A PROJECTION OPERATOR ON HARMONIC MAPPINGS

by

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1. INTRODUCTION. Throughout this paper, D will denote a simply connected domain in the xy-plane. Let $u = u(x, y)$ and $v = v(x, y)$ be a pair of real-valued functions with continuous second partial derivatives on D; and let w denote the mapping of D into the uv-plane which is defined by these functions. The Jacobian matrix of w,

$$J = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix},$$

satisfies the matrix differential equation

$$J_x e_2 - J_y e_1 = 0, \tag{1}$$

where e_1 and e_2 are the unit vectors $\binom{1}{0}$ and $\binom{0}{1}$, and where the subscripts indicate that each element of J has been replaced by the corresponding partial derivative. If w is a harmonic mapping, then the further equation

$$J_y e_2 + J_x e_1 = 0$$

is also satisfied. With the notation $\xi = Je_2$ and $\eta = Je_1$, equations (1) and (2) can be written as the vector differential equations

$$\xi_x = \eta_y, \ \xi_y = -\eta_x,$$

formally similar to the Cauchy-Riemann equations.

It should be noted that if a two-by-two matrix J of differentiable functions satisfies equation (1), it is the Jacobian matrix of a mapping.

2. DEFINITION OF PROJECTION OPERATOR. Henceforth, J will denote the Jacobian matrix of a harmonic mapping w with the components u and v. The operator P will be defined by the relation

$$\hat{J} = P[J] = (\hat{J} + KJK^{-1})/2, \text{ where } K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Received by the editors December 6, 1953.

The matrix J is itself the Jacobian matrix of some mapping; for

$$\hat{J}_x e_2 - \hat{J}_y e_1 = [(J_x e_2 - J_y e_1) + K(J_x e_1 + J_y e_2)]/2,$$

and the right member is zero by equations (1) and (2) .

Since

$$P[P[J]] = [(J+KJK^{-1}) + K(J+KJK^{-1})K^{-1}]/4 = P[J]$$

the operator P has the projection property. A simple computation shows that a matrix function is invariant under P if and only if it is of the form

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

or, in terms of the elements of J, if and only if

$$u_x = v_y \text{ and } u_y = -v_x .$$

Thus the Jacobian matrix of a harmonic mapping is invariant under P if and only if the mapping function is analytic.

In detail, the projection operator P can be written in the form

$$P[J] = \frac{1}{2}\begin{pmatrix} u_x + v_y & u_y - v_x \\ v_x - u_y & v_y + u_x \end{pmatrix};$$

by integration, an analytic function $W = U + iV$ is determined, modulo an additive complex constant, by the relations

$$U = \frac{1}{2}\int (u_x + v_y)\,dx + (u_y - v_x)\,dy ,$$

$$V = \frac{1}{2}\int (v_x - u_y)\,dx + (v_y + u_x)\,dy .$$

3. AN INEQUALITY. Let $|J|$ and $\|J\|$ denote the determinant and modulus of J, respectively, so that

$$|J| = u_x v_y - u_y v_x \text{ and } \|J\|^2 = u_x^2 + u_y^2 + v_x^2 + v_y^2 .$$

If $|P(J)|$ is the determinant of $P[J]$, a simple computation gives the result

$$|P[J]| - |J| = (\|J\|^2 - 2|J|)/4 .$$

The right-hand side is always non-negative, and therefore

$$|P[J]| \geq |J| .$$

In particular, it follows that

$$\iint_D |P[J]|\, dxdy \geq \iint_D |J|\, dxdy$$

for every domain D over which the integrals exist. This establishes the following extremal property of analytic functions.

For each harmonic mapping w that projects into an analytic function W, let

$$I(w) = \iint_D |J|\, dxdy .$$

Then $I(w) \leq I(W)$.

4. SOME TOPOLOGICAL RESULTS. Let t be a real number, and let

$$\hat{J}_t = (J + t\, KJK^{-1})/2 . \tag{3}$$

A simple computation shows that $\hat{J}_t$ satisfies equations (1) and (2); for each t, $\hat{J}_t$ is therefore the Jacobian matrix of a harmonic mapping. In particular, for $t = 1$, $\hat{J} = P[J]$; in other words, $\hat{J}_1$ is the Jacobian matrix of an analytic function. Equation (3) yields the relation

$$|\hat{J}_t| = [(1 + t^2)\, |J| + t\, \|J\|^2]/4 . \tag{4}$$

Let $\{\hat{W}_t\}$ be the class of harmonic mappings which are the integrals of $\{\hat{J}_t\}$, where the constants of integration are chosen so that, for some point p_o in D, $\hat{W}_t(p_o)$ is independent of t. Such a family $\{\hat{W}_t\}$ will be called a normalized integral of the family $\{\hat{J}_t\}$. The proof of the fact that each $\hat{W}_t$ is uniformly continuous in t on every compact subset is simple and is left to the reader.

A theorem of G.S. Young and the author which will appear elsewhere states that if a mapping has a non-negative Jacobian which is zero only where it has rank zero, then the mapping is quasi-interior; see page 9 in [1]. A theorem of Whyburn ([1] Theorem 7.2, p. 10,) states that if a sequence of quasi-interior mappings converges uniformly on every compact subset of D, then the limit mapping is quasi-interior on D. By the first theorem, W_t is quasi-interior for every $t \neq 0$ if $|J| \geq 0$. By the second theorem, $W_o = w$ is then quasi-interior. Therefore, we have the following result.

THEOREM. *If* w *is a harmonic mapping such that* $|J| \geq 0$ *on a domain* D, *then* w *is quasi-interior in* D.

It follows immediately from the quasi-interiority of w that on every compact subset S of D the maximum of $|w| = u^2 + v^2$ is achieved on the boundary of S. It would be interesting to know if the condition $|J(w)| \geq 0$ and $|J(w)| \not\equiv 0$ on D is sufficient for lightness of w and hence for interiority of w.

If w is a harmonic mapping such that $|J(w)| \geq 0$ and the rank of $J(w)$ is nowhere zero on D, then on every compact subset of D the mapping w is the uniform limit of local homeomorphisms. This follows directly from the fact that if the rank of $J(w)$ is nowhere zero, then $\|J\|$ is nowhere zero, and therefore one sees from (4) that $|\hat{J}_t(w)| > 0$ if $t \neq 0$. Thus each mapping $\hat{W}_t$ $(t \neq 0)$ is a local homeomorphism on D.

Finally, if w is light, then w and W are very closely related topologically. First of all, there exists over every compact subset of D a uniformly continuous deformation which keeps fixed the points at which $\hat{J}_t$ has rank zero. At $t = 1$, these points are the zeros of the complex derivatives of W. Thus at every point p of D there exists a neighborhood U of P over which w and W are homeomorphic to the same power of z.

BIBLIOGRAPHY

[1] G. T. Whyburn. Open mappings on locally compact spaces. Mem. Amer. Math. Soc. no. 1 (1950).

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