CONCERNING A PROBLEM OF ALEXANDROFF

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In his well-known paper [1] *On local properties of closed sets*, P. Alexandroff introduced the notion of *r-dimensional condensation* and employed it to establish the invariance of the property of regular $(n - r - 1)$-accessibility of a closed set in euclidean n-space. In Section 6, where he discussed the difficulties surrounding the attempt to set up local Betti groups, he indicated that these difficulties vanish when the space is either (1) r-lc or (2) devoid of r-dimensional condensation. In the concluding section of his paper he stated the following problem.

PROBLEM VI. *What relations are there between the absence of condensation* (in all dimensions) *and local connectedness* (also in all dimensions)?

So far as I have been able to find, no one has specifically treated this problem although, as pointed out below, it is partially settled as a corollary of certain theorems in my book *Topology of Manifolds* [2], and while the complementary parts of the solution have been known to me for some time, I have never published these (see the Remarks following the statement of Theorem 2 below). Because of the possible importance of these matters in connection with the application of local Betti groups to lc^n spaces, however, it seems desirable to publish them.

1. IMPLICATIONS OF THE lc^n PROPERTY FOR LACK OF CONDENSATION

In [2], the following theorems are proved.

A. *If the locally compact Hausdorff space* S *is* lc^n, *then* $p^r(x) \leq \omega$ *for all* $x \in S$ *and* $r \leq n$; *and if in addition* S *is semi-*$(n+1)$*-connected at some* $\bar{x} \in S$, *then* $p^{n+1}(\bar{x}) \leq \omega$. ([2], p. 211, Th. 2.26.)

B. *If* S *is a locally compact Hausdorff space such that* $p^r(x) \leq \omega$ *for some point* x *of* S, *then* S *has no* r*-dimensional condensation at* x. (([2], p. 358, Cor. 1.12.) Although the proof is given only for the case where x is of countable character—a corresponding proof was also given by Alexandroff ([1], p. 18, Cor. I)—there is little difficulty in revising the proof to remove this restriction.)

Combining these two theorems, we have

THEOREM 1. *If the locally compact Hausdorff space* S *is* lc^n, *then* S *has no* n*-dimensional condensation at any point; and if in addition* S *is semi-*$(n+1)$*-connected at some point* x, *then* S *has no* $(n+1)$*-dimensional condensation at* x.

Remark. That n-lc would be insufficient to ensure lack of n-dimensional condensation is shown, for instance, by the well-known example

$$S = \{(x, y) \mid 0 < x \leq 1,\ y = \sin 1/x\} \cup \{(0, y) \mid -1 \leq y \leq 1\}, \text{ with } n = 1.$$

Also, that lc^n at x alone is not sufficient to ensure lack of n-dimensional condensation at x is shown by the following example: Let

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$$M_0 = \{(0, y) \mid 0 \leq y \leq 1\}, \quad M_1 = \{(x, 0) \mid 0 \leq x \leq 1\},$$

$$M_n = \{(x, y) \mid x = 1/n,\ 0 \leq y \leq 1\} \quad (n = 2, 3, \cdots);$$

and let $S = \bigcup_{n=0}^{\infty} M_n$. At $x = (0, 0)$, S is lc^1 but has 1-dimensional condensation.

2. IMPLICATIONS OF LACK OF CONDENSATION FOR LOCAL CONNECTEDNESS

Simple examples show that the converse of Theorem 1 fails. Alexandroff gives the following example ([1], p. 24): Let S consist of all points on the circumferences $(x - 1/2^n)^2 + y^2 = 1/4^n$, mutually tangent at their single common point $p = (0, 0)$; S has no condensation in any dimension at p, yet is not 1-lc at p. However, it is still possible to state an important implication:

THEOREM 2. *If the locally compact, separable, metric space* S *is of finite dimension and has no* r*-dimensional condensation at any point (for all* r*), then* S *is locally connected.*

Remarks. The restriction to finite dimension in the hypothesis is necessary. This may be shown, for example, by a configuration in the fundamental cube of Hilbert space consisting of a sequence of disjoint spheres S^n $(n = 1, 2, 3, \cdots)$ converging to a point p; this configuration has no condensation in any dimension, but it is not locally connected at p. Although the configuration is not connected, it is easily modified so as to yield a continuum in Hilbert space consisting of spheres S^n which are connected in sequence by straight line intervals, and which converge to a straight line interval at all of whose points (with the possible exception of one) the configuration fails to be locally connected.

