QUASI-PROJECTIVE GEOMETRY OF TWO DIMENSIONS

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1. INTRODUCTION. If the group of automorphisms of projective geometry, that is, the group of collineations, is augmented by adjoining the correlations, the resulting extension is closed under composition, and it again forms a group. It is natural to ask for a formulation of the geometry which has this group as its group of automorphisms, since none of the familiar geometries admits the correlations as automorphisms. The name *quasi-projective geometry* is given to this theory, the two-dimensional version of which is the subject of this paper. The significance of the prefix "quasi" is to be found in the relationship between quasi-projective and projective geometries. The theory of quasi-projective geometry contains the incidence relation of projective geometry. However, it does not contain the property of being a point, since this property is not preserved by the correlations, which by definition are automorphisms of the theory. Thus quasi-projective, projective, affine, and Euclidean geometries form a series of successively richer theories.

The problem considered in this paper may be stated as follows: first, to formulate quasi-projective geometry as a deductive theory and, second, to develop some of the theorems in the elementary part of this theory. The present section will be devoted to a discussion of the nature of quasi-projective goemetry and its role in the hierarchy of geometries. A mathematical subject matter may be treated as a deductive theory or formal language, and also as a system which is characterized by a particular group of automorphisms. Both of these views of quasi-projective geometry are presented here.

It is implicit in the use of symbolic languages in the description of mathematical objects (such as a projective plane) that some, but not in general all, of the properties of the object or model are expressed in the language. This means that there is a process of abstraction involved in the relationship between the language and the object. Consequently, two languages L and L' may be so related that every model of one is a model of the other while L has greater expressive power than L'. The language L' will be said to be *more abstract* than L. Familiar instances of this relationship occur among the well-known geometries: affine and equi-affine; equiform (characterized by the group of similarity transformations) and Euclidian; and affine and Euclidian are pairs of geometries in which the first is more abstract than the second. In contrast, projective geometry as given by the minimal set of axioms of Veblen and Young is not more abstract than real projective geometry.

In order to analyze the relationship between quasi-projective and projective geometries, a set of axioms for the latter will be given.

The formulation of projective geometry employs individual variables u, v, ⋯, y, z which range over points and lines combined; a constant P for the concept, point; and the symbol I which means x is on y when it appears in the expression xIy. It should be recognized that the concept, line, can be introduced into the system by the following definition: Lx for not Px. Although the incompletely symbolized versions of the axioms presented here suffice for the present purpose, in the final analysis 1A to 7A are to designate the given statements completely symbolized in terms of P, of I, of u, ⋯, z, and of the required logical operators.

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1A. If x and y are points, there is a z on x and y.
2A. If x and y are not points, there is a z on x and y.
3A. If x and y are distinct points and x and y are on u and v, then u and v are not distinct.
4A. There are three points x, y, and z for which there is no u on x, y, and z.
5A. If x is not a point, then there are three distinct points u, v, and w which are on x.
6A. If x is on y, then y is on x.
7A. If x is on y, then either x is a point and y is not a point; or x is not a point and y is a point.

It should be noted, incidentally, that 1A to 6A alone are not adequate for ordinary projective geometry, since if T is a transformation which preserves incidence for distinct points, it could not be shown that if xIx, then $(Tx)I(Tx)$. It should be emphasized that these axioms are not essentially different from other sets of axioms for the most general two-dimensional projective geometry. This formulation was chosen on grounds of convenience in the present discussion.

Quasi-projective geometry is obtained from projective geometry by a process of abstraction in which the concept, point, is eliminated. The theorems of projective geometry which do not contain the symbol P constitute the body of theorems of quasi-projective geometry. For example, the statement, for every x there are three distinct elements u, v, and w which are incident to x, can be proved in projective geometry and is, therefore, a theorem of quasi-projective geometry. The first problem to be considered then is that of axiomatizing the set of statements which have been specified as the theorems of quasi-projective geometry.

Further consideration of the language which contains 1A to 7A will show how axioms for quasi-projective geometry can be found. There is no way in quasi-projective geometry to state that x is a point or that x is not a point. However, for two elements x and y it can be asserted that both x and y are points or both are not points: the statement that for some z the elements x and y are both incident to z expresses this relationship without any reference to points.

