ON BIORTHOGONAL SYSTEMS

by

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1. The primary purpose of this paper is a didactic one: we want to present the theory of biorthogonal systems in a more general and systematic way than it has been done before; most of our results are easy generalizations of known theorems, especially of recent work on bases in Banach spaces (see [2], [8], [10], [12], [13], [15]). The only special feature of our treatment consists in laying more emphasis on the weak topologies than is usually done, and this proves to be the unifying principle of the theory(1).

2. Let F and G be two vector spaces (over the real or the complex number field) in duality [6]. A system consisting of a family $(a_\lambda)_{\lambda \in L}$ of points of F and a family $(b_\lambda)_{\lambda \in L}$ of points of G is said to constitute a biorthogonal system if $\langle a_\lambda, b_\lambda \rangle = 1$ for all $\lambda \in L$ and $\langle a_\lambda, b_\mu \rangle = 0$ for $\lambda \neq \mu$.

PROPOSITION 1. *Let $(a_\lambda)_{\lambda \in L}$ be a family of points of* F. *In order that there exist in* G *a family $(b_\lambda)_{\lambda \in L}$ forming with (a_λ) a biorthogonal system, it is necessary and sufficient that (a_λ) be topologically free for the topology $\sigma(F,G)$* (that is, for every $\lambda \in L$, a_λ does not belong to the closed subspace generated by the a_μ of index $\mu \neq \lambda$; see [4, p. 24]). *Moreover if (b_λ) and (b'_λ) are two such families, then $b'_\lambda - b_\lambda \in A^o$, where A is the closed subspace of F generated by the family (a_λ); in particular, $b'_\lambda = b_\lambda$ for all $\lambda \in L$ if and only if $A = F$.*

The proof is an easy application of Hahn-Banach's theorem, and will therefore be omitted.

3. A biorthogonal system $(c_\mu)_{\mu \in M}$, $(d_\mu)_{\mu \in M}$ ($c_\mu \in F$, $d_\mu \in G$) is said to be an extension of a biorthogonal system $(a_\lambda)_{\lambda \in L}$, $(b_\lambda)_{\lambda \in L}$ if $L \subset M$, and $a_\lambda = c_\lambda$, $b_\lambda = d_\lambda$ for $\lambda \in L$. A biorthogonal system (a_λ), (b_λ) is maximal if it has no proper extension. From Zorn's lemma it follows immediately that

PROPOSITION 2. *Every biorthogonal system has a maximal extension.*

Maximal biorthogonal systems are characterized by the following property:

PROPOSITION 3. *Let (a_λ), (b_λ) be a biorthogonal system,* A *the closed subspace* (for $\sigma(F,G)$) *generated by (a_λ),* B *the closed subspace*

(1) We are following the terminology and notations of [4], [5], [6] and [7].

(for $\sigma(G, F)$) generated by (b_λ). The three following conditions are equivalent: 1° the system (a_λ), (b_λ) is maximal; 2° $B^o \subset A$; 3° $A^o \subset B$.

The two last conditions are obviously equivalent since $B^{oo} = B$. To say that (a_λ), (b_λ) is not maximal means that there exist elements $a \in F$, $b \in G$ such that a is orthogonal to B, b orthogonal to A, and $<a, b> = 1$. This implies that B^o, containing a, is not contained in A (prop. 1). Conversely, if there exists an element $a \in B^o$ which is not in A, there exists by Hahn-Banach's theorem, a hyperplane H of equation $<x, b> = 0$ which contains A and does not contain a; then $b \in A^o$, and $<a, b> \neq 0$; multiplying b by a convenient scalar gives $<a, b> = 1$.

It is easy to give examples of maximal biorthogonal systems in a Hilbert space $F = G$, such that A and B both have orthogonal supplementary subspaces of infinite dimension [11].

4. A t-space (French: "espace tonnelé") is a locally convex Hausdorff space E having the following property [5]: in the dual E' of E, every subset bounded for $\sigma(E', E)$ is equicontinuous (hence strongly bounded and relatively compact for $\sigma(E', E)$). We speak then of bounded sets in E' without qualification; in E (as generally in any locally convex space [14, p. 198]) bounded sets for $\sigma(E, E')$ are also bounded for the original topology of E, so no qualification is needed. The strong topology on E' has as a fundamental system of neighborhoods the polars B^o of bounded sets in E, and as fundamental system of bounded sets the polars V^o of neighborhoods in E. It follows that on the dual E'' of E', the strong topology induces on E the original topology on E, and every bounded subset of E is relatively compact in E'' for the topology $\sigma(E'', E')$.

We recall that (F)-spaces (and in particular Banach spaces) and (LF)-spaces [7] are special cases of t-spaces.

5. From now on, we shall always suppose that F is a t-space E and G its dual E'. We are going to consider especially biorthogonal denumerable systems (a_n), (b_n), with $a_n \in E$, $b_n \in E'$. The closed subspace of E (either for the strong or the weak topology $\sigma(E, E')$) generated by (a_n) will be noted A.

PROPOSITION 4. The following properties are equivalent: 1) for every $x \in E$ and every $x' \in E'$, the sums $\sum_{n=0}^{N} <a_n, x'><x, b_n>$ are bounded (in the field of scalars); 2) for every $x' \in E'$, the sums $\sum_{n=0}^{N} <a_n, x'>b_n$ are bounded in E'; 3) for every $x \in E$, the sums $\sum_{n=0}^{N} <x, b_n>a_n$ are bounded in E.

