THE LOTOTSKY METHOD FOR EVALUATION OF SERIES

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1. INTRODUCTION

A. V. Lototsky (or Lotockii) [5] has recently introduced a method for evaluation of divergent series which seems to be new and to have fundamental significance which may make it rival in importance the classic methods of Cesàro, Abel, Euler and Knopp, Borel, and others. The method involves a triangular matrix transformation of the standard form

(1.1)
$$\sigma_n = \sum_{k=1}^{n} a_{nk} s_k$$

by which a given series $u_1 + u_2 + u_3 + \cdots$ with partial sums $s_1 = u_1$, $s_2 = u_1 + u_2$, $\cdots$ is evaluable to σ if $\sigma_n \to \sigma$ as $n \to \infty$.

For each $n = 1, 2, 3, \cdots$, let $p_n(x)$ be the polynomial of degree n defined by

(1.2)
$$p_n(x) = x(x+1)(x+2)\cdots(x+n-1),$$

and let the constants $p_{n1}, p_{n2}, \cdots, p_{nn}$ be defined by

(1.3)
$$p_n(x) = p_{n1}x + p_{n2}x^2 + p_{n3}x^3 + \cdots + p_{nn}x^n.$$

To simplify our work in some places, we let $p_{nk} = 0$ when $k < 1$ and when $k > n$. Letting $a_{nk} = p_{nk}/n!$ and

(1.4)
$$\sigma_n = \sum_{k=1}^{n} \frac{p_{nk}}{n!} s_k,$$

we shall call a series $u_1 + u_2 + \cdots$ and its sequence $s_1, s_2, \cdots$ of partial sums *evaluable* L to σ if $\sigma_n \to \sigma$ as $n \to \infty$.

Numerous properties of this Lototsky method are obtained. Because the paper [5] of Lototsky appears in a periodical that is not always readily accessible, no acquaintance with it is assumed, and the connections between the present paper and [5] are explained in such a way that they can be understood without reference to [5].

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2. THE NUMBERS p_{nk}; REGULARITY

It is easy to see from (1.2) and (1.3) that

$$p_{nk} \geq 0, \qquad \sum_{k=1}^{n} \frac{p_{nk}}{n!} = 1, \tag{2.1}$$

the latter condition following from the fact that $\Sigma p_{nk} = p_n(1) = n!$. Since

$$p_{n+1}(x) = (n + x)\, p_n(x),$$

we obtain the useful recursion formula

$$p_{n+1,k} = np_{n,k} + p_{n,k-1}, \tag{2.2}$$

which is valid when $n \geq 1$ and $-\infty < k < \infty$. With the aid of (2.2), it is a short task to calculate the elements of the matrix p_{nk} for which $n \leq 13$. The first 7 rows are

(2.3)

1						
1	1					
2	3	1				
6	11	6	1			
24	50	35	10	1		
120	274	225	85	15	1	
720	1764	1624	735	175	21	1.

It is easy to see that

$$p_{n1} = (n - 1)!, \qquad p_{nn} = 1. \tag{2.31}$$

On setting

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \tag{2.4}$$

when $n > 0$, and $H_n = 0$ when $n \leq 0$, we can use (2.3), (2.2) and induction to prove that

$$p_{n2} = (n - 1)!\, H_{n-1}. \tag{2.41}$$

Using the same procedure and (2.41), we can prove that

$$p_{n3} = (n - 1)! \left(\frac{1}{2} + \frac{H_2}{3} + \frac{H_3}{4} + \cdots + \frac{H_{n-2}}{n - 1} \right) \tag{2.42}$$

when $n > 3$. The right side will be increased if the numerator of each term in brackets is replaced by H_{n-1}. This gives the formula

$$p_{n,k} \leq (n - 1)!\, H_{n-1}^{k-1} \qquad (n > k) \tag{2.43}$$

for the case $k = 3$, and continuation of the procedure proves by induction that (2.43) holds for each $k = 1, 2, 3, \cdots$. Because H_{n-1} is of order $\log n$ for large values of n, it follows from (2.43) that, for each fixed k, (2.44) $$p_{n,k}/n! \leq n^{-1} H_{n-1}^{k-1} = o(1)$$

as $n \to \infty$. Because of (2.1) and (2.44), the three standard conditions for regularity are satisfied and L is regular. In [5], most of the formulas of this section are given, and it is proved, without explicit use of the conditions for regularity, that if $s_n \to S$ then $\sigma_n \to S$.