Although no statement in quasi-projective geometry can have the interpretation, x is a point; the relation between x and y, x has the same dimension as y, (that is, both x and y are points or both are lines), can be expressed without reference to the term, point, in the form: for some u, the elements x and y are incident to u. In projective geometry it can be proved that this relation is reflexive, symmetric, and transitive; and that it leads to exactly two equivalence classes, which are the class of points and the class of lines. The first goal then in axiomatizing quasi-projective geometry is to postulate properties of the incidence relation which will guarantee that "x and y have a common incident" is an equivalence relation between x and y, and that the resulting partition is a dichotomy. The principle of duality for projective geometry implies that with respect to the categories, point and line, only the relative distribution of elements need be considered. From this point of view any statement in projective geometry can be translated into quasi-projective geometry with the aid of the relation "x has the same dimension as y." Thus an adequate set of axioms for quasi-projective geometry can be obtained by translating some set of projective axioms into the more abstract language and supplementing them by axioms from which the elementary properties of the relation "x has the same dimension as y" can be deduced.

The study of geometry by the axiomatic method can be supplemented by a consideration of transformations which are associated with it in a natural way. A one-to-one correspondence with all the individuals or primitive elements of a geometry as domain and range is said to be an automorphism with respect to a formal language for the geometry if it "preserves" all the constants appearing in the language. Here the word, constant, is used in a purely formal sense, referring to one of the two main categories into which the atomic symbols of a formal language are divided; e.g., the line at infinity, the incidence relation, the property or one-term relation of being a point. The automorphisms of a geometry form a group called the Klein group of that geometry. Within the formal language it is possible to impose conditions on an arbitrary binary relation which guarantee that it is a one-to-one correspondence and that as such it "preserves" the constants of the language. Consequently the Klein group has initially an abstract or formal character and becomes concrete only when the whole axiomatic structure is interpreted. The Klein group, of course, is not unique in this respect; the same statement applies to any other undefined or defined term in a formal system.

The language in which 1A to 7A appear has for its Klein group the collineations (that is, the incidence-preserving transformations which map points into points and lines into lines) of projective geometry. On the other hand, the correlations (that is, the incidence-preserving transformations which map points into lines and lines into points) are excluded from the Klein group because they do not preserve the property of being a point. The problem of formulating axiomatically the geometry whose Klein group includes both the collineations and correlations leads again to quasi-projective geometry. Thus two alternatives are open: *quasi-projective geometry may be regarded as the formal language abstracted from projective geometry by the elimination of the term, point; or as the geometry whose Klein group consists of the collineations and correlations.*

2. AXIOMS. The language in which two-dimensional quasi-projective geometry is formulated consists of the following symbols: constant (undefined term) I, individual variables, u, v, $\cdots$, y, z and logical symbols &, E, =, /, $\rightarrow$. The symbol I is a two-term predicate or relation, the expression xIy meaning that x is incident to y. The symbol E is the existential quantifier; e.g., (Ex)xIy means "there exists an x such that x is incident to y." The expression (x)F(x) will be interpreted as "F(x) holds for all x." The universal quantifier, e.g. (x), at the beginning of a formula is usually omitted. The symbols = and & are logical identity and conjunction (and), respectively. The symbol $\rightarrow$ represents material implication; e.g., xIy $\rightarrow$ yIx means if xIy, then yIx. The symbol /, superimposed on any relation symbol, represents the denial of that relation; e.g., $x \not{I} y$ means that x and y are not incident. The type of language employed here is called an applied first-order functional calculus. The usual postulates for such a system will be presupposed.

The following abbreviations are of use in stating the axioms: xIyIz for xIy & yIz (similarly for four or five variables), and $\neq(\cdots)$ for "the elements represented by the variables in parentheses are pairwise distinct."

The five axioms given below characterize the most general two-dimensional quasi-projective geometry which would be admitted as a geometry. These axioms allow some anomalous interpretations such as the Fano plane as well as non-Desarguean geometries and geometries over skew fields.

A1. $x \not{I} x$; i.e., incidence is irreflexive.
A2. (Exyz)[xIu & yIu & zIu & $\neq$(x,y,z)]; i.e., every element of the geometry is incident to at least three other elements.
A3. (x,y,z) $\rightarrow$ (Eu)[xIuIy or yIuIz or xIuIz]; i.e., for every triplet of distinct elements of the geometry there is an element which is incident to at least two of them.

