) = \mu(\{p_1 \circ j_1\}, \{p_2 \circ j_1\}) = \{p_1 \circ j_1\},$$

whence the map sending y into $\lambda \circ j_1(y) = \lambda(y, y_0)$ is homotopic to the identity $p_1 \circ j_1$: $(Y, y_0) \to (Y, y_0)$. Similarly, $\{\lambda \circ j_2\} = \{p_2 \circ j_2\}$, and therefore the map sending y into $\lambda(y_0, y)$ is homotopic to the identity.

The structure map λ is said to be *inversive* when there is a map i: $(Y, y_0) \to (Y, y_0)$ such that the maps $y \to \lambda(y, i(y))$ and $y \to \lambda(i(y), y)$ are null-homotopic; *associative* in case the maps ρ_1, ρ_2: $(Y \times Y \times Y, y_0^3) \to (Y, y_0)$ given by

$$\rho_1(y_1, y_2, y_3) = \lambda(y_1, \lambda(y_2, y_3)), \quad \rho_2(y_1, y_2, y_3) = \lambda(\lambda(y_1, y_2), y_3) \quad (y_1, y_2, y_3 \in Y)$$

are homotopic; *abelian* in case $(y_1, y_2) \to \lambda(y_1, y_2)$ is homotopic to $(y_1, y_2) \to \lambda(y_2, y_1)$.

THEOREM 3.2B. *λ is inversive if and only if $[Y, y_0; Y, y_0]$ is inversive.*

THEOREM 3.2C. *λ is associative if and only if $[Y \times Y \times Y, y_0^3; Y, y_0]$ is associative.*

THEOREM 3.2D. *λ is abelian if and only if $[Y \times Y, y_0; Y, y_0]$ is abelian.*

Proof of 3.2C. Suppose that $[Y \times Y \times Y, y_0^3; Y, y_0]$ is associative. Define the maps p_i: $(Y \times Y \times Y, y_0^3) \to (Y, y_0)$ by $p_i(y_1, y_2, y_3) = y_i$ $(y_i \in Y, i = 1, 2, 3)$. Then

$$\{\rho_1\} = \mu(\{p_1\}, \mu(\{p_2\}, \{p_3\})) = \mu(\mu(\{p_1\}, \{p_2\}), \{p_3\}) = \{\rho_2\}.$$

Conversely, suppose that λ is associative and that α, β, γ are elements of $[X, x_0: Y, y_0]$ $((X, x_0) \in \mathscr{C})$ with representatives f, g, h: $(X, x_0) \to (Y, y_0)$. Let $\bar{d}(y) = (y, y, y) \in Y \times Y \times Y$. Then

$$\mu(\alpha, \mu(\beta, \gamma)) = \{\lambda \circ (f \times (\lambda \circ (g \times h) \circ d) \circ d\} = \{\rho_1 \circ (f \times g \times h) \circ \bar{d}\} = \{\rho_2 \circ (f \times g \times h) \circ \bar{d}\}$$
$$= \{\lambda((\lambda \circ (f \times g) \circ d) \times h) \circ d\} = \mu(\mu(\alpha, \beta), \gamma).$$

The proofs of 3.2B and 3.2D use similar techniques.

3.3. AN EXACT SEQUENCE

Let (X, A) be a topological pair, and let A be closed in X. A space $\tilde{X}$, a point $x_0 \in \tilde{X}$, and a map F: $(X, A) \to (\tilde{X}, x_0)$ may be chosen so that F is onto, and is a homeomorphism on $X - A$. If Y is a space and $y_0 \in Y$, then the function

$$F^{\#}: [\tilde{X}, x_0; Y, y_0] \to [X, A; Y, y_0]$$

is one-to-one, onto. If λ: $Y \times Y \to Y$ is a structure map, and if d: $X \to X \times X$ and d': $\tilde{X} \to \tilde{X} \times \tilde{X}$ are diagonal maps, then a natural binary operation μ' on $[X, A; Y, y_0]$ is defined by $\mu'(\{f\}, \{g\}) = \{\lambda \circ (f \times g) \circ d'\}$ when f, g: $(X, A) \to (Y, y_0)$. We note that $F^{\#}$ is an isomorphism with respect to these binary operations.

Suppose that X is separable, A is closed in X, and Y is an absolute neighborhood retract. Let E be the space of paths in Y which start at y_0; and let the compact open topology be put on E.

LEMMA 3.3A. *Any map* f: $A \to E$ *may be extended to* g: $X \to E$.

Proof. Let $\hat{X}$ be the cone over X (that is, let $\hat{X}$ be the set $X \times I$ with identifications $(x, 0) = (x', 0)$ when $x, x' \in X$), and $\hat{A} \subset \hat{X}$ the cone over A. Then $f: A \to E$ induce $f': \hat{A} \to Y$ by $f'(x, t) = f(x)(t)$. Since $\hat{A}$ is contractible, there is a map $F: \hat{A} \times I \to Y$ with $F(x, t, 0) = f'(x, t)$ and $F(x, t, 1) = y_0$. By Borsuk's extension theorem, there exists a map $G: \hat{X} \times I \to Y$ such that

$$G(x, t, s) = \begin{cases} F(x, t, s) & (x \in A, \quad t, s \in I), \\ y_0 & (x \in X, \quad t \in I, \quad s = 1). \end{cases}$$

Then $g: X \to E$ is given by $g(x)(y) = G(x, t, 0)$.