3. POWER SERIES

In its application to power series, the L method is potent and extremely simple. We begin with the geometric series $1 + z + z^2 + \cdots$, where z is a complex number for which $z \neq 1$. In this case $u_k = z^{k-1}$,

(3.1) $$s_k = \frac{1}{1 - z} - \frac{1}{1 - z} z^k,$$

and the L transform, which we now denote by $\sigma_n(z)$, is

(3.2) $$\sigma_n(z) = (1 - z)^{-1} - (1 - z)^{-1} f_n(z),$$

where

(3.21) $$f_n(z) = \frac{1}{n!} \sum_{k=1}^{n} p_{nk} z^k.$$

Use of (1.2) and (1.3) gives

(3.22) $$f_n(z) = \frac{z(z + 1)(z + 2)\cdots(z + n - 1)}{n!}.$$

In case z is 0 or a negative integer, it is obvious that $f_n(z) \to 0$ as $n \to \infty$ and hence that $\sigma_n(z) \to (1 - z)^{-1}$, so that Σz^n is evaluable L to $(1 - z)^{-1}$. In case $z \neq 0, -1, -2, \cdots$, we use the functional equation of the factorial (or gamma) function to put (3.22) in the form

(3.23) $$f_n(z) = \frac{1}{(z - 1)!} \frac{(n + z - 1)!}{n!}.$$

In case $\Re(z - 1) < 0$ or $\Re z < 1$, it follows that $f_n(z) \to 0$ and $\sigma(z) \to (1 - z)^{-1}$. Thus the geometric series Σz^n, which converges to $(1 - z)^{-1}$ only when $|z| < 1$, is evaluable L to $(1 - z)^{-1}$ over the whole half-plane $\Re z < 1$. By use of the Cauchy integral formula it can be shown that *if* $\Sigma a_n z^n$ *is a power series which has a positive radius of convergence and if* F(z) *is the analytic function which it generates by analytic extension along radial lines from the origin of the complex plane, then* $\Sigma a_n z^n$ *is evaluable* L *to* F(z) *at each point inside the Borel polygon of* F(z).

By a method which is different from that given above, and which will be presented in Section 6, it is shown in [5] that if z is a negative integer then the series $0 + 1 + z + z^2 + \cdots$ is evaluable L to $(1 - z)^{-1}$.

4. EXPONENTIAL INTEGRAL SERIES

In this section we show that *the classic rapidly divergent series*

$$0! - \frac{1!}{z} + \frac{2!}{z^2} - \frac{3!}{z^3} + \frac{4!}{z^4} - \cdots \tag{4.1}$$

is evaluable L *for each complex number* z *for which* $\Re z \geq \log 2 = 0.69315 \cdots$. One reason this series is of interest is the fact that the right member of the formula

$$\int_z^\infty \frac{e^{-t}}{t}dt \sim \frac{e^{-z}}{z}\left(0! - \frac{1!}{z} + \frac{2!}{z^2} - \frac{3!}{z^3} + \cdots\right) \tag{4.11}$$

is the standard asymptotic expansion of the exponential integral in the left member. In case z is a complex number which is not real, the integrals in (4.11) and some of the formulas below are to be interpreted as line integrals over the line in the complex plane extending from the point z to ∞ in the direction of the positive real axis. Of course the case $z = 0$ is always excluded from consideration, and we shall not be concerned with the case in which z is real and negative because in this case the series (4.1) is a divergent series of positive terms which is evaluable L to $+\infty$. There are some methods, more powerful than the simpler ones ordinarily treated in textbooks, by which the series (4.1) is evaluable for at least some values of z. For a discussion of some of these methods by which (4.1) is evaluable, particularly when $z = 1$, and for references to earlier literature, see Good [3]. The value $V(z)$ to which the series (4.1) is evaluable by these methods is