A4. $wIxIyIz \ \& \neq (w,x,y,z) \rightarrow w \not{I} z$; i.e., if four distinct elements are such that each is incident to the next, the first is not incident to the last.

A5. $uIvIxIyIz \ \& \neq (u,v,x,y,z) \rightarrow u \not{I} z$; i.e., if five distinct elements are such that each is incident to the next, then the first is not incident to the last.

The significance of an axiom is sometimes illuminated by a model in which it is not satisfied and in which all of the other axioms are satisfied. The models to be described below incidentally show the consistency and independence of each of the eight axioms for two-dimensional quasi-projective geometry.

M. *A projective plane.* The range of the individual variables includes the points and also the lines of the projective plane. The symbol I is interpreted as incidence in the projective plane. All the axioms are satisfied by M. The consistency of the axioms becomes manifest when a finite projective plane is taken as the model.

M1. *A projective plane in which the incidence relation has been extended so that* x *is incident to* x *for all* x. As before, the range of the individual variables includes the points and the lines of the projective plane. The symbol I is interpreted as the extended incidence relation defined above. A1 is violated and A2 to A5 are satisfied.

M2. *Figure* 1. In this case the individual variables range over the vertices of the hexagon and xIy means that x and y are on the same side of the figure. All the axioms are satisfied except A2.

M3. *Two disjoint projective planes.* The individual variables range over all points and lines in the two planes, and the symbol I denotes incidence in one or the other plane. A3 fails when a point of one plane and a point and a line of the other plane are the x,y, and z of the axiom.

M4. *The points and planes of projective three-space.* In this instance the individual variables range over the points and the planes of the model, and the symbol I means incidence in the projective geometry. A4 fails since there are many planes on two distinct points.

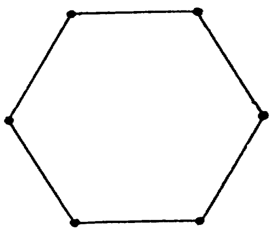

Figure 1.

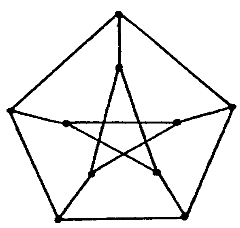

Figure 2.

M5. *Figure* 2. The vertices of the graph constitute the range of the individual variables, and the relation I holds for two vertices when they are on the same one-cell of the graph. A5 is violated by many cycles of five elements which appear in the figure. The graph of figure 2 can be obtained by identifying opposite vertices of a dodecahedron. A study of the figure will show that two elements which are not incident have a common incident, so that A3 is satisfied in a rather trivial way. The validity of the other axioms is easily verified.

This section concludes with the definitions of xDy (read: x and y have the same dimension) and xCy (read: x and y are complementary) which play an important role in the ensuing theory.

DEFINITION 1. xDy *stands for* (Eu)xIuIy.

In the next section it will be shown that D is an equivalence relation which effects a dichotomy of the primitive elements. The two equivalence classes of D are the quasi-projective analogues of the classes of points and lines of projective geometry, although it is not possible to define either of these classes in quasi-projective geometry separately. The role of D is analogous to that of equality of area in affine and equi-affine geometry. Two triangles may be said to be equal in area in affine geometry in which the area of a triangle cannot be defined. On the other hand, area is significant in equi-affine space.

DEFINITION 2. xCy *stands for* x$\not{D}$y & x$\not{I}$y.

In projective terms this means that two elements are regarded as complementary if one is a point, the other is a line, and the line is not incident to the point. These two terms are fundamental to the theory developed in the next section.

3. ELEMENTARY THEOREMS. The exposition of quasi-projective geometry begins with the derivation of a preliminary result, the symmetry of the incidence relation. The next step is to show that D is an equivalence relation whose partition is a dichotomy. Then follows a series of theorems which are quasi-projective analogues of familiar theorems of projective geometry.

The following lemma can easily be established by reference to the indicated axioms:

L1. (Ex & y, x $\neq$ y)[xIuIy & yIuIx].