It is obvious that 2) and 3) imply 1). The fact that 1) implies 3) follows from the equivalence of weak and strong boundedness in E; the fact that 1) implies 2), from the equivalence of strong boundedness and boundedness for $\sigma(E', E)$ in E'.

We shall say that a biorthogonal system $(a_n), (b_n)$ is <u>quasi-regular</u> if it possesses any one (hence all three) of the properties stated in prop. 4.

PROPOSITION 5. <u>If the biorthogonal system</u> $(a_n), (b_n)$ <u>is quasi-regular, for every</u> $x \in A$, <u>the series of general term</u> $\langle x, b_n\rangle a_n$ <u>converges strongly to</u> x. <u>If</u> B_1 <u>is the strongly closed subspace of</u> E' <u>generated by</u> (b_n), <u>for every</u> $x' \in B_1$, <u>the series of general term</u> $\langle a_n, x'\rangle b_n$ <u>converges strongly to</u> x'.

The proofs are the same as for Banach spaces [2, p. 107]. Consider $s_N(x) = \sum_{n=0}^{N} \langle x, b_n\rangle a_n$; s_N is a continuous linear mapping of E into E, and the assumption implies that in the space $\mathcal{L}(E)$ of linear continuous mappings of E into itself, the sequence $(s_N)_{N=1,2,\ldots}$ is bounded for the topology of pointwise convergence, hence equicontinuous [5, p. 7 th. 1]. As $s_N(x)$ converges towards x (for the strong topology of E) for every x which is a finite linear combination of the a_n, it converges also to x for any x in the closure of the set of these linear combinations [3, p. 29, prop. 3]. Similarly, consider $s'_N(x') = \sum_{n=0}^{N} \langle a_n, x'\rangle b_n$; s'_N is a weakly continuous linear mapping of E' into itself (for the topology $\sigma(E', E)$), and for every $x' \in E'$, the sequence $(s'_N(x'))$ is bounded in E'. It follows that the sequence (s'_N) is equicontinuous when E' is given the strong topology [5, p. 13, prop. 8]; the end of the proof is similar.

PROPOSITION 6 (see [15, p. 795, lemmas 1 and 2]). <u>Let</u> $(a_n), (b_n)$ <u>be a quasi-regular biorthogonal system</u>.

1° <u>For every sequence</u> (λ_n) <u>of scalars such that the sums</u> $\sum_{n=0}^{N} \lambda_n b_n$ <u>are bounded in</u> E', <u>there exists</u> $x' \in E'$ <u>such that</u> $\langle a_n, x'\rangle = \lambda_n$ <u>for every</u> n.

2° <u>For every</u> $x'' \in E''$, <u>the sums</u> $\sum_{n=0}^{N} \langle x'', b_n\rangle a_n$ <u>are bounded in</u> E.

3° <u>Conversely, for every sequence</u> (μ_n) <u>of scalars such that the sums</u> $\sum_{n=0}^{N} \mu_n a_n$ <u>are bounded in</u> E, <u>there exists</u> $x'' \in E''$ <u>such that</u> $\langle x'', b_n\rangle = \mu_n$ <u>for every</u> n.

1° The sequence of the sums $\sum_{n=0}^{N} \lambda_n b_n$ being bounded in E', has a cluster value x' for the weak topology $\sigma(E', E)$; as $\langle a_n, \sum_{k=0}^{N} \lambda_k b_k\rangle = \lambda_n$ for every $N \geq n$, this implies that $\langle a_n, x'\rangle = \lambda_n$ for every n.

2° For every $x' \in E'$,

$$\langle \sum_{n=0}^{N} \langle x'', b_n\rangle a_n, x'\rangle = \langle x'', \sum_{n=0}^{N} \langle a_n, x'\rangle b_n\rangle$$

is bounded, since the sums $\sum_{n=0}^{N} \langle a_n, x'\rangle b_n$ are bounded in E', and x''

is strongly continuous in E' This shows that the sums $\Sigma_{n=0}^{N} \langle x'', b_n \rangle a_n$ are bounded in E.

3^o The proof is the same as in 1^o, using the fact that a bounded set in E is relatively compact in E'' for $\sigma(E'', E')$.

PROPOSITION 7. The biorthogonal system $(a_n), (b_n)$ being quasi-regular, let ϕ be the natural homomorphism of E' onto E'/A^o; for every $x' \in E'$, the series of general term $\langle a_n, x' \rangle \phi(b_n)$ converges to $\phi(x')$ for the weak topology $\sigma(E'/A^o, A)$. Similarly, let ψ be the natural homomorphism of E'' onto E''/B_1^o (B_1^o being the subspace orthogonal to B_1 in E''); for every $x \in E$, the series of general term $\langle x, b_n \rangle \psi(a_n)$ converges to $\psi(x)$ for the topology $\sigma(E''/B_1^o, B_1)$.