On the other hand, it seems likely that the restriction "separable, metric" is unnecessary. It is used here solely in order to be able to imbed the space S in euclidean space. Our proof, then, is not intrinsic. However, the intrinsic type of proof seems to run into unsolved problems concerning the dimensions of the carriers of (Čech) cycles. On the other hand, the present proof depends on a theorem which has considerable interest in its own right, since it is a direct generalization of the classical theorem of Schoenflies concerning the positional character of a Peano continuum in the plane (see [2], p. 116). We refer to Theorem 3 below. [Theorem 3 was announced in an abstract [3], but not published heretofore since it did not fit into the framework of [2], in which the other results (such as Theorem A above) of the cited paper were published.]

THEOREM 3. *Let* M *be a closed, connected subset of the euclidean space* E^n *such that* (1) *the diameters of the domains complementary to* M *form a null sequence, and* (2) *if* B *is the boundary of a domain* D *complementary to* M, *then* B *is regularly* s*-accessible from* D *for* $s = 0, 1, \cdots, n - 2$. *Then* M *is locally connected.*

Proof. Suppose M is not locally connected. Then there exist concentric $(n - 1)$-spheres K_1, K_2 of radii r_1, r_2 respectively, such that $r_1 > r_2$, and such that if I is the domain bounded by $K_1 \cup K_2$, there are infinitely many components M_i of $M \cap (K_1 \cup K_2 \cup I)$ that meet both K_1 and K_2. Let K be a third $(n - 1)$-sphere, concentric with K_1 and of radius $(r_1 + r_2)/2$. Let Z^0 be a nontrivial cycle carried by $x_1 \cup x_2$ (see [2], p. 142, Def. 11.4) such that $x_1 \in K_1$ and $x_2 \in K_2$. Then Z^{n-1}, the fundamental $(n - 1)$-cycle of K based on any subdivision of K, is linked with Z^0. Consequently, there exists a positive number $e < (r_1 - r_2)/2$ such that any e-approximation to Z^{n-1} is also linked with Z^0. We now arrange the proof in three steps.

(1) If D is a domain complementary to M, meeting $S(K, e)$ but neither K_1 nor K_2, then the boundary of D, being a continuum, lies wholly in one set M_i. Hence if we add to M all such domains D, the resulting set M' has the property that those of its complementary domains that meet $S(K, e)$ also meet $K_1 \cup K_2$ and are therefore finite in number (by condition (1) of the hypothesis); the domains of this type we denote hereafter by D_m $(m = 1, 2, \cdots, k)$, and the boundary of D_m we denote by B_m. (Each M_i, with the added domains D for which $F(D) \subset M_i$, will be denoted by M_i'.)

(2) We next define a sequence of positive numbers e_j $(j = 1, 2, \cdots, n)$. First, $e_1 = e$, where e is defined as above. In general, having defined e_i for $1 \leq i < n - 1$, we define e_{i+1} as follows: If $x \in K \cap \bigcup B_m$, there exists a number $d_x > 0$ such that any $(n - i - 1)$-cycle of $D_m \cap S(x, d_x)$ $(m = 1, 2, \cdots, k)$ bounds an $(n - i)$-chain on $x \cup D_m$ of diameter less than $e_i/8$; this is a result of condition (2) of the hypothesis. Similarly, if $y \in K - K \cap \bigcup B_m$, there exists a number $d_y > 0$ such that either $S(y, d_y) \subset M' \cap S(y, e_i/8)$ or $S(y, d_y) \subset (E^n - M') \cap S(y, e_i/8)$. The set of all neighborhoods of types $S(x, d_x)$, $S(y, d_y)$ covers K and hence has a finite subset G_{i+1} which covers K. The elements of G_{i+1} of type $S(x, d_x)$ we denote generically by $U_{x(i+1)}$, and those of type $S(y, d_y)$ by $U_{y(i+1)}$. The number e_{i+1} is selected so that any point set which contains a point of K and is of diameter less than e_{i+1} lies wholly in at least one element of G_{i+1}, and so that $e_{i+1} < e_i/8$.