COROLLARY 3.3B. *If* $f_0, f_1: X \to E$ *are maps such that* $f_0 | A$ *is homotopic to* $f_1 | A$, *then the homotopy may be extended to a homotopy between* f_0 *and* f_1.

Apply Lemma 3.3A, with A replaced by $X \times \dot{I} \cup A \times I$, and with X replaced by $X \times I$.

Let Ω be the space of loops in Y based at y_0, and let w_0 be the constant loop. The maps $i_1: (X, x_0) \to (X, A)$ and $i_2: (A, x_0) \to (X, x_0)$ are inclusion maps.

THEOREM 3.3C. *If either* (1) X *is a* CW-*complex*, A *is a subcomplex and* Y *is arbitrary, or* (2) X *is separable*, A *is closed in* X, *and* Y *is an* ANR, *then a function* $\delta: [A, x; \Omega, w_0] \to [X, A; Y, y_0]$ *may be defined in such a way that the sequence*

$$\cdots \to [X, x_0; \Omega, w_0] \xrightarrow{i_2^{\#}} [A, x; \Omega, w_0] \xrightarrow{\delta} [X, A; Y, y_0] \xrightarrow{i_1^{\#}} [X, x_0; Y, y_0] \xrightarrow{i_2^{\#}} [A, x_0; Y, y_0]$$

is exact.

Proof. The function δ is defined as follows. Any map $f: (A, x_0) \to (\Omega, w_0)$ may be extended to $f': X \to E$, by Lemma 3.3A in case (2), and in case (1) by the observation that all obstructions vanish. By using Corollary 3.3B, or by observing that all obstruction cocycles vanish, we see that $\{f'\}$ is determined by $\{f\}$. Let $p: E \to Y$ be the map which assigns to each path its terminal point. Set $\delta\{f\} = \{p \circ f'\}$ (this form of the function δ was suggested to the author by W. Hurewicz). The exactness of the sequence is readily verified.

THEOREM 3.3D. *If* $\lambda: (Y \times Y, y_0^2) \to (Y, y_0)$ *induces the binary operation* μ *on* $[X, A; Y, y_0]$, *then a binary operation* μ *may be defined on* $[A, x_0; \Omega, w_0]$ *in such a fashion that* δ *is a* μ-*homomorphism*.

Proof. Define $\lambda': (E \times E, \Omega \times \Omega, w_0^2) \to (E, \Omega, w_0)$ by $\lambda'(a, b)(t) = \lambda(a(t), b(t))$ for $a, b \in E$ and $t \in I$. If $f, g: (A, x_0) \to (\Omega, w_0)$, set $\mu(\{f\}, \{g\}) = \{\lambda' \circ (f \times g) \circ d\}$. If $f', g': X \to E$ are extensions of f, g, then

$$\begin{aligned} \delta\mu(\{f\}, \{g\})) &= \{\lambda' \circ (f \times g) \circ d\} = \{p \circ \lambda' \circ (f' \times g') \circ d\} \\ &= \{\lambda \circ ((p \circ f') \times (p \circ g')) \circ d\} = \mu(\delta\{f\}, \delta\{g\}). \end{aligned}$$

3.4. RELATIONS BETWEEN COHOMOTOPY THEORIES

Suppose Y and Y' are H-spaces with structure maps $\lambda: (Y\times Y, y_0^2) \to (Y, y_0)$ and $\lambda': (Y'\times Y', y_0'^2) \to (Y, y_0')$. Let $h: (Y, y_0) \to (Y' y_0')$ be a map, and $\mathscr{C}$ a collection of pairs containing $(Y\times Y, y_0)$.

THEOREM 3.4A. *The function* $h_\#: [X, x_0; Y, y_0] \to [X, x_0; Y' y_0']$ *is a homomorphism for all* $(X, x_0) \in \mathscr{C}$ *if and only if* $h \circ \lambda$ *is homotopic to*

$$\lambda' \circ (h\times h): (Y\times Y, y_0) \to (Y', y_0').$$

Proof. Suppose $h\circ\lambda$ is homotopic to $\lambda' \circ (h\times h)$. If $(X, x_0) \in C$ and

$$f, g: (X, x_0) \to (Y, y_0)$$

are given, then

$$h_\#\mu(\{f\}, \{g\}) = \{h\circ\lambda\circ(f\times g)\circ d\} = \{\lambda'\circ(h\times h)\circ(f\times g)\circ d\}$$

$$= \{\lambda'\circ(h\circ f\times h\circ g)\circ d\} = \mu(h_\#\{f\}, h_\#\{g\}).$$

Conversely, suppose that the relation $h_\#\mu(\{f\}, \{g\}) = \mu(h_\#\{f\}, h_\#\{g\})$ holds for all $(X, x_0) \in \mathscr{C}$ and for all maps $f, g: (X, x_0) \to (Y, y_0)$. In particular, the relation then holds when $(X, x_0) = (Y\times Y, y_0^2)$ and when $f, g: (Y\times Y, y_0^2) \to (Y, y_0)$ are the projections $f(y_1, y_2) = y_1$ and $g(y_1, y_2) = y_2$ $(y_1, y_2 \in Y)$. But

$$(f\times g)\circ d(y_1, y_2) = (f\times g)((y_1, y_2), (y_1, y_2)) = (y_1, y_2).$$

Thus

$$\{h\circ\lambda\} = \{h\circ\lambda\circ(f\times g)\circ d\} = h_\#\mu(\{f\}, \{g\}) = \mu(h_\#\{f\}, h_\#\{g\})$$

$$= \{\lambda'\circ(h\times h)\circ(f\times g)\circ d\} = \{\lambda'\circ(h\times h)\},$$

whence $h\circ\lambda$ is homotopic to $\lambda'\circ(h\times h)$. (This result was suggested to the author by E. H. Spanier.)

