INTERFERENCE PHENOMENA FOR ENTIRE FUNCTIONS

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1. Let $f(z)$ be an entire function of exponential type, bounded at the positive and negative integers. According to Cartwright's theorem [2, p. 180], if the type τ of $f(z)$ is less than π, the function is bounded on the real axis; and there exists a number C, depending only on τ, such that $|f(x)| \leq C \sup |f(n)|$ for all real x. If, however, $\tau = \pi$, it is evident that $f(n)$ can be bounded while $f(x)$ is unbounded (example: $f(z) = z \sin \pi z$). The possibility remains that the values of $f(x)$ may interfere with each other in such a way that a certain combination will be bounded even though $f(x)$ itself is not. S. Bernstein [1] showed that if $f(x) = o(|x|)$, the boundedness of $f(n)$ implies

$$|f(x + \tfrac{1}{2}) + f(x - \tfrac{1}{2})| \leq C \sup |f(n)|; \tag{1.1}$$

the best value for C in (1.1) is $8/\pi$ (misprinted in [2], p. 219). (To see that (1.1) is really significant, we must observe that $f(n)$ bounded and $f(x) = o(|x|)$ do not necessarily imply $f(x) = O(1)$ when $\tau = \pi$; see §5.) Timan [4] showed that $o(|x|)$ can be replaced by $o(|x|^2)$ provided that (1.1) is weakened to

$$f(x + \tfrac{1}{2}) + f(x - \tfrac{1}{2}) = O(1).$$

He also generalized Bernstein's interference operator to

$$L[f] = \int_{-\infty}^{\infty} f(x + t)\, d\rho(t), \tag{1.2}$$

with

$$\int_{-\infty}^{\infty} e^{\sigma |x|} |d\rho(x)| < \infty \quad \text{for some } \sigma > \pi, \tag{1.3}$$

and proved corresponding results; in particular, $L[f(x)]$ is bounded, whenever $f(n)$ is bounded, $f(x) = o(|x|)$, and $\tau = \pi$, if and only if $\int_{-\infty}^{\infty} e^{\pm i\pi t}\, d\rho(t) = 0$.

I shall replace (1.2) by the still more general operator $L = \lambda(D)$, where $D = d/dx$ and $\lambda(t)$ is regular on the closed segment $[-i\pi, i\pi]$ of the imaginary axis. When $\lambda(t)$ is regular in the disk $|t| \leq \pi$, the operator $\lambda(D)$ can be defined by

$$\lambda(D) f(z) = \sum_{n=0}^{\infty} \lambda_n f^{(n)}(z) \quad \left(\lambda(t) = \sum_{n=0}^{\infty} \lambda_n t^n\right), \tag{1.4}$$

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for all entire functions of exponential type π. In the general case, $\lambda(D)$ has to be defined differently (3.2), and is applicable only to functions of exponential type π whose indicator diagrams reduce to segments of the imaginary axis, i.e. which are $O(e^{\epsilon |x|})$ on the real axis for every positive ϵ. The result now reads as follows: for functions of exponential type π, the condition $\lambda(\pm i\pi) = 0$ is necessary and sufficient for $f(n) = O(1)$ and $f(x) = o(|x|)$ to imply

$$|\lambda(D)f(x)| \leq C \sup |f(n)| ,$$

and for $f(n) = O(1)$ and $f(x) = o(x^2)$ to imply

$$\lambda(D)f(x) = O(1).$$

(This is more than is stated in [2], p. 221.) In Timan's case $\lambda(t) = \int_{-\infty}^{\infty} e^{tu} d\rho(u)$, which under (1.3) is regular in the strip $|\Re(t)| \leq \pi$. Another special case, which indeed antedates Bernstein's interference theorem (it was given, not quite completely, by Macintyre [3]), corresponds to $\lambda(t) = \pi^2 + t^2$: if $f(z)$ is of type π, then $f(n) = O(1)$ and $f(x) = o(|x|)$ imply