The sequence $(s'_N(x'))$ is bounded in E', hence relatively compact for $\sigma(E', E)$, and therefore the sequence $(\phi(s'_N(x'))$ has at least one cluster value $\phi(y')$ for the topology $\sigma(E'/A^o, A)$. But for any cluster value $\phi(z')$ of that sequence, $\langle a_n, \phi(s'_N(x')) - \phi(z') \rangle = \langle a_n, s'_N(x') - z' \rangle$ tends to 0 when N runs through a convenient increasing sequence of integers. As soon as $N \geq n$, $\langle a_n, s'_N(x') - z' \rangle = \langle a_n, x' - z' \rangle$, hence $\langle a_n, x' - z' \rangle = 0$ for all n, which proves that $\phi(z') = \phi(x')$. The relatively compact sequence $(\phi(s'_N(x'))$, having only one cluster value $\phi(x')$, converges therefore to $\phi(x')$ for the topology $\sigma(E'/A^o, A)$. A similar proof applies to the second part of the proposition, using the fact that the sequence $(s_N(x))$, being bounded in E, is relatively compact for $\sigma(E'', E')$ in E''.

COROLLARY. If B is the closed subspace of E', for the weak topology $\sigma(E', E)$, generated by (b_n), then, if the system $(a_n), (b_n)$ is quasi-regular, one has $E' = B + A^o$, hence $A \cap B^o = \{0\}$ in E.

The preceding argument shows that, for any $x' \in E'$, the sequence $(s'_N(x'))$ has a cluster value y' in E' for $\sigma(E', E)$, and that $\phi(y') = \phi(x')$; in other words $y' - x' \in A^o$. On the other hand, every $s'_N(x')$ belongs to B, hence also y', which proves our corollary.

7. Example 1. Let $E = E'$ be a separable Hilbert space, (e_n) an orthonormal basis for E. Let us take $a_n = e_n - e_{n+1}$ for all $n \geq 0$, $b_n = e_0 + e_1 + \cdots + e_n$ for all $n \geq 0$; then it is immediately verified that $(a_n), (b_n)$ is a biorthogonal system such that $A = B_1 = E$. Let us show that that system, although maximal, is not quasi-regular. For $x = \Sigma_{n=0}^{\infty} \xi_n e_n$, $x' = \Sigma_{n=0}^{\infty} \eta_n e_n$, it is readily verified that $\Sigma_{n=0}^{N} \langle a_n, x' \rangle \langle x, b_n \rangle = \Sigma_{n=0}^{N} \xi_n \eta_n - \eta_{N+1} (\xi_0 + \xi_1 + \cdots + \xi_N)$; as $\Sigma_{n=0}^{N} \xi_n \eta_n$ converges to $\langle x, x' \rangle$, we have only to choose x and x' such that the sequence of real numbers $\eta_{N+1} (\xi_0 + \xi_1 + \cdots + \xi_N)$ is unbounded. Take $\xi_n = 1/(n+1)$, so that $\Sigma_{n=0}^{\infty} \xi_n^2 < +\infty$, and $\xi_0 + \xi_1 + \cdots + \xi_N \sim \log N$; on the other hand,

take $\eta_{N+1} = 0$ except for $N+1 = 2^{2^n}$, when $\eta_{N+1} = 1/(\log\log N) \sim 1/(n \log 2)$. Then $\sum_{n=0}^{\infty} \eta_n^2 < +\infty$, and $\eta_{N+1}(\xi_0 + \xi_1 + \cdots + \xi_N) \sim \log N/\log\log N$ for $N = 2^{2^n}$, which satisfies our conditions.

Example 2 (see [8, p. 188]). In all spaces whose elements are sequences of real numbers, e_n will denote the sequence having all terms equal to 0, except for the n-th term equal to 1. Let E be the space (ℓ^1) of Banach, E' its dual (m), and take $a_n = e_n - e_{n+1}$ $(n \geq 0)$, $b_n = e_0 + e_1 + \cdots + e_n$ $(n \geq 0)$. This is a maximal biorthogonal system, since $B = E'$. For $x = \sum_{n=0}^{\infty} \xi_n e_n \in E$, $x' = \sum_{n=0}^{\infty} \eta_n e_n \in E'$, we have, as in example 1, $\sum_{n=0}^{N} \langle x, b_n\rangle\langle a_n, x'\rangle = \sum_{n=0}^{N} \xi_n \eta_n - \eta_{N+1}(\xi_0 + \xi_1 + \cdots + \xi_N)$. The series of general term ξ_n converges absolutely, and the sequence (η_n) is bounded, hence the series of general term $\xi_n\eta_n$ converges, and the sums $\sum_{n=0}^{N} \langle x, b_n\rangle\langle a_n, x'\rangle$ are bounded; this shows that the system (a_n), (b_n) is quasi-regular. However, it is easy to give examples of vectors x, x' for which the series of general term $\langle x, b_n\rangle\langle a_n, x'\rangle$ does not converge: take (ξ_n) such that the (absolutely convergent) series $\sum_{n=0}^{\infty} \xi_n$ has a sum $\neq 0$, and take for (η_n) a bounded sequence having no limit, for instance $\eta_n = (-1)^n$.