4. RESULTS INVOLVING BOTH HOMOTOPY AND COHOMOTOPY

THEOREM 4A. *If* $\operatorname{cat}(X, x_0) \leq 2$ *and* Y *is an* H-*space with identity element* y_0, *then the binary operation on* $[X, x_0; Y, y_0]$ *induced by the categorical decomposition of* X *is the same as that induced by the* H-*structure map on* Y.

The proof is routine. Some immediate consequences of this result are:

COROLLARY 4B. *Any two categorical decompositions of* (X, x_0) *induce the same binary operation on* $[X, x_0; Y, y_0]$.

COROLLARY 4C. *If* λ *and* λ' *are* H-*structure maps on* Y, *both having* y_0 *as their identity element, then they induce the same binary operation on* $[X, x_0; Y, y_0]$.

COROLLARY 4D. *The binary operation on* $[X, x_0; Y, y_0]$ *is commutative.*

If $\operatorname{cat}(X, x_0) \leq 2$ and (Y, y_0) is not an H-space, then the binary operation on $[X, x_0; Y, y_0]$ may depend on the choice of the categorical decomposition. This is demonstrated by the following example. Let S^n, S^m and S^{n+m-1} be spheres of dimensions n, m and $n + m - 1$ $(2 < n < m)$. The space $X = S^n \vee S^m \vee S^{n+m-1}$ is a suspension of the suspended space $S^{n-1} \vee S^{m-1} \vee S^{n+m-2}$. Let Ω be the space of loops on Y based at y_0, and let ω_0 be the constant loop. It is easily verified that $\delta: [S^{n-1} \vee S^{m-1} \vee S^{n+m-2}, x_0; \Omega, \omega_0] \to [X, x_0; Y, y_0]$ is an isomorphism (see Section 3.3) with respect to the binary operation induced by the suspensions described above. Note that $[S^{n-1} \vee S^{m-1} \vee S^{n+m-1}, x_0; \Omega, \omega_0]$ is abelian, whence $[X, x_0; Y, y_0]$ is also abelian.

A. J. Goldstein pointed out to the author that $X^* = S(S^{n-1} \times S^{m-1})$ has the same homotopy type as X. The argument runs as follows. The space X^* may be regarded as a CW-complex having four cells E^0, E^n, E^m and E^{n+m-1}. The first three of these cells form $S^n \vee S^m = S(S^{n-1} \vee S^{m-1})$. The characteristic map

$$e^{n+m-1}: I^{n+m-1} \to X^*$$

defines a map f_1 from, say, the upper hemisphere of S^{n+m-1} onto E^{n+m-1}. Let $i_1: S^{n-1} \subset S^{n-1} \vee S^{m-1}$ and $i_2: S^{m-1} \subset S^{n-1} \vee S^{m-1}$ be inclusion maps. Then $\{e^{n+m-1} \mid \dot{I}^{n+m-1}\}$ in $\pi_{n+m-2}(S^n \vee S^m)$ is the suspension of the bracket product $[\{i_1\}, \{i_2\}] \in \pi_{n+m-3}(S^{n-1} \vee S^{m-1})$, and hence it is trivial [5]. Thus f_1 restricted to the diameter of S^{n+m-1} may be extended to a map from the lower hemisphere of S^{n+m-1} into $S^n \vee S^m$. These maps, together with the identity map on $S^n \vee S^m$, induce a map $f: X \to X^*$ such that the induced homology maps $f_*: H_i(X) \to H_i(X^*)$ $(i = 0, 1, \cdots)$ are isomorphisms. Both spaces are 1-connected; hence they have the same homotopy type [8, p. 1135].

The binary operation which the suspension of $S^{n-1} \times S^{m-1}$ induces on $[X^*, x_0; Y, y_0]$ need not be abelian. In fact, let $Y = S^n \vee S^m$. Then

$$[X^*, x_0; Y, y_0] = [S^{n-1} \times S^{m-1}, x_0; \Omega(Y), y_0].$$

Let $f: S^{n-1} \times S^{m-1} \to \Omega(Y)$ be the projection of $S^{n-1} \times S^{m-1}$ onto S^{n-1}, followed by the natural homeomorphism of S^{n-1} into $\Omega(Y)$; and let $g: S^{n-1} \times S^{m-1} \to S^{n-1} \to \Omega(Y)$ be similarly defined. Note that $f_*: H_{n-1}(S^{n-1} \times S^{m-1}) \approx H_{n-1}(\Omega(Y))$ and g^* carries $H_{m-1}(S^{n-1} \times S^{m-1})$ isomorphically onto a direct summand of $H_{m-1}(\Omega(Y))$. Let $a \in H_{n-1}(S^{n-1} \times S^{m-1})$, $b \in H_{m-1}(S^{n-1} \times S^{m-1})$, and $c \in H_{n+m-2}(S^{n-1} \times S^{m-1})$ be generators, and let $h \in \mu(\{f\}, \{g\})$, $k \in \mu(\{g\}, \{f\})$. If $(a, b) \to a \cdot b$ is the Pontrjagin product on the ring $H_*(\Omega(Y))$, then $h_*(c) = \pm(f_*a) \cdot (g_*b)$, and $k_*c = \pm(g_*b) \cdot (f_*a)$. Since $H_*(\Omega(Y))$ is the tensor ring of the free abelian group $H_*(S^{n-1} \vee S^{m-1})$ (see [2, p. 334]), $(f_*a) \cdot (g_*b)$ is not equal to $\pm(g_*b) \cdot (f_*a)$, $k_* \neq h_*$, and therefore $[X^*, x_0. Y, y_0]$ is not abelian. Thus X has categorical decompositions leading to distinct homotopy theories.