$$|\pi^2 f(x) + f''(x)| \leq C \sup |f(n)|;$$

$f(n) = O(1)$ and $f(x) = o(|x|^2)$ imply

$$\pi^2 f(x) + f''(x) = O(1).$$

It is easily verified (§2) that if $f(n)$ is bounded and $f(x) = o(|x|^q)$ for some $q > 1$, then $f(z)$ is of the form $g(z) + P(z) \sin \pi z$, where $P(z)$ is a polynomial of degree less than q, and $g(z)$ is an entire function of exponential type π which is $o(|x|)$ on the real axis. Hence the extension of our results to $q > 1$ involves only the consideration of functions of the form $P(z) \sin \pi z$. The conclusion (§5) (stated in part by Timan for operators (1.2)) is that $L[f(x)] = O(1)$ for all f such that $f(n) = O(1)$ and $f(x) = o(|x|^q)$, if and only if $\lambda(t)$ has at least $(q-1)$-fold zeros at $\pm i\pi$; if $\lambda(t)$ has at least q-fold zeros, an inequality of the form

$$\lambda(D)f(x) \leq C \sup |f(n)|$$

holds.

It is possible to use still more general operators, corresponding to cases where $\lambda(t)$ is not even analytic, but this introduces further complications and will not be considered here.

2. Our first lemma is essentially known (cf. [4]).

LEMMA 1. *If* $\{a_n\}$ *is a bounded sequence and*

$$g(z) = a_0 \frac{\sin \pi z}{\pi z} + z \sin \pi z \sum_{n=-\infty}^{\infty}{}' \frac{(-1)^n a_n}{n\pi(z-n)}, \tag{2.1}$$

where Σ' *omits the term corresponding to* $n = 0$, *then* $g(z)$ *is an entire function of exponential type* π, *takes the values* a_n *at* $z = n$, *and is* $o(|x|)$ *on the real axis; if in addition* $a_0 = 0$, *then* $|g(x)| \leq A|x|$ *for real* x, *where* A *depends only on* $\sup |a_n|$.

Since the series on the right of (2.1) converges uniformly in any bounded region, $g(z)$ is an entire function. If $|a_n| \leq A$ and $z = x + iy$, then

$$\left|\sin \pi z \sum{}' \frac{(-1)^n a_n}{\pi n(z - n)}\right| \leq A\sum{}' \left|\frac{\sin \pi(z - n)}{\pi(z - n)}\right| \cdot \frac{1}{|n|}\,.$$

The term (or terms) for which n is closest to z contributes $o(1)$ for real z, and $O(e^{\pi|z|})$ in general. For real x this contribution is, in fact, at most $2/(|x| - \frac{1}{2})$ for $|x| \geq 1$ and at most 1 otherwise, since the term with $n = 0$ is omitted. Thus this part of the sum contributes at most $4A|x|$ to $g(x)$. The remaining sum is termwise less than

$$A\pi^{-1}e^{\pi|y|}\sum{}'' \frac{1}{|n(x - n)|}\,,$$

where Σ'' omits $n = 0$ and also the term or terms with $|n - x| \leq \frac{1}{2}$. It is readily calculated that $\Sigma'' = o(1)$ as $|x| \to \infty$, and so Lemma 1 is established.

LEMMA 2. *If* $f(z)$ *is an entire function of exponential type* π *which is* $o(|x|^q)$ *as* $|x| \to \infty$ *and has* $f(n) = 0$ $(n = 0, \pm 1, \pm 2, \cdots)$, *then* $f(z) = P(z)\sin \pi z$, *where* $P(z)$ *is a polynomial of degree less than* q.

This is an easy consequence of a theorem of Pólya and Valiron (see [2], p. 156, 9.4.2), which asserts that $f(z) = P(z)\sin \pi z$, with the degree of P not exceeding q, under the hypothesis that $|f(z)| \leq \epsilon(|z|)e^{\pi|z|}$ with $\epsilon(r) = O(r^q)$. In fact, if $f(z)$ is of exponential type π and is $o(|x|^q)$ on the real axis, put $F(z) = z^{-q}\{f(z) - Q(z)\}$, where $Q(z)$ is the polynomial consisting of the Maclaurin expansion of $f(z)$ through z^{q-1}; then $F(z)$ is of exponential type π and $F(x) = o(1)$. Consequently ([2], p. 82, 6.2.4), $|F(x + iy)| \leq Me^{\pi|y|}$ with $M = \sup|F(x)|$, whence

$$|f(z)| \leq Me^{\pi|y|}|z|^q + |Q(z)|\,.$$

Therefore by the theorem of Pólya and Valiron, $f(z) = P(z)\sin \pi z$ with $P(z)$ of degree at most q. Since $f(x) = o(|x|^q)$, $P(z)$ must actually be of degree at most $q - 1$.