Having defined e_{n-1}, we define the number e_n as follows: With $i = n - 1$, we proceed as in the preceding paragraph, with the additional stipulation, however, that d_x is chosen so that any point of $D_m \cap S(x, d_x)$ $(m = 1, 2, \cdots, k)$ may be joined to x by an arc of $x \cup D_m$ of diameter less than $e_{n-1}/16$ (see [2], pp. 353-354.) We thus obtain the following property: If $a \in D_1 \cap S(x, d_x)$ and $b \in D_2 \cap S(x, d_x)$, for example, then the nontrivial cycle based on the pair of points a, b bounds a chain of $x \cup D_1 \cup D_2$ of diameter less than $e_{n-1}/8$. Finally, e_n is chosen less than $e_{n-1}/8$.

(3) We now establish the existence of a cycle $\overline{Z}^{n-1}$ which e-approximates Z^{n-1} and yet fails to meet a certain set M_i. This will contradict the fact that Z^0 bounds on $K_1 \cup K_2 \cup M_i$.

Let S be a barycentric subdivision of K of mesh less than e_n. Let H denote a set M_i' such that (a) no vertex of S lies on H, (b) no point x corresponding to a $U_{xi} \in G_i$ $(i = 2, 3, \cdots, n)$ lies on H, and (c) no point y corresponding to a $U_{yi} \in G_i$ lies on H (and hence no U_{yi} contains points of H).

Now let ab be a 1-simplex of S (we denote simplexes by a sequence of symbols of their vertices). If ab fails to meet H, we retain it; otherwise, there exist finitely many closed intervals $a_r b_r$ of ab whose endpoints a_r, b_r (with the possible exception of a, b) lie in $E^n - M'$, and such that all points of $H \cap ab$ lie on the intervals $a_r b_r$. [This follows from Theorem 1.3, p. 100 of [2], if there we let $A = H \cap ab$, $B = a \cup b$, $S = M' \cap ab$. Since components of $H \cap ab$ are also components of $M' \cap ab$, the set $C = \bigcup_{x \in A} C(x)$ is A itself, and by the conclusion of the cited theorem, $M' \cap ab = E \cup F$, where E and F are separated, and where $A \subset E$, $B \subset F$. Let $d = d(A, B)$; then $S(H \cap ab, d/2) \cap ab$ is an open subset of ab, a finite number of whose components $a_r b_r$ cover $H \cap ab$.] Since the diameter of ab is less than e_n, ab lies wholly in one element of G_n, which is necessarily a U_{xn}. Hence the nontrivial cycle on each a_r, b_r bounds a 1-chain in this U_{xn}, of diameter less than $e_{n-1}/8$, which meets M' in at most the point x. It follows that there exists a broken line with end-points a, b not meeting H, and of diameter less than $e_{n-1}/8$. This broken line may be used as an approximation replacing ab. And in this manner, proceeding through the set of all 1-simplexes of S, we arrive at an approximating set of broken lines that fails to meet H.