$$V(z) = ze^z \int_z^\infty \frac{e^{-t}}{t} dt = z \int_0^\infty \frac{e^{-t}}{z + t} dt, \tag{4.12}$$

the last member of (4.12) being obtained from the middle one by a change of the path of integration. When $z = 1$, the series (4.1) reduces to the series

$$0! - 1! + 2! - 3! + 4! - \cdots, \tag{4.13}$$

and the value Euler [2] obtained for this series is

$$V(1) = 0.59634\,73621\,237. \tag{4.14}$$

Viewing (4.1) as a power series in z^{-1}, we observe the well-known fact that the radius of convergence is 0 and that each point z^{-1} is regarded as being outside the Borel polygon.

As we shall see, the application of L to the series (4.1) is simplified by the fact that the factorials have integral representations (the Euler integrals) which enable us to make use of the simplicity of the L transform of the geometric series. For the series (4.1) we have

$$(4.2)\qquad u_j = \left(\frac{-1}{z}\right)^{j-1}(j-1)! = \int_0^\infty e^{-t}\left(\frac{-t}{z}\right)^{j-1} dt,$$

and hence

$$(4.21)\qquad s_k = \sum_{j=1}^{k} u_j = z\int_0^\infty \frac{e^{-t}}{z+t}\left[1-\left(\frac{-t}{z}\right)^k\right]dt.$$

Using (1.4), (4.21), and (4.12), we find that the L transform $\sigma_n(z)$ of the series (4.1) is

$$(4.3)\qquad \sigma_n(z) = V(z) - \frac{z}{n!}\int_0^\infty \frac{e^{-z}}{z+t}\sum_{k=1}^{n} p_{nk}\left(\frac{-t}{z}\right)^k dt.$$

Using (1.2) and (1.3) gives

$$(4.31)\qquad \sigma_n(z) = V(z) + z f_{n-1}(z),$$

where

$$(4.32)\qquad f_n(z) = \frac{-1}{(n+1)!}\int_0^\infty \frac{e^{-t}}{z+t}\left(\frac{-t}{z}\right)\left(\frac{-t}{z}+1\right)\left(\frac{-t}{z}+2\right)\cdots\left(\frac{-t}{z}+n\right) dt.$$

To show that the series (4.1) is evaluable L to V(z), it is therefore necessary as well as sufficient to show that $f_n(z) \to 0$ as $n \to \infty$.

Changing the path of integration in (4.32) by setting $t = zu$ and $u = z^{-1}t$ gives

$$(4.33)\qquad f_n(z) = \frac{1}{(n+1)!}\int_0^\infty \frac{ue^{-zu}}{1+u}(-u+1)(-u+2)\cdots(-u+n)\, du,$$

where the path of integration is the line running from 0 to ∞ through the point z^{-1}. When z is a complex number having a positive real part, it can be shown with the aid of the Cauchy integral formula that the path of integration in (4.33) can be replaced by the real line $u \geq 0$. Hence, when $\Re z = x > 0$,

$$(4.34)\qquad f_n(z) = \frac{(-1)^n}{(n+1)!}\int_0^\infty \frac{te^{-zt}}{1+t}(t-1)(t-2)\cdots(t-n)\, dt$$

and

(4.35) $$|f_n(z)| \leq \frac{1}{(n+1)!}\int_0^\infty e^{-xt}|(t-1)(t-2)\cdots(t-n)|\,dt.$$

Letting $g_n(z)$ denote the result of replacing ∞ by n on the integral in (4.35) gives

(4.36) $$g_n(z) = \frac{1}{(n+1)!}\sum_{k=1}^{n}\int_{k-1}^{k} e^{-xt}|(t-1)(t-2)\cdots(t-n)|\,dt.$$