8. PROPOSITION 8. The following properties are equivalent:

1) for every $x \in E$ and every $x' \in E$, the series of general term $\langle x, b_n\rangle\langle a_n, x'\rangle$ converges in the field of scalars;

2) for every $x' \in E'$, the series of general term $\langle a_n, x'\rangle b_n$ converges to $p(x') \in E'$ for the weak topology $\sigma(E', E)$;

3) for every $x \in E$, the series of general term $\langle x, b_n\rangle a_n$ converges to $q(x) \in E''$ for the weak topology $\sigma(E'', E')$.

It is clear that 2) and 3) imply 1). The fact that 1) implies 2) follows from the fact that the sequence $(s'_N(x'))$ is a Cauchy sequence for the topology $\sigma(E', E)$, hence converges because it is relatively compact; a similar argument applies to the sequence $(s_N(x))$, which is relatively compact in E'' for the topology $\sigma(E'', E')$.

We shall say that a biorthogonal system $(a_n), (b_n)$ is weakly regular if it has any one (hence all three) of the properties listed in prop. 8. Example 2 of no. 7 shows there exists quasi-regular systems which are not weakly regular.

PROPOSITION 9. Let $(a_n), (b_n)$ be a weakly regular biorthogonal system. Then:

1° The mapping p defined in prop. 8 is a strongly continuous projection of E' onto a strongly closed subspace B_2, having as its kernel the

<u>subspace</u> A^o (so that E' is the topological direct sum ([4], p. 15) of B_2 and A^o for the strong topology).

2^o <u>The mapping</u> q <u>defined in prop. 8 is a strongly continuous mapping of</u> E <u>into</u> E'', <u>coinciding on</u> E <u>with the transposed mapping of</u> p.

1^o We have first to prove that, for any strong neighborhood of 0 in E', which can be taken of the form M^o, where M is a bounded, symmetric and convex subset of E, there is a strong neighborhood V of 0 in E' such that the relation $x' \in V$ implies $p(x') \in M^o$, that is, $|\langle x, p(x')\rangle| \leq 1$ for every $x \in M$. Now, from the definitions, it follows that $\langle x, p(x')\rangle = \lim_{N\to\infty} \sum_{n=0}^{N} \langle x, b_n\rangle\langle a_n, x'\rangle = \lim_{N\to\infty} \langle s_N(x), x'\rangle$. But the set of all elements of the form $s_N(x)$, where $x \in M$ and N is an arbitrary integer, is a <u>bounded</u> set P in E, since the sequence (s_N) is equicontinuous in $\mathcal{L}(E)$, as we have seen in the proof of prop. 5; if we take $V = P^o$, we will then have $|\langle x, p(x')\rangle| \leq 1$ for all $x \in M$ and all $x' \in V$.

Moreover, if $p(x') = 0$, then $\lim_{N\to\infty} \langle s_N(x), x'\rangle = 0$ for all $x \in E$. Taking $x = a_n$ shows that $\langle a_n, x'\rangle = 0$ for all indices n, and therefore the kernel of p is A^o. That $p(p(x')) = p(x')$ follows from the fact that $\langle a_n, p(x')\rangle = \langle a_n, x'\rangle$ for all indices n.

2^o The relation $\langle x, p(x')\rangle = \lim_{N\to\infty} \langle s_N(x), x'\rangle = \langle q(x), x'\rangle$ shows that q is the restriction to E of the transposed mapping p' of p. Now it is known that p' is a continuous mapping of E'' into itself, when E'' is given, either the weak topology $\sigma(E'', E')$, or the strong topology [1, p. 790, th. 1].

9. The three subspaces B, B_1, B_2 in E' are obviously such that $B_1 \subset B_2 \subset B$; it will be seen later (n^o 14) that they can be all distinct. We are now going to investigate the cases in which two of these subspaces coincide.

PROPOSITION 10. <u>Let</u> $(a_n), (b_n)$ <u>be a weakly regular biorthogonal system. The following properties are equivalent:</u>

1) $B = B_2$;

2) $A + B^o$ <u>is dense in</u> E ;

3) <u>for every</u> $x \in E$, <u>the series of general term</u> $\langle x, b_n\rangle a_n$ <u>converges strongly to</u> q(x) (<u>in</u> E'').

The relation $B = B_2$ implies $B \cap A^o = \{0\}$ by prop. 9, hence $(B \cap A^o)^o = E$, and $(B \cap A^o)^o$ is the strong and weak closure in E of $A + B^o$. Conversely, if $(B \cap A^o)^o = E$, then $B \cap A^o = \{0\}$, and as $B \supset B_2$ and $B_2 \cap A^o = \{0\}$, $B = B_2$; this proves that assumptions 1) and 2) are equivalent. For every $x \in A + B^o$, $s_N(x)$ converges strongly to a point $q(x) \in A$, since $x = y + z$, with $y \in A$, $z \in B^o$, and therefore $s_N(x) = s_N(y)$, which converges strongly to y (prop. 5). Let now x be an arbitrary element in E; there exists y in $A + B^o$ such that x-y is arbitrarily small; in order to prove that $s_N(x)$ converges strongly to q(x), it is enough to remark that $s_N(z)$ is arbitrarily small with $z \in E$, uniformly in N, since the sequence (s_N) is equicontinuous in $\mathcal{L}(E)$ (see proof of prop. 5). This shows that 2) implies 3). Conversely, suppose that for every $x \in E$, the sequence $(s_N(x))$ converges strongly to $q(x) \in E''$. As $s_N(x) \in A$, q(x) belongs to the strong closure $\bar{A}$ of A in E'', and for every n, $\langle q(x), b_n \rangle$ is the limit of $\langle s_N(x), b_n \rangle = \langle x, b_n \rangle$ as soon as $n \leq N$; this proves that q(x) - x is orthogonal to B in E''. Now, if $x' \in B \cap A^o$, $\langle q(x) - x, x' \rangle = 0$, and on the other hand, $\langle s_N(x), x' \rangle = 0$ for all N, hence $\langle q(x), x' \rangle = 0$, which shows that $\langle x, x' \rangle = 0$ for all $x \in E$, and therefore $B \cap A^o = \{0\}$ in E'; we have already seen that this is equivalent to $B = B_2$, hence 3) implies 1).