5. APPLICATIONS

5.1. CONTRACTIBLE FIBRES

Let $p: E \to B$ be a fibre map with fibre $F = p^{-1}(b_0)$ $(b_0 \in B)$. In [4], E. H. Spanier and J. H. C. Whitehead proved that if F is either a locally finite CW-complex, or a compactum which is an ANR and is contractible in E, then F is an H-space. This result may be improved as follows:

THEOREM 5.1A. *Suppose that* $F \times F$ *is smooth about some point* y_0^2 $(y_0 \in F)$, *and that* p *satisfies the homotopy lifting condition for maps from* $F \times F \times I$. *If* F *is contractible in* E, *by means of a contraction leaving* y_0 *fixed, then* F *is an* H-*space.*

Proof. Let $\mathscr{C}$ consist of the pairs (F, y_0) and $(F\times F, y_0^2)$. If $(X, x_0) \in \mathscr{C}$, then from Theorem 2.4B it follows that the sequence

$$\cdots \to [SX, x_0; B, b_0] \xrightarrow{\partial} [X, x_0; F, y_0] \xrightarrow{i_\#} [X, x_0, E, y_0]$$

is exact, where $i: (F, y_0) \to (E, y_0)$ is the inclusion map. By hypothesis, $i_\#$ is trivial, so that the map ∂ is onto. The remainder of the proof consists in using the group operation on $[SX, x_0; B, b_0]$ to establish a cohomotopy theory on $F, y_0, \mathscr{C}$.

Let $\phi: (F\times I, y_0\times I) \to (E, y_0)$ be a contraction of F, with $\phi(y, 0) = y$ and $\phi(y, 1) = y_0$ when $y \in F$. Suppose that $\alpha \in [X, x_0; F, y_0]$ is represented by the map f, and that $\theta: X\times I \to SX$ is the usual identification. It is easily verified that $\bar{f}(\theta(x, t)) = p(\phi(f(x), t))$ defines a map $\bar{f}: (SX, x_0) \to (B, b_0)$. Note that $\{\bar{f}\}$ is independent of the representative $f \in \alpha$. Define the function $\Phi: [X, x_0; F, y_0] \to [SX, x_0; B, b_0]$ by $\Phi(\alpha) = \{\bar{f}\}$. The binary operation μ' on $[X, x_0; F, y_0]$ is now given as follows. If α and β are in $[X, x_0; F, y_0]$ and μ is the binary operation on $[SX, x_0; B, b_0]$, then

$$\mu'(\alpha, \beta) = \partial\mu(\Phi\alpha, \Phi\beta).$$

That condition C1 holds is readily verified. As for C2, suppose that (X, x_0) and (X', x_0') are in $\mathscr{C}$ and $f: (X, x_0) \to (X', x_0')$. Note that the diagram

$$\begin{array}{ccc} [SX, x_0; B, b_0] & \overset{\Phi}{\underset{\partial}{\rightleftarrows}} & [X, x_0; F, y_0] \\ \uparrow Sf^\# & & \uparrow f^\# \\ [SX', x_0'; B, b_0] & \overset{\Phi}{\underset{\partial}{\rightleftarrows}} & [X', x_0'; F, y_0] \end{array}$$

is commutative in both ways (in other words, that $\partial Sf^\# = f^\#\partial$ and $\Phi f^\# = Sf^\#\Phi$), and that $\partial\Phi$ is the identity on $[X, x_0; F, y_0]$. If $\alpha, \beta \in [X', x_0'; F, y_0]$, then

$$f^\#\mu'(\alpha, \beta) = \partial\mu(Sf^\#\Phi\alpha, Sf^\#\Phi\beta) = \mu'(f^\#\alpha, f^\#\beta).$$

The result now follows from 3.2A.

If the kernel of ∂ is a normal subgroup of $[SX, x_0; B, b_0]$ when $(X, x_0) = (F, y_0)$ and $(F\times F\times F, y_0^3)$, then ∂ is a homomorphism and (by 3.2B and 3.2C) F is an associative, inversive H-space. If B is an H-space, then $[SX, x_0; B, b_0]$ is abelian for any (X, x_0), whence $[X, x_0; F, y_0]$ is abelian, and F is a commutative H-space.

The kernel of ∂ need not be normal, as the following example shows. Let S_1 and S_2 be 1-spheres, $B = S_1 \vee S_2$, $b_0 = S_1 \cap S_2$; let E be the space of paths in B beginning in S_1 (with the compact-open topology), and let $p: E \to B$ be the map which assigns to each path its terminal point. Then p is a fibre map, and the fibre $F = p^{-1}(b_0)$ is contractible in E. The image of $[X, x_0; E, y_0]$ under $p_\#$ is the same as the image of $[X, x_0; S_1, b_0]$ under inclusion. In particular, the image of the fundamental group $\pi_1(S_1, b_0)$ is not normal in $\pi_1(B, b_0)$. Thus $\pi_0(F, y_0)$ is not a group, and therefore F is not an associative, inversive H-space under the structure map constructed in 5.1A. However, F is the union of disjoint contractible spaces, and therefore an associative, inversive structure map is easily constructed.