But when $k - 1 \leq t \leq k$ we have

(4.37) $$|(t-1)(t-2)\cdots(t-n)|/n! \leq (k-1)!\,(n-k+1)!/n!\,,$$

and this cannot exceed 1. Therefore, when $x = \Re z > 0$,

(4.38) $$g_n(z) \leq \frac{1}{n+1}\sum_{k=1}^{n}\int_{k-1}^{k} e^{-xt}dt \leq \frac{1}{n+1}\int_0^\infty e^{-xt}dt$$

and $g_n(z) \to 0$ as $n \to \infty$. This and (4.35) imply that, when $x = \Re z > 0$,

(4.4) $$|f_n(z)| \leq o(1) + h_n(z)\,,$$

where

(4.41) $$h_n(z) = \frac{1}{(n+1)!}\int_n^\infty e^{-xt}(t-1)(t-2)\cdots(t-n)\,dt.$$

Changing the variable of integration in (4.41) gives

(4.42) $$h_n(z) = \frac{e^{-nx}}{(n+1)!}\sum_{k=0}^{\infty}\int_k^{k+1} e^{-xt}(n+t-1)(n+t-2)\cdots(n+t-n)\,dt$$

$$\leq \frac{e^{-nx}}{(n+1)!}\sum_{k=0}^{\infty}\int_k^{k+1} e^{-kx}(n+k)(n+k-1)\cdots(k+1)\,dt$$

$$= \frac{e^{-nx}}{n+1}\sum_{k=0}^{\infty}\binom{n+k}{k}(e^{-x})^k = \frac{e^{-nx}}{n+1}(1-e^{-x})^{-n-1}$$

$$= \frac{1}{(n+1)(1-e^{-x})}\frac{1}{(e^x-1)^n}.$$

From this we see that we will have $h_n(z) = o(1)$ if $e^x - 1 \geq 1$, that is, if $x \geq \log 2$. Thus, when $x = \Re z \geq \log 2$, (4.4) implies that $f_n(z) = o(1)$ and (4.31) implies that $\sigma_n(z) \to V(z)$, and the series (4.1) is evaluable L to $V(z)$.

It is of particular interest that the series Σu_n for which

$$u_n = (-1)^n n!/(\log 2)^n \tag{4.5}$$

is evaluable L, because it will be shown in Section 5 that *a series* Σu_n *cannot be evaluable* L *unless there is a constant* M' *for which* $|u_n| \leq M' n!/(\log 2)^n$. Thus there is a sense in which (4.5) is the most rapidly divergent series that is evaluable L. In particular the series (4.1) cannot be evaluable L when $|z| < \log 2$. Our work does not show whether (4.1) is evaluable L when z is a nonreal number for which $\Re z < \log 2$ and $|z| > \log 2$.

The results and methods of this section are completely different from those of [5], where it is shown that the series

$$0 + 0! - 1! + 2! - 3! + 4! - \cdots \tag{4.6}$$

is evaluable L to the value

$$\frac{1}{2}\int_0^\infty e^{-t}\,dt + \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+2)!}\int_0^\infty e^{-t}(t-1)(t-2)\cdots(t-n)\,dt. \tag{4.61}$$

If we start with a suspicion that the series in (4.61) converges to the value

$$V(1) = \int_0^\infty \frac{e^{-t}}{1+t}\,dt, \tag{4.62}$$

it is possible to prove that this is true.

5. THE INVERSE OF L

The relation between the elements $s_1, s_2, \cdots$ of a sequence and the elements $\sigma_1, \sigma_2, \cdots$ of its L transform can be put in the form

$$n!\,\sigma_n = \sum_{k=1}^{n} p_{nk} s_k = p_n(s) = s(s+1)(s+2)\cdots(s+n-1) \tag{5.1}$$

where, in (5.1) and formula (5.21) below, a polynomial in s stands for the result of expressing the polynomial in the form $\Sigma a_k s^k$ and then replacing s^k by s_k. For each $n = 1, 2, 3, \cdots$, let constants $q_{n1}, q_{n2}, \cdots, q_{nn}$ be defined by