We shall say that a biorthogonal system $(a_n), (b_n)$ is strongly regular if it satisfies property 3) of proposition 10; this obviously implies weak regularity, as well as properties 1) and 2) of prop. 10.

10. PROPOSITION 11. *Let $(a_n), (b_n)$ be a weakly regular biorthogonal system. The following properties are equivalent:*

1) $A + B^o = E$;

2) *for every* $x \in E$, $q(x) \in E$;

3) *the mapping* p *defined in prop. 8 is continuous for the weak topology* $\sigma(E', E)$.

The space E *is then the strong topological direct sum of* A *and* B^o, *and the space* E' *is the topological direct sum (both for the strong topology and the weak topology* $\sigma(E', E)$) *of* B *and* A^o.

As $\langle x, p(x') \rangle = \langle q(x), x' \rangle$, the equivalence of 2) and 3) is immediate [6, p. 118]. If $E = A + B^o$, then we have seen in the proof of prop. 10 that this implies that $q(x) \in A$. Conversely, if $q(x) \in E$ for all $x \in E$, q is a strongly continuous projection of E on A, and therefore E is the topological direct sum (for the strong topology) of A and B^o; moreover, p is the transposed mapping of q, hence E' is the topological direct sum of B and A^o for $\sigma(E', E)$.

We shall say that a biorthogonal system $(a_n), (b_n)$ is completely regular when it satisfies any one (hence all three) of the properties of prop. 11.

COROLLARY 1. *Every quasi-regular biorthogonal system such that* $A = E$ *is completely regular*.

For such a system, the sequence (a_n) is usually called a *basis* for E [2, p. 110].

We shall say that a locally convex space F is semi-complete for its topology $\mathcal{T}$ if any Cauchy sequence in F, relative to the topology $\mathcal{T}$, converges to a point of F for $\mathcal{T}$.

COROLLARY 2. 1° If E is strongly semi-complete, every strongly regular biorthogonal system is completely regular.

2° If E is weakly semi-complete (i.e., for the topology $\sigma(E, E')$) every weakly regular biorthogonal system is completely regular.

The first part follows from the proof of prop. 10 and the fact that the sequence $(s_N(x))$ is then a Cauchy sequence in E for the strong topology. A similar argument proves the second part, since then $(s_N(x))$ is a Cauchy sequence in E for the topology $\sigma(E, E')$, hence $q(x) \in E$ for every $x \in E$.

COROLLARY 3. If a maximal biorthogonal system is strongly regular, then $A = E$ and the system is completely regular.

This follows from prop. 10 and 11, since for a maximal system $B^o \subset A$, and A is closed in E.

11. PROPOSITION 12. Let $(a_n), (b_n)$ be a weakly regular biorthogonal system. In order that $B_1 = B_2$, it is necessary and sufficient that, for every $x' \in E'$, the series of general term $\langle a_n, x'\rangle b_n$ converges strongly to $p(x')$. When this condition is satisfied, for every $x'' \in E''$, the series of general term $\langle x'', b_n\rangle a_n$ converges to $p'(x'')$ for the topology $\sigma(E'', E')$.

It is clear that if, for every $x' \in E'$, the sequence $(s'_N(x'))$ converges strongly to a point of E', this limit is in B_1 by definition, and is equal to $p(x')$ (prop. 8), hence $B_1 = B_2$. Conversely, if $B_1 = B_2$, every element $x' \in E'$ can be written $x' = y' + z'$, with $y' \in B_1$ and $z' \in A^o$ (prop. 9), hence $s'_N(x') = s'_N(y')$ converges strongly to $p(y') = p(x')$. As E' is then the direct topological sum of B_1 and A^o for the strong topology, E'' is (for $\sigma(E'',E')$) the direct topological sum of B_1^o and A^{oo}, which is the closure of A in E'' for the topology $\sigma(E'', E')$. For any $x'' \in E''$, the sequence of the $s_N(x'') = \Sigma_{n=0}^{N}\langle x'', b_n\rangle a_n$, being bounded in E (prop. 6) has at least a cluster point in E'' for the topology $\sigma(E'', E')$; if a'' and b'' are any two such cluster points, it is readily seen that $\langle a'', b_n\rangle = \langle b'', b_n\rangle = \langle x'', b_n\rangle$ for any n (see proof of prop. 7), hence $a'' - b''$ is orthogonal to B_1; but as a'' and b'' obviously belong to A^{oo}, $a'' = b''$, and therefore, as the sequence $(s_N(x''))$ is contained in a subset of E'' compact for $\sigma(E'', E')$, it converges to a limit in that set. From the relation $\langle s_N(x''), x'\rangle = \langle x'', s'_N(x')\rangle$, it follows immediately that the limit of $s_N(x'')$ is $p'(x'')$, where p' is the transposed mapping of p.