LEMMA 3. *If* $f(z)$ *is an entire function of exponential type* π *which is* $o(|x|)$ *on the real axis and has* $f(n)$ *bounded, then*

$$f(z) = f'(0)\frac{\sin \pi z}{\pi} + f(0)\frac{\sin \pi z}{\pi z} + z \sin \pi z \sum{}' \frac{(-1)^n f(n)}{n\pi(z - n)}, \tag{2.2}$$

where Σ' *extends over* $1 \leq |n| < \infty$.

This is a theorem of Valiron [5]; cf. [2], p. 221, where a factor τ^{-1} should be supplied in the first term on the right.

If $f(z)$ satisfies the hypotheses of Lemma 2 except that the values $f(n)$ are bounded instead of 0, form the function $g(z)$ of (2.1) with $a_n = f(n)$. By Lemma 1, $g(z) - f(z)$ satisfies the hypotheses of Lemma 2 and so is of the form $P(z)\sin \pi z$. This justifies the remarks about the case $q > 1$ near the end of §1.

LEMMA 4. *If* $f(z)$ *is an entire function of exponential type* π, *if* $f(0) = 0$ *and* $|f(x)| < A|x|$ *for real* x, *and if* $\phi(x)$ *is the Laplace-Borel transform of* $f(z)$, *then* $|\phi(z)| \leq A(|z| - \pi|\sin\theta|)^{-2}$ *for* $|z| > \pi$.

The function $\phi(z)$ is defined as the Laplace transform of $f(z)$ for $\Re(z) > 0$ and as the analytic continuation of this outside the segment $[-i\pi, i\pi]$ of the imaginary axis. The analytic continuation may be effected by rotating the line of integration in

$$\phi(z) = \int_0^\infty f(t)\, e^{-zt}\, dt,$$

so that

$$\phi(re^{i\theta}) = e^{-i\theta}\int_0^\infty f(te^{-i\theta})\, e^{-rt}\, dt$$

for $r > \pi|\sin\theta|$. (See, e.g., [2], p. 74.) If $g(z) = f(z)/z$, we have $|g(x)| \le A$, hence [2, pp. 82-83] $|g(z)| \le Ae^{\pi|y|}$, and $|f(z)| \le A|z|\,e^{\pi|y|}$. Thus

$$|\phi(re^{i\theta})| \le A\int_0^\infty t\, e^{\pi t|\sin\theta| - rt}\, dt$$

$$= A(r - \pi|\sin\theta|)^{-2} \quad (r > \pi|\sin\theta|).$$

LEMMA 5. *If* $\{g_k(x)\}$ *is a set of entire functions of exponential type* π *which satisfy* $|g_k(x)| \le A|x|$ *for real* x, *if* $\phi_k(t)$ *is the Laplace-Borel transform of* $g_k(x)$, *and if* $g_k(x) \to f(x)$ *uniformly in every bounded domain, where* $f(x)$ *is an entire function of exponential type* π *with Laplace-Borel transform* $\phi(t)$, *then* $\phi_k(t) \to \phi(t)$, *uniformly for* t *at a positive distance from* $[-i\pi, i\pi]$.

For real positive x, $|g_k(x)e^{-xt}| \le Axe^{-xt}$, and so $g_k(x)e^{-xt}$ converges dominatedly to $f(x)e^{-xt}$ for each real positive t. Hence $\phi_k(t) \to \phi(t)$ for $0 < t < \infty$. By Lemma 4, the transforms $\phi_k(z)$ are bounded uniformly in any domain at a positive distance from $[-i\pi, i\pi]$. Vitali's theorem now establishes the conclusion of the lemma.

3. If $f(z)$ is an entire function of exponential type π and is $O(e^{\epsilon|x|})$ on the real axis for every positive ϵ, we have Pólya's representation

$$f(z) = \frac{1}{2\pi i}\int_C e^{zt}\phi(t)\, dt, \tag{3.1}$$

where $\phi(t)$ is the Laplace-Borel transform of $f(x)$ and C is a contour surrounding $[-i\pi, i\pi]$. (See [2], p. 74.)