Proof: For every u there are distinct elements x, y, z such that xIu & yIu & zIu (A2). For some element w, xIwIy or yIwIz or xIwIz (A3). Any one of these conditions would involve a violation of A4 if w and u were not equal. Therefore, uIy or uIz; suppose that uIz holds. A repetition of the argument with y, z, x in that order in the application of A3 yields uIx or uIz. Suppose that uIx occurs. Then xIuIy & yIuIx. Q. E. D.

The first theorem is an analogue, for three elements, of the fourth and fifth axioms.

T1. xIyIz $\rightarrow$ x$\not{I}$z & x$\not{I}$x.

Proof: If $\neq$ (x, y, z) fails, the theorem follows immediately (A1). The proof is completed by a contrapositive argument in two cases from the assumptions that xIyIz & (xIz or zIx) & $\neq$ (x, y, z).

Case 1. (xIy $\overline{\text{or}}$ yIx) & (yIz $\overline{\text{or}}$ zIy) & (xIz or zIx), where $\overline{\text{or}}$ is exclusive disjunction (i.e., one or the other alternative, but not both).
There exist elements u, v, w which satisfy the conditions (a) uIxIu & vIyIv & wIzIw and (b) $\neq$ (u, y, z) & $\neq$ (v, x, z) & $\neq$ (w, x, y) (L1). Then, $\neq$ (u, v, w, x, y, z) follows (b, A1, A4). There is an element r which satisfies the condition (c) xIrIy or yIrIz or xIrIz. Furthermore, $\neq$ (r, u, v, w, x, y, z) holds (A1, A4, A5). Therefore, each of the alternatives in (c) is contradictory (A5).

Case 2. xIyIzIyIxIz.
There exist elements u, v, w satisfying the conditions (a) uIx & vIy & wIz and (b) $\neq$ (u, y, z) & $\neq$ (v, x, z) & $\neq$ (w, x, y) (A2). Then, $\neq$ (u, v, w, x, y, z) holds (b, A1, A4). There is an element r which satisfies the condition (c). zIrIy or yIrIx or zIrIx (A3). Furthermore, $\neq$ (r, u, v, w, x, y, z) holds (A1, A4, A5). Therefore, each alternative of (c) is contradictory (A5). Q. E. D.

The symmetry of I can now be established.

T2. $uIx \rightarrow xIu$.

Proof: There are elements y, v such that yIx & vIx and $\neq (u, v, y)$ (A2). Then, $\neq (u, v, x, y)$ holds (A1). There is an element z such that zIy & $z \neq x$ (A2). It follows that $\neq (x, y, z, u, v)$ is valid (A1, T1). There exists an element w such that $vIwIu$ or $uIwIz$ or $vIwIz$ (A3). Moreover, $\neq (w, y, z, u, v)$ holds (T1, A1). Consequently, $uIwIz$ is impossible and $vIwIz$ is likewise impossible (A5). If $x \neq w$, then $vIwIuIx$ & vIx & $\neq (u, v, w, x)$ would hold. Since this is impossible (A5), $x = w$ and xIu. Q. E. D.

The symmetry of I is so closely interwoven with the development which follows that T2 will often be tacitly assumed in the course of the proof.

The remark which follows is a strengthened version of A5, in that it shows that distinctness is not required.

R1. $uIvIxIyIz \rightarrow u\not{I}z$ (A1, T1).

T3. $uIvIwIxIy \rightarrow (Ez)yIzIu$.

Proof: $\neq (u, v, y)$ holds (A1, T1). Therefore, for some element z, $yIzIu$ or $uIzIv$ or $yIzIv$ (A3). Only $yIzIu$ is possible (T1, R1). Q. E. D.

From the preceding results it is easy to deduce that D is an equivalence relation.

T4. *The relation* D *is reflexive, symmetric, and transitive.*

Proof: $(Ex)xIu$ (A2). Therefore D is reflexive; i.e., $(Ex)uIxIu$ (T2). Also, D is symmetric; i.e., $uIvIwIxIy \rightarrow (Ez)yIzIu$ (T3). Q. E. D.

T5. $(Eux)(z)[u\not{D}x$ & $(zDu$ or $zDx)]$.