In general, for any weakly regular biorthogonal system, the fact that E' is the strong direct topological sum of A^o and B_2 (prop. 9) implies that for the strong topology as well as for the weak topology $\sigma(E'', E')$, E'' is the direct topological sum of A^{oo} and of B_2^o (orthogonal subspace of B_2 in E''); note that $B_2^o \cap E = B_1^o \cap E = B^o$ (orthogonal subspace of B in E)). In particular, A^{oo} is strongly isomorphic to the dual of the (strong) subspace B_2 of E'.

12. The convergence of $s_N(x'')$ in E'' for the topology $\sigma(E'', E')$, for every $x'' \in E''$, does not imply $B_1 = B_2$ (see no. 14, example 3). But we have the following propositions:

PROPOSITION 13. Let $(a_n), (b_n)$ be a quasi-regular biorthogonal system; if B_1 is semi-complete for the topology $\sigma(E', E'')$, and if for every $x'' \in E''$, $s_N(x'')$ converges (to $p'(x'')$) for the topology $\sigma(E'', E')$, then the system is weakly regular and $B_1 = B_2$.

From the weak convergence of $s_N(x)$ in E'' for every $x \in E$ it follows that the system is weakly regular (prop. 8); moreover, for every $x' \in E'$, the sequence $(s'_N(x'))$ is then a Cauchy sequence for the topology $\sigma(E', E'')$; as it belongs to B_1, it has a limit in B_1 for $\sigma(E', E'')$, hence $p(x') \in B_1$, and this proves $B_1 = B_2$.

It can be shown that in the assumptions of prop. 13 the assumption on the convergence of $s_N(x'')$ cannot be deleted, even if the system is originally supposed to be completely regular (no. 14, example 6).

PROPOSITION 14. Let $(a_n), (b_n)$ be a biorthogonal system such that for every $x'' \in E''$, the series of general term $\langle x'', b_n \rangle a_n$ converges to an element $q(x'') \in E$ for the topology $\sigma(E'', E')$. Then the system $(a_n), (b_n)$ is completely regular, and q is a strongly continuous projection of E'' onto A, so that E'' is the strong topological direct sum of A and B_1^o. Moreover, A, with the topology induced by the strong topology of E, is then naturally isomorphic to the strong dual of the subspace B_1 of E' (B_1 being given the topology induced by the strong topology of E') [13, p. 978, th. 9].

The assumption immediately shows that, for every index n, $\langle q(x''), b_n \rangle = \langle x'', b_n \rangle$, hence $s_N(x'') = s_N(q(x''))$; moreover, as $s_N(x'')$ is in A and converges to an element $q(x'')$ in E for the weak topology $\sigma(E, E')$, $q(x'')$ belongs to A, which is closed in E for $\sigma(E, E')$. From prop. 5 it follows that $s_N(q(x'')) = s_N(x'')$ converges strongly to $q(x'')$, and then prop. 11 shows that the system $(a_n), (b_n)$ is completely regular. To prove $q(x'')$ strongly continuous, it suffices to show that, for every bounded set M in E' there exists another bounded set P in E' such that the relation $x'' \in P^o$ implies $q(x'') \in M^o$. But $\langle s_N(x''), x' \rangle = \langle x'', s'_N(x') \rangle$, and we know that when $x' \in M$, the set of all elements $s'_N(x')$ (N arbitrary) is a bounded set P in E', the sequence (s'_N) being strongly equicontinuous in $\mathcal{L}(E')$ (proof of prop. 5); this shows that $|\langle q(x''), x'\rangle| = \lim_{N\to\infty} |\langle x'', s'_N(x')\rangle| \leq 1$ for $x' \in M$ and $x'' \in P^o$.

As the dual of B_1 is (algebraically) naturally isomorphic to the quotient space E''/B_1^o, there is a natural 1-1 mapping of that dual onto A. We have still to prove that the intersections with A of strong neighborhoods of 0 in E are identical with the polars (in A) of bounded subsets of B_1. This will certainly be the case, if we prove that every bounded subset M of B is contained in the weak closure (for $\sigma(E', E)$) of a bounded subset of B_1. Now, when $x' \in M$, we have seen above that the set of all $s'_N(x')$ is a bounded subset $P \subset B_1$, and on the other hand, $s'_N(x')$ converges to x' for the topology $\sigma(E', E)$ (prop. 8 and 9), hence M is contained in the weak closure of P.

Conversely, if the natural mapping of A into E''/B_1^o is onto, this means that $E'' = A + B_1^o$ (algebraically at least), and therefore that for every $x'' \in E''$, there is an $x \in A$ such that $\langle x'', b_n\rangle = \langle x, b_n\rangle$; the convergence of the series of general term $\langle x'', b_n\rangle a_n$ to x then follows immediately from prop. 5, if the system $(a_n), (b_n)$ is supposed to be quasi-regular.