Consider next a 2-simplex abc of S. Its boundary 1-simplexes have been replaced by broken lines constituting a polygon P, and the original fundamental 1-cycle Z^1 on the boundary of abc may be replaced by the obvious 1-cycle $\overline{Z}^1$ on the approximating polygon P. Since P is of diameter less than e_{n-1}, $\overline{Z}^1$ bounds a chain K^2 of diameter less than e_{n-1}, where $|K^2|$ consists of a finite set of closed 2-simplexes of some subdivision of E^n. If $|K^2|$ fails to meet H, we retain it. Suppose, however, that $|K^2|$ meets H. Since $|K^2|$ is of diameter less than e_{n-1} and contains a point of K (the point a, for instance), it must lie in an element $U_{x(n-1)}$ of G_{n-1}. And since there exists a separation of $M' \cap \|K^2\|$ (if L is a chain, then $|L|$ denotes a complex and $\|L\|$ denotes a point set; see [2], pp. 53, 55) into separated sets E and F such that $E \supset H \cap \|K^2\|$ and $F \supset \|\overline{Z}^1\|$, (see Theorem 1.3, p. 100 of [2]), a suitable subdivision of $|K^2|$ yields a 2-chain $\overline{K}^2$ such that $\|\overline{K}^2\| \supset H \cap \|K^2\|$ and $\|\partial\overline{K}^2\|$ lies in $E^n - M'$. Let $\partial\overline{K}^2 = L^1$. Since L^1 lies in $U_{x(n-1)}$, there exists a 2-chain L^2 of $(E^n - M') \cup x$, of diameter less than $e_{n-2}/8$, whose boundary is L^1. We may assume that the carrier of L^2 is either a finite polyhedron of some subdivision of E^n, or an infinite polyhedron (together with x) which has only a finite number of simplexes exterior to any neighborhood of x ([1], pp. 15-16). And since $x \notin H$, we may replace $|L^2|$ by a finite polyhedron $|\overline{L}^2|$ which differs from $|L^2|$ only in a small neighborhood of x, satisfies the relation $\partial\overline{L}^2 = \partial\overline{K}^2$, and does not meet H. Finally, the simplex abc is to be replaced by $|L^2| \cup |K^2 - \overline{K}^2| = C^2$. Each 2-simplex of S is treated similarly, of course.

The procedure for replacing the 3-, 4-, $\cdots$, (n - 2)-simplexes by approximations that do not meet H is carried out in a manner analogous to that above. And in the final step of the process, each (n - 1)-simplex is replaced by an approximating polyhedron $|K^{n-1}|$ obtained from a chain K^{n-1}, just as abc was replaced by C^2 above, where K^{n-1} lies in $E^n - H$. The union of the chains K^{n-1} is a cycle $\overline{Z}^{n-1}$ that forms an e-approximation of Z^{n-1} which fails to meet H. As stated above, this constitutes a contradiction.

Remark. The fact that regular (n - 2)-accessibility alone would not have been sufficient in Theorem 3 may be shown by the following example, with $n = 3$: Let S_1 be the surface of the unit cube in the first quadrant of 3-space having vertices (0, 0, 0), (1, 1, 1), etc. For $n = 2, 3, 4, \cdots$, let S_n denote the set of all points in this cube such that $x = 1/n$. Let K denote the set of all points on the top of the cube (that is, in the plane $z = 1$) which do not lie on the (edges of) the sets S_n $(n \geq 2)$. Finally, let $S = \bigcup_{n=1}^{\infty} S_n - K$. Then S is not locally connected, although it is regularly 1-accessible at all points.

3. PROOF OF THEOREM 2

Since S is separable, metric and of finite dimension, we may consider it to be imbedded in a euclidean space E^n (see [4], p. 60, for instance). We may also assume that n is large enough so that $E^n - S$ is a single domain. Let C be a component of S. Then C is open in S; for suppose $p \in C$ is a limit point of S - C. Then there exists a sequence $\{x_n\}$ of points x_n of S - C, having p as a sequential limit point, and such that if C_n denotes the component of S containing x_n, then $C_n \cap C_m$ is empty, for $n \neq m$. Since S has no 0-dimensional condensation at p, there exists, corresponding to each open set P containing p, an open set Q such that $p \in Q \subset P$ and such that every cycle Z^0 mod S - P carried by a compact subset of S - x is homologous to zero mod S - Q (see [2], p. 356, Lemma 1.6). Since each x_n carries a 0-cycle mod S - P, it follows that the component C_n must meet S - Q. If then we take concentric spheres

K_1, K_2 with center p and lying in Q, we may show by the methods used in proving Theorem 3 (with components of the sets $C_n \cap \bar{I}$ playing the roles of the sets M_i) that this situation leads to a contradiction, and we conclude that the components of S are open. That the accessibility conditions of the hypothesis of Theorem 3 are satisfied follows from the duality between lack of r-dimensional condensation and regular (n - r - 1)-accessibility (see [2], p. 356, Theorem 1.9).

Theorem 3 can now be applied to show that each component of S is locally connected, and hence S is itself locally connected.

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