5.2. CLASSIFICATION THEOREMS. STATEMENT OF RESULTS.

Sections 5.2 to 5.4 are concerned with the determination of $[X, x_0; Y, y_0]$ (up to group extensions) for certain choices of (X, x_0) and (Y, y_0). The technique used is basically this. Assume that X is the suspension of a CW-complex, and that Y has only n nontrivial homotopy groups. Decompose Y by fiberings

$$Y_1 \underset{p_1}{\rightarrow} Y_2 \underset{p_2}{\rightarrow} \cdots \underset{p_{n=1}}{\rightarrow} Y_n = Y,$$

where each p_i is a fibre map with fibre F_i. These may be so arranged that $Y_1, F_1, \cdots, F_{n-1}$ are all Eilenberg-MacLane spaces. Next, apply Theorem 2.4B to each fibre map, and note that the groups $[X; Y_1], [X; F_1], \cdots, [X; F_{n-1}]$ are known. This leads to an inductive procedure which gives information about $[X, x_0; Y, y_0]$. Certain other conditions on X and Y permit a reduction to the above situation.

Throughout, it is assumed that X is a CW-complex, and that Y is 1-connected. Note that this assumption permits the use of $[X; Y]$ in place of $[X, x_0; Y, y_0]$. The notation $\pi_n = \pi_n(Y)$ is used. The following results are obtained.

THEOREM 5.2A. *Let* X *be the suspension of a* CW-*complex, and let* m *be an integer. If* π_n *is trivial for* $n > m$, *and* $H^n(X; \pi_n)$ *has finite order* N_n *for* $n \leq m$, *then* $[X; Y]$ *is finite, and its order does not exceed the product* $N_2 N_3 \cdots N_m$.

THEOREM 5.2B. *If* X *is the suspension of a* CW-*complex, and* Y *has only two nontrivial homotopy groups* π_n, *and* π_m $(1 < n < m)$, *then the sequence*

$$\cdots \rightarrow H^n(SX; \pi_n) \xrightarrow{\gamma} H^{m+1}(SX; \pi_m) \rightarrow [X; Y] \rightarrow H^n(X; \pi_n) \xrightarrow{\gamma} H^{m+1}(X; \pi_m) \rightarrow \cdots$$

is exact. The homomorphism γ *is the cohomology operation corresponding to the* k-*invariant of* Y.

In Section 5.4, a description of $[X; Y]$ is presented for the case in which X is an $(n-1)$-connected CW-complex of dimension $n+2$.

5.3. PROOFS OF THEOREMS 5.2A AND 5.2B

LEMMA 5.3A. *If* (X, A) *is a relative* CW-*complex, if* Y *has only one nontrivial homotopy group* $\pi_n = \pi_n(Y)$ $(n > 1)$, *and if* $y_0 \in Y$, *then there exists an isomorphism* $\Theta: [X, A; Y, y_0] \rightarrow H^n(X, A; \pi_n)$.

This is a familiar result in obstruction theory (see, for example, [7]). The isomorphism is given by $\Theta(\{f\}) = f^*d$ when $f: (X, A) \rightarrow (Y, y_0)$ is a map, where $f^*: H^n(Y, y_0; \pi_n) \rightarrow H^n(X, A; \pi_n)$ is the cohomology homomorphism induced by f, and where $d \in H^n(Y, y_0; \pi_n)$ is the basic cohomology class on Y. Note that if $h: \pi_n \rightarrow H_n(Y)$ is the Hurewicz isomorphism, then d corresponds, under the Universal Coefficient Theorem, to $h^{-1} \in \mathrm{Hom}(H_n(Y); \pi_n)$.

In the work that follows, it is frequently necessary to simplify a space by "killing off" some of its homotopy groups. The following method is used, among others, for this purpose. Let Y be a topological space, and n a positive integer. A space $\Lambda_n(Y) \supset Y$ is constructed by appending $(j+1)$-cells to Y in such a manner that $\pi_j(\Lambda_n(Y)) = 0$ for $j > n$ [9]. The inclusion map $i: Y \rightarrow \Lambda_n(Y)$ induces isomorphisms $i_\#: \pi_j(Y) \approx \pi_j(\Lambda_n(Y))$ for $j \leq n$. The pair $(\Lambda_n(Y), Y)$ is a relative CW-complex.