$$s^n = q_{n1} p_1(s) + q_{n2} p_2(s) + \cdots + q_{nn} p_n(s). \tag{5.2}$$

From (5.1) and (5.2) we obtain

$$\sum_{k=1}^{n} q_{nk}k!\sigma_k = \sum_{k=1}^{n} q_{nk}p_k(s) = s^n = s_n, \tag{5.21}$$

and *we have the formula*

$$s_n = \sum_{k=1}^{n} (k!\, q_{nk})\sigma_k, \tag{5.3}$$

which gives the inverse of the transformation L. The first 7 rows of the matrix q_{nk} are

(5.31)

1						
-1	1					
1	-3	1				
-1	7	-6	1			
1	-15	25	-10	1		
-1	31	-90	65	-15	1	
1	-63	301	-350	140	-21	1 .

Supposing that $q_{nk} = 0$ when $k < 1$ and when $k > n$, we see that the elements of this matrix seem to satisfy the recursion formula

$$q_{n+1,k} = -kq_{n,k} + q_{n,k-1}. \tag{5.32}$$

Validity of (5.32) follows from the formula

$$\sum_{k=1}^{n+1} q_{n+1,k}p_k(s) = s^{n+1} = s \cdot s^n$$

$$= s\sum_{k=1}^{n} q_{nk}\, p_k(s) = \sum_{k=1}^{n} q_{nk}\, s(s+1)\cdots(s+k-1)(-k+s+k) \tag{5.33}$$

$$= \sum_{k=1}^{n} (-kq_{nk})p_k(s) + \sum_{k=2}^{n+1} q_{n,k-1}\,p_k(s) = \sum_{k=1}^{n+1} (-kq_{nk} + q_{n,k-1})p_k(s)$$

and the fact that the functions $p_k(s)$ are linearly independent.

Using (5.3) and (5.32), we now show that *if a series* $u_1 + u_2 + \cdots$ *with partial sums* $s_1, s_2, \cdots$ *is evaluable* L, *then there exist constants* M *and* M' *such that*

$$|s_n| \leq Mn!/(\log 2)^n \tag{5.4}$$

and

$$|u_n| \leq M'n!/(\log 2)^n. \tag{5.41}$$

When we have proved (5.4), (5.41) will follow from (5.4) and the inequality

$$|u_n| = |s_n - s_{n-1}| \leq |s_n| + |s_{n-1}|.$$

Whenever the sequence $s_1, s_2, \cdots$ is evaluable L, and indeed whenever the sequence $s_1, s_2, \cdots$ has a bounded transform $\sigma_1, \sigma_2, \cdots$, we see from (5.3) that there exists a constant M_1 for which $|s_n| \leq M_1 Q_n$, where

$$Q_n = \sum_{k=1}^{n} k!\,|q_{nk}|\,. \tag{5.42}$$

It suffices therefore to show that there exists a constant M_2 for which

$$Q_n \leq M_2 n!/(\log 2)^n\,. \tag{5.43}$$

Since q_{nk} and $q_{n,k-1}$ have opposite signs when both differ from 0, it follows from (5.32) that

$$|q_{n+1,k}| = k|q_{n,k}| + |q_{n,k-1}|\,. \tag{5.5}$$

Letting $q_n(x)$ be defined by

$$q_n(x) = |q_{n1}|x + |q_{n2}|x^2 + \cdots + |q_{nn}|x^n \tag{5.51}$$

and using (5.5) gives

$$q_{n+1}(x) = x[q_n'(x) + q_n(x)] \tag{5.52}$$

and hence

$$e^x q_{n+1}(x) = x\frac{d}{dx}e^x q_n(x)\,. \tag{5.53}$$

This and the fact that the formula

$$e^x q_n(x) = \sum_{k=1}^{\infty} \frac{k^{n-1}x^k}{(k-1)!} \tag{5.54}$$

holds when $n = 1$ imply that (5.54) holds for each $n = 1, 2, 3, \cdots$. Starting with (5.41), we find that

$$Q_n = \sum_{k=1}^{n} |q_{nk}| \int_0^\infty e^{-x}x^k dx = \int_0^\infty e^{-x} q_n(x)\,dx\,. \tag{5.55}$$