Definition. If $\lambda(w)$ *is regular on* $[-i\pi, i\pi]$, *we define the operator* $L = \lambda(D)$ *by*

$$L[f(z)] = \frac{1}{2\pi i}\int_C e^{zt}\lambda(t)\,\phi(t)\, dt, \tag{3.2}$$

when $f(z)$ *has the representation* (3.1).

If $\lambda(w)$ is regular for $|w| \le \pi$, (3.2) can be written in the form (1.4).

LEMMA 6. *If the functions* $g_k(z)$ *are as in Lemma* 5, *then as* $k \to \infty$, $L[g_k(z)] \to L[f(z)]$.

For, if C is as in (3.1), we have

$$L[g_k(z)] = \frac{1}{2\pi i}\int_C e^{zt}\lambda(t)\phi_k(t)\,dt,$$

and we can take the limit under the integral sign, by bounded convergence (Lemma 4).

LEMMA 7. *If*

$$g(z) = \int_{-\pi}^{\pi} e^{izt}\,d\alpha(t), \tag{3.3}$$

with $\alpha(t)$ *of bounded variation, and if* $L[g]$ *is defined as in* (3.2), *then*

$$L[g(z)] = \int_{-\pi}^{\pi} e^{izt}\lambda(it)\,d\alpha(t). \tag{3.4}$$

In fact, if $\phi(z)$ is the Laplace-Borel transform of $g(z)$, we have

$$\begin{aligned}\phi(re^{i\theta}) &= e^{-i\theta}\int_0^{\infty} g(te^{-i\theta})e^{-rt}\,dt \qquad (r > \pi\,|\sin\theta|)\\ &= e^{-i\theta}\int_0^{\infty} e^{-rt}\,dt\int_{-\pi}^{\pi} e^{ite^{-i\theta}u}\,d\alpha(u)\\ &= e^{-i\theta}\int_{-\pi}^{\pi} d\alpha(u)\int_0^{\infty} e^{-(r-ie^{-i\theta}u)t}\,dt\\ &= e^{-i\theta}\int_{-\pi}^{\pi}\frac{d\alpha(u)}{r-ie^{-i\theta}u} = \int_{-\pi}^{\pi}\frac{d\alpha(u)}{re^{i\theta}-iu}.\end{aligned}$$

Hence

$$\begin{aligned}\frac{1}{2\pi i}\int_C e^{zt}\lambda(t)\phi(t)\,dt &= \frac{1}{2\pi i}\int_C e^{zt}\lambda(t)\,dt\int_{-\pi}^{\pi}\frac{d\alpha(u)}{t-iu}\\ &= \frac{1}{2\pi i}\int_{-\pi}^{\pi} d\alpha(u)\int_C\frac{e^{zt}\lambda(t)}{t-iu}\,dt\\ &= \int_{-\pi}^{\pi} e^{iuz}\lambda(iu)\,d\alpha(u).\end{aligned}$$

LEMMA 8. *If* $\lambda(it)$ *is such that*

$$e^{ist}\lambda(it) = \sum_{-\infty}^{\infty} c_n(s)\,e^{int} \qquad (-\pi \le t \le \pi), \tag{3.5}$$

with $\Sigma|c_n(s)| < \infty$ *for* $s = 0$ *and* $s = s_0$, *where* s_0 *is real and not an integer, then* $\Sigma|c_n(s)|$ *is a bounded function of* s $(-\infty < s < \infty)$.

Let $\lambda(it) = \Sigma c_n(0) e^{int}$; then

$$2\pi\, c_n(0) = \int_{-\pi}^{\pi} e^{-int}\lambda(it)\,dt = F(n),$$

where $F(z) = \int_{-\pi}^{\pi} e^{-izt}\lambda(it)\,dt$. Then, for any real s_0 that is not an integer,

$$2\pi\, c_n(s_0) = \int_{-\pi}^{\pi} e^{-int}e^{is_0t}\lambda(it)\,dt = F(n - s_0).$$