Proof: By means of the laws of logic which were presupposed, it may be inferred that $(Eux)xIu$ (A2). Therefore $u\not{D}x$ (Definition of D). Also, zDu or zDx (A3, T1). Q. E. D.

The customary differentiation of points from lines in geometry is reflected in the concepts and terminology which have evolved in this branch of mathematics; e.g., quadrangle, quadrilateral, collinearity, concurrence, etc. In projective geometry various concepts and relations involve some elements which are specified to be points and others which are specified to be lines, although the duality of points and lines makes it apparent that such a distinction is not essential. Three relations which are especially suitable to formulation in quasi-projective geometry will be defined.

DEFINITION 3. Dep.(x, y, z) *stands for* $(Eu)[uIx$ & uIy & $uIz]$; that is, (x, y, z) *are dependent.*

DEFINITION 4. Ind.(x, y, z) *stands for* $(u)[u\not{I}x$ or $u\not{I}y$ or $u\not{I}z]$ & xDy & yDz; that is, (x, y, z) *are independent.* Note that (x, y, z) are neither dependent nor independent if two of them are not in the same equivalence class of D.

Harmonic sets are usually defined as quadruplets of points (or perhaps lines) which are related in a certain way to a complete quadrangle (or quadrilateral).

The harmonic relationship can best be described without the use of a formal definition by reference to eight elements which satisfy the following condition: $sItIuIvIwIxIyIzIs$ & $\neq (s, t, u, v, w, x, y, z)$. If $xImIw$ & $tInIx$ & $uIpIy$ & $vIqIz$, then (m, n, p, q) can be defined as an harmonic set.

The statement "there exist three distinct independent elements in the quasi-projective plane," which is the analogue of the projective axiom 4A, is an immediate logical consequence of the next theorem.

T6. (Eyz)[Ind.(x, y, z)].

Proof: There exist elements u, v, for which uIxIv & $\neq$ (x, u, v) (A2, A1). Similarly, there exist elements y, z, such that yIu & $\neq$ (u, x, y) & zIv & $\neq$ (x, v, z) (A2, A1). If, for some w, xIw & yIw & zIw, then w = u & w = v (A4) which is contradicted by the distinctness of u and v. Therefore every w is not incident to one of the elements x, y, z. Since x, y, z are in the same equivalence class of D (Def. 1, T3), they are independent. Q. E. D.

The remaining theorems in this section are designed to establish for the relation C some of the properties of the primitive, I.

T7: x$\not{C}$x.

Proof: xDx (T4). Therefore, x$\not{C}$x (Def. 2). Q. E. D.

DEFINITION 5. (w, x, y, z) *is a quadrifigure stands for* Ind.(x, y, z) & Ind.(w, y, z) & Ind.(w, y, z) & Ind.(w, x, z) & Ind.(w, x, y).

L2. (Ew, x, y)[(w, x, y, z) *is a quadrifigure*].

Proof: There are elements x, y for which Ind.(x, y, z) holds (T6). There are elements s, t, r, such that xItIy & yIrIz & xIsIz (Def. 5, T3). There is an element u such that uIx & $\neq$ (u, s. t) (A1). Only one element is incident to both r and u (A4). Therefore there is an element w which is incident to u and not incident to r (A2). Then, wDx & wDy & wDz (T4). Finally, Ind.(w, x, y) & Ind.(w, y, z) & Ind.(w, x, z) (A4). Also, (w, x, y, z) (A4). Q. E. D.

The following remark is an immediate consequence of the definition of independence:

R2. Ind.(x, y, z) & yIuIz $\rightarrow$ xCu.

The next theorem should be compared with A2.

T8. (Exyz)[xCu & yCu & zCu & $\neq$ (x, y, z)].

Proof: There exist r, s, t for which (r, s, t, u) is a quadrifigure (L2). For some elements x, y, z, rIzIsIxItIyIr (Def. 4, T3). Then $\neq$ (x, y, z) holds (A4). Since (u, x, y, z) is a quadrifigure, xCu & yCu & zCu (Def. 5, R2). Q. E. D.

A characteristic difference between projective geometry and Boolean Algebra is that two elements of a Boolean algebra cannot have the same complement while every pair of points (or lines) have a common complement. Although the existence of a common complement is sufficient for the deduction of T10 below, a stronger result is obtained as

T9. xDy$\rightarrow$(Eu,v)[uCx & uCy & vCx & vCy & u $\neq$ v].