This and (5.54) give

$$Q_n = \int_0^\infty e^{-2x} \sum_{k=1}^{\infty} \frac{k^{n-1}x^k}{(k-1)!}\,dx\,. \tag{5.56}$$

Since the terms of the series are all positive, we can reverse the order of integration and summation to obtain

(5.6) $$Q_n = \frac{1}{2}\sum_{k=1}^{\infty}\frac{k^n}{2^k}.$$

To appraise the last sum, we let $\lambda = \log 2$ and

(5.61) $$f_n(x) = x^n/2^x = x^n e^{-\lambda x}.$$

We note that $f_n(0) = 0$, $f_n(x) \to 0$ as $x \to \infty$, $f_n(x)$ is increasing over $0 \le x \le \lambda^{-1}n$, and $f_n(x)$ is decreasing over $x \ge \lambda^{-1}n$; and we see that elementary geometrical or analytical considerations analogous to proofs of the integral test for convergence of series yield existence of numbers $\theta_1, \theta_2, \cdots$ such that $-1 < \theta_n < 1$ and

(5.62) $$\sum_{k=0}^{\infty} f_n(k) = \theta_n f_n(\lambda^{-1}n) + \int_0^{\infty} f_n(x)\,dx.$$

This gives

(5.63) $$\sum_{k=1}^{\infty}\frac{k^n}{2^k} = \theta_n\frac{n^n e^{-n}}{(\log 2)^n} + \frac{n!}{(\log 2)^{n+1}}.$$

Since Stirling's formula shows that

(5.64) $$0 < n^n e^{-n} < (2n\pi)^{-1/2} n!,$$

we obtain

(5.65) $$\sum_{k=1}^{\infty}\frac{k^n}{2^k} = O\left(\frac{n!}{n^{1/2}(\log 2)^n}\right) + \frac{n!}{(\log 2)^{n+1}}.$$

Hence

(5.66) $$Q_n = \frac{1}{2}\sum_{k=1}^{\infty}\frac{k^n}{2^k} = O\left(\frac{n!}{(\log 2)^n}\right),$$

and the conclusions (5.43) and (5.4) follow.

Most of the results of this section are obtained in [5] by use of more notation and substantially more complicated proofs.

6. SERIES-TO-SERIES VERSION OF L

Supposing that $\sigma_1, \sigma_2, \cdots$ is the L transform of a sequence $s_1, s_2, \cdots$, let

(6.1) $$s_n = u_1 + u_2 + \cdots + u_n,$$

(6.11) $$\sigma_n = U_1 + U_2 + \cdots + U_n.$$

We see that $U_1 = \sigma_1 = s_1 = u_1$. When $n > 1$ we have $U_n = \sigma_n - \sigma_{n-1}$ and hence

$$U_n = \sum_{k=1}^{n} \left(\frac{p_{n,k}}{n!} - \frac{p_{n-1,k}}{(n-1)!} \right) \sum_{j=1}^{k} u_j = \frac{1}{n!} \sum_{j=1}^{n} \left(\sum_{k=j}^{n} (p_{n,k} - np_{n-1,k}) \right) u_j. \tag{6.2}$$

But the identity

$$\sum_{k=j}^{n} (p_{nk} - np_{n-1,k}) = p_{n-1,j-1} \tag{6.21}$$

results from adding the identities

$$p_{n,k} - np_{n-1,k} = p_{n-1,k-1} - p_{n-1,k} \qquad (j \le k \le n), \tag{6.22}$$

which are obtained by replacing n by $n - 1$ in (2.2). Therefore

$$U_n = \frac{1}{n!} \sum_{j=1}^{n} p_{n-1,j-1} u_j \tag{6.23}$$

and, since $p_{n-1,0} = 0$, we obtain the transformation formulas

$$U_1 = u_1, \tag{6.3}$$

$$U_n = \frac{1}{n!} \sum_{k=1}^{n-1} p_{n-1,k} u_{k+1} \qquad (n > 1). \tag{6.31}$$