Suppose that Y is a 1-connected space whose Hurewicz homotopy groups in dimensions greater than m are trivial. Let Z denote the space $\Lambda_{m-1}(Y)$, and $i: Y \to Z$ the inclusion map. The pair (Z, Y) is m-connected, and $\pi_{m+1}(Z, Y) \approx \pi_m = \pi_m(Y)$. If K' is a space of type $(\pi_m, m+1)$, and $k_0 \in K'$, then $[Z, Y; K', k_0] \approx H^{m+1}(Z, Y; \pi_m)$. On the other hand, the class of the obstruction to retracting Z onto Y is in $H^{m+1}(Z, Y; \pi_m)$. Let $\xi: (Z, Y) \to (K', k_0)$ be a map corresponding to this obstruction (that is, let $\Theta(\{\xi'\}) \in H^{m+1}(Z, Y; \pi_m)$ be the class of the obstruction). Let K be the mapping cylinder of ξ', and $\xi: Z \to K$ the inclusion map. Note that K has the homotopy type of K'. Let E be the space consisting of paths in K which terminate in Y, and with the compact-open topology. The space E is contractible. (Proof: Contract the paths to the constant paths corresponding to their endpoints. The portion of the mapping cylinder induced by $(\xi' \mid Y)$ is a cone over Y, and hence contractible. Using this contraction, we can set up a contraction of the constant paths.) The map $q: E \to K$ which sends each path into its initial point is a fibre map whose fibre F is of type (π_m, m). Let $W \subset E$ be the paths in Z terminating in Y, and let $p = (q \mid W)$. Then p is a fibre map, W has the homotopy type of Y, and the fibre F' of p is of type (π_m, m). Let $\eta: W \to E$ and $\zeta: F' \to F$ be inclusion maps. The retraction of K onto K' and the map $\xi': (Z, Y) \to (K', k_0)$ induce maps of F and F', respectively, into the loop space over K'. It is immediate that these maps induce isomorphisms of the homotopy groups.

If X is the suspension of a CW-complex X', then we have the diagram

$$\begin{array}{ccccccccccc}
\cdots \to & [SX; Z] & \xrightarrow{\partial} & [X; F'] & \to & [X; W] & \xrightarrow{p_\#} & [X; Z] & \xrightarrow{\partial} & [X'; F'] & \to \cdots \\
& \downarrow \xi_\# & & \downarrow \zeta_\# & & \downarrow \eta_\# & & \downarrow \xi_\# & & \downarrow \zeta_\# & \\
\cdots \to & [SX; K] & \xrightarrow[\partial']{} & [X; F] & \to & [X; E] & \xrightarrow[q_\#]{} & [X; K] & \xrightarrow[\partial']{} & [X'; F] & \to \cdots .
\end{array}$$

The diagram is commutative. By Theorem 2.4B, the horizontal sequences are exact. The sets $[X'; E]$, $[X; E]$, $[SX; E]$, $\cdots$ each consist of one element, and therefore the homomorphisms ∂' are isomorphisms. It has been noted that the homomorphisms $\zeta_\#$ are isomorphisms. This yields the following exact sequence:

$$\text{(5.3B)} \qquad \cdots \to [SX; Z] \xrightarrow{\xi_\#} [SX; K] \to [X; Y] \xrightarrow{i_\#} [X; Z] \xrightarrow{\xi_\#} [X; K] \to \cdots$$

Proof of 5.2A. If $m = 2$, then $[X; Y] = H^2(X; \pi_2)$, and this group is of order N_2. Suppose the result has been established for spaces whose homotopy groups of dimension above $m - 1$ are trivial. In the notation established above, $[X; Y]$ is an extension of $[SX; K]/\xi_\#[SX; Z]$ by a subgroup of $[X; Z]$. The order of $[SX; K]/\xi_\#[SX; Z]$ does not exceed N_m, which is the order of $[SX; K] \approx H^{m+1}(SX; \pi_m) \approx H^m(X; \pi_m)$, and by the inductive hypothesis, the order of $[X; Z]$ is at most $N_2 \cdots N_{m-1}$. Thus

$$\text{order } ([X; Y]) \leq (\text{order } [SX; K]) \cdot (\text{order } [X; Z]) \leq N_2 \cdots N_{m-1} \cdot N_m .$$

Proof of 5.2B. It is assumed that Y has only two nontrivial homotopy groups, π_n and π_m, and that $1 < n < m$. The space Z is now of type (π_n, n). The diagram,

$$\begin{array}{ccccccccc}
\cdots \to & [SX; Z] & \xrightarrow{\xi_\#} & [SX; K] & \to [X; Y] \to & [X, Z] & \xrightarrow{\xi_\#} & [X; K] & \to \cdots \\
& \downarrow \Theta & & \downarrow \Theta & & \downarrow \Theta & & \downarrow \Theta & \\
& H^n(SX; \pi_n) & \xrightarrow[\gamma]{} & H^{m+1}(SX; \pi_m) & & H^n(X; \pi_n) & \xrightarrow[\gamma]{} & H^{m+1}(X; \pi_m) &
\end{array}$$

is obtained by combining the sequence (5.3B) with Lemma 5.3A; the operation γ is defined to be $\Theta^{-1}\xi_{\#}\Theta$. Since $\Theta(\{\xi'\})$ is the obstruction to retracting Z onto Y, $\Theta(\{\xi\})$ is the k-invariant of Y, and γ the cohomology operation corresponding to the k-invariant. The result now follows easily.