Thus the function $G(z) = F(z/2)$ is entire and of exponential type $\pi/2$, with $\Sigma|G(2n)| < \infty$ and $\Sigma|G(2n - 2s_0)| < \infty$. If we arrange the points $2n$, $2n - 2s_0$ in a single increasing sequence $\{k_n\}$, we have $|k_n - n| < C$ for some fixed C. Consequently [2, p. 197],

$$\int_{-\infty}^{\infty} |G(x)|\,dx \le C_1\Sigma|G(k_n)|$$

with a universal constant C_1, and this in turn implies [2, p. 101] that, for all real s,

$$\Sigma|F(n - s)| \le C_2\Sigma(|F(n)| + |F(n - s_0)|)$$

with a universal C_2. Thus $\Sigma|c_n(s)| \le 2\pi C_2$.

4. We can now prove our main theorem.

THEOREM 1. *If* $f(z)$ *is an entire function of exponential type* π *which is* $o(|x|)$ *on the real axis and bounded on the integers, and if* L *is the linear operator associated with* $\lambda(t)$ *by* (3.2), *with* $\lambda(\pm i\pi) = 0$ *and* $\lambda(t)$ *regular on* $[-i\pi, i\pi]$, *then*

$$|L[f(x)]| \le C \sup|f(n)|, \tag{4.1}$$

with

$$C = \sup_s \Sigma|c_n(s)|, \quad e^{ist}\lambda(it) = \sum_{-\infty}^{\infty} c_n(s)\,e^{int};$$

and no smaller C *can be used in* (4.1), *even if we require that* $f(x) = O(1)$ *instead of* $f(x) = o(|x|)$.

Suppose to begin with that $f(0) = 0$. By Lemma 3 we have $f(z) = \lim_{k\to\infty} g_k(z)$, uniformly in any bounded region, where

$$g_k(z)\ f'(0)\frac{\sin \pi z}{\pi} + z \sin \pi z \sum_{|n|\le k}{}' \frac{(-1)^n f(n)}{n\pi(z - n)}. \tag{4.2}$$

Since $g_k(n) = f(n)$ for $|n| \le k$, and $g_k(n) = 0$ otherwise, Lemma 1 implies that $|g_k(x)| \le A|x|$, where A is independent of k. By Lemmas 5 and 6, this implies

$$L[g_k(z)] \rightarrow L[f(z)].$$

However, each $g_k(z)$ is of the form

$$g_k(z) = \int_{-\pi}^{\pi} e^{izt}\, d\alpha_k(t), \tag{4.4}$$

where α_k is of bounded variation. Indeed, $\sin \pi z$ and $z^{-1} \sin \pi z$ have this form; so does the function $(z - n)^{-1} \sin \pi z = \pm(z - n)^{-1} \sin \pi(z - n)$, whence also the function $(z \sin \pi z)/(z - n) = \sin \pi z + (n \sin \pi z)/(z - n)$. Now, using Lemma 7, we have

$$L[g_k(z)] = \int_{-\pi}^{\pi} e^{izt} \lambda(it)\, d\alpha_k(t). \tag{4.5}$$

Since $e^{ist}\lambda(it)$ is regular on $[-\pi, \pi]$ and vanishes at both ends, its Fourier coefficients are $O(n^{-2})$, and consequently it has an absolutely convergent Fourier series on $(-\pi, \pi)$ for every real s. Let

$$e^{ist} \lambda(it) = \sum_{-\infty}^{\infty} c_n(s)\, e^{int}.$$

Using this in (4.5) and integrating term by term, we obtain

$$L[g_k(x)] = \sum_{-\infty}^{\infty} c_n(s) \int_{-\pi}^{\pi} e^{i(n+x-s)t} d\alpha_k(t)$$

$$= \sum_{-\infty}^{\infty} c_n(s)\, g_k(n + x - s);$$

taking $s = x$, we have

$$L[g_k(s)] = \sum_{-\infty}^{\infty} c_n(s)\, g_k(n) = \sum_{|n| \leq k} c_n(s) f(n).$$

Letting $k \rightarrow \infty$ we have, by Lemma 6,

$$L[f(s)] = \sum_{-\infty}^{\infty} c_n(s) f(n). \tag{4.6}$$

Up to this point we have supposed that $f(0) = 0$. However, (4.6) still holds without this restriction. For, applied to $f(z) - f(0)$, it yields, by the linearity of L,

$$L[f(s)] - f(0)L[1] = \sum_{-\infty}^{\infty} c_n(s) f(n) - f(0) \sum_{-\infty}^{\infty} c_n(s).$$

We have

$$L[1] = \frac{1}{2\pi i} \int_C \lambda(t) e^{zt} t^{-1} dt = \lambda(0),$$

from (3.2); on the other hand, $\Sigma c_n(s) = \lambda(0)$ by (3.5).