Proof: For some w, z xIw & yIw & zIw & z $\neq$ x & z $\neq$ y (Def. 1, A2). There are elements u, v such that uIx & vIz & $\neq$ (u, v, w) (A2). Obviously, (x, y) and (u, v) are in different equivalence classes of D. Finally, x$\not{I}$u & x$\not{I}$v & y$\not{I}$u & y$\not{I}$v (A4). Q. E. D.

The substitution of C for I in A3 leads to

T10. (Eu)[(xCu & yCu) or (yCu & zCu) or (xCu & zCu)].

Proof: The theorem follows easily by the use of A3 and T9.

Again, the substitution of C for I in A5 leads to

T11. uCv & vCx & xCy & yCz $\rightarrow$ u$\not{C}$z.

Proof: The theorem follows easily from T5 and Def. 2.

T12. xIy if and only if x₵y & not (Eu)[xCuCy].

Proof: x₵y if and only if xIy or (Eu)[xIuIy] (Def. 2). x₵y if and only if xIy or (Eu)[xCuCy] (T11). Therefore, xIy if and only if x₵y & not (Eu)[xCuCy]. Q. E. D.

Note the resemblance between T12 and Definition 2. Definition 2, expressed without the abbreviation D, is converted into Theorem 12 by the interchange of C and I. Axioms for quasi-projective geometry can be introduced with the use of only C as a primitive or undefined term. In this case T12 can serve as a definition of incidence. In such a system T11 is the key to the definition of the equivalence relation which divides the elements into the two classes in each of which all the elements have the same dimension.

The theorems involving "C" lead to a principle of deduction from which new theorems can be generated. Let T represent any theorem in quasi-projective geometry, expressed without the use of abbreviations, and let T* represent the result of replacing all occurrences of "I" in T by "C". Then, if T can be deduced from A1, A2, A3 and A5 without the use of A4, it is known that T* is also a theorem in quasi-projective geometry. The principle can easily be extended to include the case where T can be deduced from the same four axioms together with other statements involving I and valid also for C. For example, since both I and C are symmetric, T* is a theorem if T can be deduced from the axioms other than A4 plus T2.

4. CONCLUSION. Quasi-projective geometry as presented in Section 2 is aimed at only the most basic aspects of the theory, and consequently it includes many uncommon geometries along with the familiar ones. However, with the theory developed here, the quasi-projective counterpart of any special projective geometry can be constructed. The quasi-projective counterpart of any statement expressed in the language of projective geometry can be obtained by a simple procedure. The variables for elements are those used for points and for lines in the projective statement, and I is placed between two variables if it was between them in the original statement. Finally, if two variables represent two points or two lines, the existence of a third element incident to both of them is asserted. Axioms for the corresponding quasi-projective geometries are obtained if this process is applied to projective axioms such as those concerned with the Fano configuration, the Desargues configuration, the Pappus configuration, the density of points on a line, continuity, etc. The point of view of quasi-projective geometry is advantageous for the study of the above-mentioned configurations. However, since the theory of self-dual (or more precisely, symmetric) configurations of quasi-projective geometry has been developed in terms of linear graphs, its presentation is reserved for a later paper.

The exact relationship between quasi-projective and projective geometry may be formulated along the following lines. The sub-group of automorphisms of quasi-projective geometry which have one equivalence class of the equal-dimension relation as an invariant must be the group of automorphisms of projective geometry. If this is the case, projective geometry should be logically equivalent to A5 plus B, where B denotes certain additional axioms which will make a new primitive P the name of an equivalence class of the "equal-dimension" relation. The two theories already have the same primitives or constants, and it is clear, from an examination of the appropriate theorems in Section 3, that each axiom of one theory is a theorem of the other. Note the analogy between this situation and that which occurs in the comparison between projective and affine geometry.

The study of quasi-projective geometry is only part of a larger investigation in metamathematics. The terms "logical equivalence" and "abstraction," as applied to formal mathematical theories, require explication. One of the procedures being followed is to develop in some detail certain examples for what light they will shed on the general metamathematical theory.

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