5.4. CLASSIFICATION THEOREMS, CONTINUED

LEMMA 5.4A. *If* X *is an* $(n-1)$-*connected* CW-*complex* $(n > 1)$ *of dimension* $k < 2n$, *then* X *has the homotopy type of a suspended space.*

Proof. We may assume, without loss of generality, that the $(n-1)$-skeleton of X is a point. The proof proceeds by induction on k, the dimension of X. It is immediate that the result holds for $k = n$. Suppose the result holds for complexes of dimension at most $k-1$, and that X is of dimension $k < 2n$. The $(k-1)$-skeleton X^{k-1} of X has the homotopy type of some suspended space SY. Let $f\colon X^{k-1} \to SY$ be a homotopy equivalence. Let the k-cells of X be denoted by E^k_α (α running over a suitable index set) with characteristic maps $e^k_\alpha\colon I^k \to X$. Then $f \circ (e^k_\alpha \mid \dot{I}^k)$ is defined and represents an element of $\pi_{k-1}(SY)$. Since the suspension homomorphism from $\pi_{k-2}(Y)$ into $\pi_{k-1}(SY)$ is an isomorphism when $k < 2n$, there is a map $g_\alpha\colon \dot{I}^{k-1} \to Y$ whose suspension is homotopic to $f \circ (e^k_\alpha \mid \dot{I}^k)$. A space

$$Z = Y \cup \bigcup_{\alpha \in A} E^{k-1}_\alpha$$

is obtained by attaching $(k-1)$-cells E^{k-1}_α by means of the maps g_α (that is, the characteristic map of E^{k-1}_α restricted to $\dot{I}^{k-1}$ is g_α). The map $f\colon X^{k-1} \to SY$ may be extended to a map of $X \to SZ$; for the problem of extending f over E^k_α is that of showing that $\{f \circ (e^k_\alpha \mid \dot{I}^k)\}$ is zero in $\pi_{k-1}(SZ)$. But the cell E^{k-1}_α supports a null-homotopy of g_α, and hence its suspension supports a null-homotopy of $f \circ (e^k_\alpha \mid \dot{I}^k)$. Note that the extension may be taken to be a homeomorphism on E less its boundary. The proof will be complete as soon as it is shown that the extension $f'\colon X \to SZ$ of f induces, for each i, an isomorphism of the homology group $H_i(X)$ onto $H_i(SZ)$ (see [8, p. 1135]).

As for this last point, consider the commutative diagram

$$\begin{array}{ccccccccc}
H_{i+1}(X, X^{k-1}) & \to & H_i(X^{k-1}) & \to & H_i(X) & \to & H_i(X, X^{k-1}) & \to & H_{i-1}(X^{k-1}) \\
\downarrow f''_* & & \downarrow f_* & & \downarrow f'_* & & \downarrow f''_* & & \downarrow f_* \\
H_{i+1}(SZ, SY) & \to & H_i(SY) & \to & H_i(SZ) & \to & H_i(SZ, SY) & \to & H_{i-1}(SY) .
\end{array}$$

The horizontal sequences are both exact, the f_* are isomorphisms by the inductive hypothesis; and the f''_* are isomorphisms, since $(f' \mid X - X^{k-1})$ is a homeomorphism. Thus f'_* is an isomorphism, from the Five-Lemma. This concludes the proof of Lemma 5.4A.

Let us suppose that X is an $(n-1)$-connected CW-complex of dimension $n+2$. In addition, assume that $n \geq 3$, so that X is the suspension of a CW-complex X'. Given a 1-connected space Y, set $Y_1 = \Lambda_{n+2}(Y)$.

LEMMA 5.4B. $[X; Y] \approx [X; Y_1]$.

Proof. The sequence $[\hat{X}, X; Y_1, Y] \to [X; Y] \to [X; Y_1] \to [\hat{X}', X'; Y_1, Y]$ is exact. The first and last sets are trivial.

Thus we may assume that the homotopy groups of Y are trivial in dimensions greater than $n+2$. Let $p\colon Y_2 \to Y$ be an $(n-1)$-connective fibre map [5]. The fibre F will have $\pi_i(F) = 0$ for $i \geq n-1$.

LEMMA 5.4C. $[X; Y] \approx [X; Y_2]$.

Proof. The sequence $[X; F] \to [X; Y_2] \to [X; Y] \to [X'; F]$ is exact. The first and last sets are trivial.

We may therefore assume that Y has only three nontrivial homotopy groups $\pi_i = \pi_i(Y)$ $(i = n, n+1, n+2)$. Let $r\colon P \to Y$ be an n-connective fibre map. The fibre Q is a space of type $(\pi_n, n-1)$. Let $k = k_n^{n+2}(Y)$ in $H^{n+2}(\pi_n, n; \pi_{n+1})$ be the k-invariant, and α the corresponding cohomology operation.

THEOREM 5.4D. *A homomorphism* β *may be chosen such that the sequence*

$$0 \to [X: P] \xrightarrow{r_\#} [X; Y] \xrightarrow{\beta} H^n(X; \pi_n) \to H^{n+2}(X; \pi_{n+1}) \to \cdots \text{ is exact.}$$

Proof. The exact sequence of the fibre map r is

$$\cdots \to [X; Q] \to [X; P] \xrightarrow{r_\#} [X; Y] \xrightarrow{\partial} [X'; Q] \to [X'; P] \to \cdots$$

It is immediate that $[X; Q] = 0$, that $[X'; Q] \approx H^{n-1}(X'; \pi_n) \approx H^n(X; \pi_n)$ and that $[X'; P] \approx H^{n+1}(X'; \pi_{n+1}) \approx H^{n+2}(X; \pi_{n+1})$. It remains only to demonstrate that the homomorphism from $H^n(X; \pi_n)$ to $H^{n+2}(S; \pi_{n+1})$ is α.