Hence we have, by Lemma 8,

$$|L[f(s)]| \leq \sup_s \Sigma |c_n(s)| \cdot \sup_n |f(n)|,$$

and this is (4.1).

To show that no smaller C can be used, let ϵ be positive and find an s and a k such that $\Sigma_{-k}^{k} |c_n(s)| > C - \epsilon$. Then construct the function g(z) of Lemma 1 with $a_n = \operatorname{sgn} c_n(s)$ for $|n| \leq k$, $a_n = 0$ for $|n| > k$. Then $g(x) = O(1)$, and we have

$$L[g(s)] = \sum_{-k}^{k} |c_n(s)| = \sum_{-k}^{k} |c_n(s)| \cdot \sup |g(n)| > C - \epsilon.$$

5. We now show that the condition $\lambda(\pm i\pi) = 0$ is essential.

THEOREM 2. *Let* $\lambda(t)$ *be regular on* $[-i\pi, i\pi]$, *with either* $\lambda(i\pi) = \lambda(-i\pi) \neq 0$, *or else* $\lambda(i\pi) \neq \lambda(-i\pi)$. (i) *There exists no* C *such that* (4.1) *is true for every entire* f *of exponential type* π *which is bounded on the real axis;* (ii) *there exists an entire function of exponential type* π *which is* $o(|x|)$ *on the real axis and bounded on the integers, but with* $L[f(x)]$ *not bounded on the real axis.*

In the first case the function $\lambda(t) - \lambda(i\pi)$, and in the second case the function $\lambda(t) + ibt + c$, with suitable b and c, has the properties of $\lambda(t)$ of Theorem 1. Hence, respectively,

$$|\lambda(D) f(x) - \lambda(i\pi) f(x)| \leq C \sup |f(n)|$$

or

$$|\lambda(D) f(x) + ibf'(x) + cf(x)| \leq C \sup |f(n)|.$$

Thus (4.1) cannot be satisfied unless it is satisfied, respectively, for the operator I (identity) or the operator $ibD + cI$ (where D denotes differentiation). However, it fails for both, as the example $f(z) = \sin \pi z$ shows. This establishes the first assertion of Theorem 2.

The second assertion of Theorem 2 is less immediate. We have to construct, first of all, an entire function f(z) of exponential type π with $f(x) = o(|x|)$, $f(n) = O(1)$, and $f(x) \neq O(1)$. Such a function is given by

$$f(z) = z \sin \pi z \sum_{n=1}^{\infty} \frac{1}{n(z-n)},$$

as was shown by Timan [4]. In fact, we have $f(n) = 0$ for $n \leq 0$, $f(n) = (-1)^n$ for $n > 0$, and $f(x) = o(|x|)$ by Lemma 1. For $z = m + \delta$, $m > 0$, $0 < \delta \leq 1/2$,

$$\pm f(z) = (m + \delta) \sin \pi\delta \sum_{n=1}^{\infty} \frac{1}{n(m - n + \delta)}.$$

Now

$$(m + \delta) \sum_{n=1}^{\infty} \frac{1}{n(m - n + \delta)} = \left(\sum_{n=1}^{m} + \sum_{n=m+1}^{\infty}\right)\left(\frac{1}{n} + \frac{1}{m - n + \delta}\right)$$

$$= 2 \log m + \int_{m+1}^{\infty} \left(\frac{1}{x} - \frac{1}{x - m - \delta}\right) dx + O(1)$$

$$= \log m + O(1),$$

so that $f(x)$ is certainly unbounded. Moreover, for $z = m + \delta$, $0 < \delta < 1/2$,

$$cf(z) + ibf'(z) = \pm(m + \delta)\{c \sin \pi\delta + \pi ib \cos \pi\delta\} \sum_{n=1}^{\infty} \frac{1}{n(z - n)}$$

$$+ ib \sin \pi z \sum_{n=1}^{\infty} \frac{1}{n(z - n)} - ib \sin \pi z \sum_{n=1}^{\infty} \frac{1}{n(z - n)}.$$