13. We shall say that a quasi-regular system $(a_n), (b_n)$ is <u>perfect</u> if $B = B_1$. The system is then weakly regular by prop. 5 and corollary of prop. 7, which show that for every $x' \in E'$, the sequence $(s'_N(x'))$ is strongly convergent. As we have then moreover $B = B_2$, the system is also strongly regular, and, from the relation $B_1 = B_2$, it follows that $s_N(x'')$ converges in E'' for the topology $\sigma(E'', E')$. However, the system may fail to be completely regular if E is not strongly semi-complete (see no. 14, example 5). In a reflexive space E, every quasi-regular biorthogonal system is perfect and completely regular, since strong and weak closure of a vector subspace of E' coincide, and $E'' = E$.

PROPOSITION 15. *Let $(a_n), (b_n)$ be a perfect biorthogonal system satisfying the assumptions of prop. 14; then the system is completely regular, and A and $B_1 = B$ are reflexive spaces, which are dual to each other.*

The first assertion has been proved in prop. 14, and prop. 14 also shows that A is the dual of $B_1 = B$; on the other hand, E being the strong topological direct sum of A and B^o, the dual of A is (weakly and strongly) isomorphic to the subspace $B^{oo} = B$ of E', hence A and B are reflexive.

14. <u>Example</u> 3. Let E be the space (ℓ^1), E' its dual (m), and take $a_n = e_n$ in E, $b_n = e_n$ in E'. The system $(a_n), (b_n)$ is obviously completely regular, with $A = E$, $B = B_2 = E'$, whereas B_1 is the space (c_o) of sequences converging to 0. We notice that although $B_1 \neq B_2$, $s_N(x'')$ converges to $q(x'') \in E$ for every $x'' \in E''$ (in other words, the assumptions of prop. 14 are satisfied), for the restriction of x'' to B_1 being a contin-

uous linear form, is identified with an element $x \in E$, and $\langle x'', b_n \rangle = \langle x, b_n \rangle$, which proves our assertion (see prop. 18).

Example 4. Let E be the space (m) of bounded sequences, E' its dual, and take $a_n = e_n$ in E, $b_n = e_n$ in $(\ell^1) \subset E'$. Then every $x' \in E'$, restricted to the subspace $A = (c_0)$ of E, is a continuous linear form on (c_0), hence can be identified with an element $x' \in B_1 = (\ell^1)$; from this it follows immediately that $B_1 = B_2$, whereas $B = E'$, since E', being the bidual of B_1, is such that B_1 is dense in it for $\sigma(E', E)$.

From this and the preceding example, it is easy to define an example of a weakly regular system for which all three of the subspaces B, B_1, B_2 are distinct; take for E the product $(\ell^1) \times (m)$, with $a_{2n} = (e_n, 0)$, $a_{2n+1} = (0, e_n)$.

Example 5. Let H be a separable Hilbert space, (e_n) an orthonormal basis of H. It is possible, using a method of Hausdorff [9, p. 303], to define a subspace F of H, containing the e_n, which is distinct from H, but is a Baire space, hence [5] a t-space. In the product $H \times H$, consider the space E generated by $F \times F$ and one more element of the form (a, a), where $a \notin F$; then E is still a Baire space by the Hausdorff argument, hence a t-space; moreover, its dual is $H \times H$. Take $a_n = (e_n, 0)$ in E, $b_n = (0, e_n)$ in E'. Then it is obvious that $A = F$ (considered as being in the first factor of $H \times H$) and $B = B_1 = B_2 = H$ (considered as the second factor of $H \times H$); in other words, the given system is perfect, but $E \neq A + B^o$, for B^o is the second factor F, and $E \neq F \times F$.

Example 6. Let E be the (strongly closed) subspace of (ℓ^1) generated by the sequence (a_n), where $a_n = e_n - e_{n+1}$ $(n \geq 0)$; it is easy to verify that E is a hyperplane of (ℓ^1), hence its dual E' can be identified with a weakly closed hyperplane of (m). If we take $b_n = e_0 + e_1 + \cdots + e_n$ $(n \geq 0)$, then the system $(a_n), (b_n)$ is quasi-regular (no. 7, example 2), hence completely regular (cor. 1 of prop. 11). For any y in the dual of (m), the restriction of y to E' is a continuous linear form, hence equal to some $x'' \in E''$; therefore $\langle y, b_n \rangle = \langle x'', b_n \rangle$ for all n. But if we take $y \in (\ell^1)$ and not belonging to E, we know (no. 7, example 2) that the series of general term $\langle y, b_n \rangle \langle a_n, x' \rangle$ is not convergent for convenient choices of y in (ℓ^1) and x' in (m) (x' can be chosen in E', since its projection on E', parallel to E^o in (m), will give the same value to the $\langle a_n, x' \rangle$). This shows that when $B_1 \neq B_2$, the series of general term $\langle x'', b_n \rangle a_n$ may fail to be convergent for the topology $\sigma(E'', E')$, although E is semi-complete for $\sigma(E, E')$.

We may notice here, on the other hand, that in the example of a basis given by R. C. James [10], the space E is not semi-complete for the weak topology $\sigma(E, E')$, but however the series of general term $\langle x'', b_n \rangle a_n$ is convergent to x'' for the topology $\sigma(E'', E')$.