First, we note that the highest homotopy group of Y seems to have no relation to α. This suggests eliminating this group in the usual manner. Let $\bar{Y} = \Lambda_{n+1}(Y)$, and let $\bar{r}\colon \bar{P} \to \bar{Y}$ be an n-connective fibre map. If P is set equal to $\bar{r}^{-1}(Y)$, and $r = (\bar{r} \mid P)$, then $r\colon P \to Y$ is an n-connective fibre map for Y. The fibre, in both cases, will be called Q. The diagram

$$\begin{tikzcd}
\cdots \arrow[r] & {[X; Y]} \arrow[r, "\partial"] \arrow[d, "i_{1\#}"] & {[X'; Q]} \arrow[r, "i_{4\#}"] \arrow[d, "i_{2\#}"] & {[X'; P]} \arrow[r] \arrow[d, "i_{3\#}"] & \cdots \\
\cdots \arrow[r] & {[X; \bar{Y}]} \arrow[r, "\bar{\partial}"'] & {[X'; Q]} \arrow[r, "i_{5\#}"'] & {[X'; \bar{P}]} \arrow[r] & \cdots
\end{tikzcd}$$

is commutative, and the horizontal sequences are exact. The maps $i_1, \cdots, i_5$ are inclusion maps. It is clear that $i_{2\#}$ and $i_{3\#}$ are isomorphisms.

The space Y has only two nontrivial homotopy groups. Such spaces have been discussed in the proof of Theorem 5.2B. With the notation of that theorem (save that Y is now called $\bar{Y}$ and $m = n+1$), $\bar{P}$ may be taken to be the space of paths in Z beginning at some point $y_0 \in \bar{Y}$ and terminating in Y (see, for example, [6]). That is to say, $\bar{P} = F$. The fibre map r sends each path into its terminal point. The fibre Q of r is the space of loops on Z. We have the commutative diagram

$$\begin{tikzcd}
{} \arrow[r, "\bar{\partial}"] & {[X'; Q]} \arrow[r, "i_{5\#}"] & {[X'; \bar{P}]} \arrow[r] & \cdots \\
{} \arrow[r, "p_\#"'] & {[X; Z]} \arrow[r, "\partial"'] \arrow[u, "\partial_1"] & {[X'; F]} \arrow[r] \arrow[u, "j_\#"'] & \cdots .
\end{tikzcd}$$

In this diagram, j: $F = \bar{P}$ is the identity map, and ∂_1 is an isomorphism (Q is the fibre of a contractible fibre space over Z). The result now follows from the corresponding result in Theorem 5.2B.

This result has also been obtained, in a different way, by Barratt [1].

5.5. H-STRUCTURE MAPS

Let Y be an H-space with an identity element $y_0 \in Y$. Let

$$i: (Y \vee Y, y_0^2) \to (Y \times Y, y_0^2) \qquad \text{and} \qquad j: (Y \times Y, y_0^2) \to (Y \times Y, Y \vee Y)$$

be inclusion maps. Use $HS(Y) \subset [Y \times Y, y_0^2; Y, y_0]$ to denote the set of homotopy classes of H-structures on Y. The problem of describing $HS(Y)$ is suggested in [3].

THEOREM 5.5A. *If* $[Y \times Y, y_0^2; Y, y_0]$ *is a group, then* $HS(Y)$ *is in one-to-one correspondence with* $j^{\#}[Y \times Y, Y \vee Y; Y, y_0]$.

Proof. Let K: $(Y \vee Y, y_0^2) \to (Y, y_0)$ be the map $K(y, y_0) = K(y_0, y) = y$, for $y \in Y$. A map f: $(Y \times Y, y_0^2) \to (Y, y_0)$ is a structure map if and only if $(f | (Y \vee Y, y_0^2))$ is homotopic to K. Thus $HS(Y) = (i^{\#})^{-1}(\{K\})$. Since $[Y \times Y, y_0^2; Y, y_0]$ is a group, this coset is in one-to-one correspondence with the kernel of $i^{\#}$, which is $j^{\#}[Y \times Y, Y \vee Y; Y, y_0]$.

THEOREM 5.5B. *If* Y *is an associative, inversive* H-*space with two nontrivial homotopy groups* $\pi_n = \pi_n(Y)$ *and* $\pi_m = \pi_m(Y)$ $(1 < n < m)$, *then* $HS(Y)$ *is in one-to-one correspondence with* $H^m(Y \times Y, Y \vee Y; \pi_m)$.

Proof. Consider the diagram

$$\begin{array}{ccccc} [Y \vee Y, y_0^2; \Omega(Y), w_0] & \xrightarrow{\delta} & [Y \times Y, Y \vee Y; Y, y_0] & \to & [Y \times Y, y_0^2; Y, y_0] \\ \downarrow \Theta_1 & & \downarrow \Theta_2 & & \\ H^{m-1}(Y \vee Y; \pi_m) & \to & H^m(Y \times Y, Y \vee Y; \pi_m), & & \end{array}$$

where δ is the cohomotopy coboundary, and δ' is the cohomology coboundary of the pair $(Y \times Y, Y \vee Y)$. Since $Y \vee Y$ is $(n - 1)$-connected and $\pi_{m-1}(\Omega(Y))$ is the only nontrivial homotopy group of $\Omega(Y)$ with dimension greater than $n - 1$, there is an isomorphism Θ_1 as indicated. The pair $(Y \times Y, Y \vee Y)$ is $(2n - 1)$-connected, and π_m is the only nontrivial homotopy group of Y with dimension larger than n. Thus Θ_2 is an isomorphism. In short, Y, in its role as image space, may be replaced by a space of type (π_m, m). Thus the square is commutative. But δ' is trivial, whence δ is trivial; and therefore $j^{\#}$ is a monomorphism. The result now follows from Theorem 5.5A.

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