Since $c \sin \pi\delta + \pi ib \cos \pi\delta$ vanishes for at most one δ in $0 < \delta < 1/2$, the first term on the right is unbounded, by what we have just proved; the second term is bounded by Lemma 1; and the third term is also bounded. For, with x real and positive, but not an integer,

$$\sum_{n=1}^{\infty} \frac{1}{n(x - n)^2} = \frac{1}{x^2}\left(\sum_{n<x} + \sum_{n>x}\right)\left(\frac{1}{n} - \frac{1}{n - x} + \frac{x}{(n - x)^2}\right)$$

$$= \frac{1}{x^2}\left\{2 \log x + x \cdot O(1) + O(1) + \int_{x+1}^{\infty} \left(\frac{1}{t} - \frac{1}{t - x} + \frac{x}{(t - x)^2}\right) dt\right\}$$

$$= O(1/x).$$

6. We now investigate the extent to which the hypothesis $f(x) = o(|x|)$ can be relaxed to $f(x) = o(|x|^q)$ with $q > 1$. Since $f(x) + Ax^{q-1} \sin \pi x$, with arbitrary A, satisfies the same hypotheses as $f(x)$, in Bernstein's case (for example) we cannot have an inequality $|L[f(x)]| \leq C \sup |f(n)|$ when $q = 2$; for $L[x \sin \pi x]$, although bounded, is not identically zero. As we saw in §2, if $\{f(n)\}$ is bounded and $f(x) = o(|x|^q)$, we have $f(x) = g(x) + P(x) \sin \pi x$, where $g(n) = f(n)$, $g(x) = o(|x|)$, and $P(x)$ has degree at most $q - 1$. Since Theorem 1 applies to $g(x)$, what we can say about $L[f(x)]$ depends on what we can say about $L[P(x) \sin \pi x]$. Now we have

$$P(z)\sin \pi z = \int_C e^{zw}\psi(w)\,dw,$$

where $\psi(w)$ is analytic everywhere, except for poles of the same order (at most q) at $\pm i\pi$. Since

$$L[P(z)\sin \pi z] = \int_C \lambda(w)\psi(w)e^{zw}\,dw$$

and $\lambda(\pm i\pi) = 0$, the operator L converts $P(z)\sin \pi z$ into $Q_1(z)\sin \pi z + Q_2(z)\cos \pi z$, where the degrees of Q_1 and Q_2 are at least one less than that of $P(z)$. If $\lambda(w)$ has zeros of order at least $q - 1$, Q_1 and Q_2 are constants and $L[P(z)\sin \pi z]$ is bounded; if $\lambda(w)$ has zeros of order at least q, $L[P(z)\sin \pi z] \equiv 0$. We then have the following theorem.

THEOREM 3. *If* $f(z)$ *is an entire function of exponential type* π *which is bounded on the integers and is* $o(|x|^q)$ $(q > 1)$ *as* $|x| \to \infty$, *if* $\lambda(t)$ *is regular on* $[-i\pi, i\pi]$ *and has zeros of order at least* p *at* $\pm i\pi$, *and if* L *is the operator defined by* (3.2), *then*

$$|L[f(x)]| \leq C \sup |f(n)| \quad \textit{if } p \geq q;$$

$$L[f(x)] = O(1) \quad \textit{if } p \geq q - 1;$$

and more generally

$$L[f(x)] = O(|x|^k) \quad \textit{if } p \geq q - k - 1 \quad (k = 1, 2, \cdots, [q] - 1).$$

For the Bernstein and Macintyre cases, $\lambda(t) = 2\cosh t/2$ and $\lambda(t) = \pi^2 + t^2$, respectively, and $p = 1$. In particular, the original interference theorem can be generalized to read: If $f(x) = o(|x|^q)$, $q > 1$, and $f(n) = O(1)$, then

$$f(x + 1/2) + f(x - 1/2) = O(|x|^k)$$

if k is any integer not less than $q - 2$.

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