Example 7. Let E be the space (C) of continuous functions in the interval $I = \{0 \leq t \leq 1\}$ of the real line; its dual E' is the space of all Stieltjes measures μ on I, and it contains as a subspace the Banach space L^1 of all Lebesgue integrable function on I (such a function f, or rather the whole class of functions equivalent to f, being identified with the measure $d\mu(x) = f(x)dx$ as usual). Now take for (a_n) an orthogonal system in (C) such that the development of any function $f \in (C)$ with respect to (a_n) converges strongly to f in (C); such a system is for instance the Franklin orthogonal system [12, p. 122]; and take b_n to be equal to a_n, but considered as an element of L^1. It is then clear that $A = E$ and $B_1 = L^1$; on the other hand, $B = E' = B_2$ by corollary 1 to prop. 11; thus, although L^1 is semi-complete for $\sigma(E', E'')$ [2, p. 141], $B_1 \neq B_2$.

15. For any finite subset J of the set of integers, we shall write $s_J(x) = \Sigma_{n \in J} <x, b_n> a_n$ and $s'_J(x') = \Sigma_{n \in J} <a_n, x'> b_n$, and a similar definition for $s_J(x'')$ when $x'' \in E''$.

PROPOSITION 16. For a biorthogonal system $(a_n), (b_n)$, the following properties are equivalent:

1) For every $x \in E$ and every $x' \in E'$, the sums $\Sigma_{n \in J} <x, b_n><a_n, x'>$ are bounded (by a number depending on x and x') when J runs through all finite subsets of the set of integers.

2) For every $x \in E$ and every $x' \in E'$, the series of general term $<x, b_n><a_n, x'>$ is absolutely convergent.

3) For every $x' \in E'$, the series of general term $<a_n, x'> b_n$ converges unconditionally for the topology $\sigma(E', E)$.

4) For every $x \in E$, the series of general term $<x, b_n> a_n$ converges unconditionally in E'', for the topology $\sigma(E'', E')$.

The proof is immediate, and will be omitted.

We shall say that a biorthogonal system $(a_n), (b_n)$ is absolute [13, p. 971] if it satisfies the properties stated in prop. 16. An absolute biorthogonal system is weakly regular, but it is still possible that for such a system the three subspaces B, B_1, B_2 be distinct (see examples 3 and 4 in no. 14, which are absolute systems).

PROPOSITION 17. Let $(a_n), (b_n)$ be an absolute biorthogonal system. Then, for every $x \in A$, the series of general term $<x, b_n> a_n$ converges unconditionally to x for the strong topology, and for every $x' \in B_1$, the series of general term $<a_n, x'> b_n$ converges unconditionally to x' for the strong topology.

The proof is similar to that of prop. 5: the set of all s_J is bounded in $\mathcal{L}(E)$ for the topology of pointwise convergence, hence equicontinuous;

similarly, the set of all s'_J is equicontinuous when E' is given the strong topology, and the result follows at once from these two properties.

PROPOSITION 18. Let $(a_n), (b_n)$ be an absolute biorthogonal system. Then:

1^o For every $x'' \in E''$, the series of general term $<x'', b_n> a_n$ converges unconditionally to $p'(x'')$ for the topology $\sigma(E'', E')$.

2^o If in addition E is semi-complete for $\sigma(E, E')$, then the assumptions of prop. 14 are verified.

1^o We have $<s_J(x''), x'> = <x'', s'_J(x')>$ for every $x' \in E'$ and every $x'' \in E''$. But from prop. 16 it follows that the set of elements $s'_J(x')$ is bounded in E' when J runs through all finite subsets of integers; hence the sums $\Sigma_{n \in J} <x'', b_n><a_n, x'> = <s_J(x''), x'>$ are bounded, and this shows that the series of general term $<x'', b_n><a_n, x'>$ is absolutely convergent. The mapping $J \to s_J(x'')$ therefore transforms the filter of sections of the directed set formed by the finite sets of integers, into a Cauchy filter base (for $\sigma(E'', E')$) consisting of subsets of a bounded set in E; such a bounded set being relatively compact in E'' for $\sigma(E'', E')$, the unconditional convergence of the series of general term $<x'', b_n><a_n, x'>$ follows.

2^o From 1^o it follows that the sum $p'(x'')$ of the series of general term $<x'', b_n> a_n$ is the limit (for $\sigma(E'', E')$) of the sequence $(s_N(x''))$, which is a Cauchy sequence in E for $\sigma(E, E')$; therefore, if E is semi-complete for that topology, $p'(x'') \in E$.

16. S. Karlin has shown [13, p. 980, th. 12] that there exist non-absolute bases in the spaces L^p for $p > 1$ and $p \neq 2$; we propose to show that the same is true for L^1. Take $E = L^1$, $E' = L^\infty$, and consider the biorthogonal system $(a_n), (b_n)$ where a_n is the n^{th} function of the Franklin orthogonal system (see no. 14, example 7) considered as an element of L^1, and b_n the same function considered as an element of $(C) \subset L^\infty$. It follows from example 7 that $(a_n), (b_n)$ is a completely regular system, and as $A = E$, it is a basis for L^1. However it is not an absolute basis in L^1; for if it were, as E is semi-complete for $\sigma(E, E')$, the result of prop. 18, 2^o would apply, hence, by prop. 14, E would be the dual of $B_1 = (C)$, which is absurd.

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