From (6.11) we see that Σu_k is evaluable L to σ if and only if ΣU_k converges to σ. When the series $u_1 + u_2 + \cdots$ is evaluable L, we may denote the value by the left member of the formula

$$L\left\{ \sum_{k=1}^{\infty} u_k \right\} = u_1 + \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{k=1}^{n-1} p_{n-1,k} u_{k+1}, \tag{6.4}$$

and we see from (6.3) and (6.31) that (6.4) *is valid if and only if the series on the right is convergent.* The formula (6.4) can be put in the form

$$L\left\{ \sum_{k=1}^{\infty} u_k \right\} = u_1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \sum_{k=1}^{n} p_{nk} u_{k+1}. \tag{6.41}$$

The right members of (6.4) and (6.41) are

$$u_1 + \frac{u_2}{2!} + \frac{u_2 + u_3}{3!} + \cdots + \frac{p_{n1} u_2 + \cdots + p_{nn} u_{n+1}}{(n+1)!} + \cdots. \tag{6.42}$$

Applying (6.41) to the series $0 + u_1 + u_2 + \cdots$ gives the formula

(6.5) $$L\{0 + u_1 + u_2 + \cdots\} = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \sum_{k=1}^{n} p_{nk} u_k,$$

which has a more attractive right member than the previous formulas. The right member of (6.5) is

(6.51) $$\frac{u_1}{2!} + \frac{u_1 + u_2}{3!} + \cdots + \frac{p_{n1} u_1 + \cdots + p_{nn} u_n}{(n+1)!} + \cdots,$$

and this is systematically used in [5] in place of $L\{u_1 + u_2 + \cdots\}$, when series are being evaluated. Unfortunately this gives a confusion of the L transforms and L values of the two different series $u_1 + u_2 + \cdots$ and $0 + u_1 + u_2 \cdots$. Because of this circumstance, we introduce a class $L_0, L_1, L_2, \cdots$ of methods for evaluation of series which are related to L in the same way that the Borel-Sannia methods $B_0, B_1, B_2, \cdots$ are related to the Borel method B. We will say that the series $u_1 + u_2 + \cdots$ is evaluable L_r to σ if the series

(6.6) $$0 + 0 + \cdots + 0 + u_1 + u_2 + \cdots$$

obtained by prefixing r zeros to the series $u_1 + u_2 + \cdots$ is evaluable L to σ.

In particular, $L_0 = L$. The formula

(6.7) $$L_1\{u_1 + u_2 + \cdots\} = \sum_{n=1}^{\infty} \frac{1}{(n+1)!} \sum_{k=1}^{n} p_{nk} u_k$$

is then correct, and many of the statements in [5] concerning L should be converted into statements involving L_1. As we shall show in the next section, *the conclusion that a given series is evaluable* L_1 *is weaker than the conclusion that it is evaluable* L.

7. THE SEQUENCE $0, s_1, s_2, \cdots$

The relation between the L transforms of the two sequences $s_1, s_2, \cdots$ and $0, s_1, s_2, \cdots$ is so simple that fundamental theorems concerning them are completely obvious. These theorems are analogous to well-known but more difficult theorems involving Borel transforms of the two sequences. Let $\sigma_1, \sigma_2, \cdots$ denote the L transform of the sequence $s_1, s_2, \cdots$ so that, with the definition $s_0 = 0$,

(7.1) $$\sigma_n = \frac{1}{n!} \sum_{k=1}^{n} p_{nk} s_k = \frac{1}{n!} \sum_{k=1}^{n+1} p_{n,k-1} s_{k-1},$$

and let $\sigma_1^{(1)}, \sigma_2^{(1)}, \cdots$ denote the L transform of the sequence $0, s_1, s_2, \cdots$, so that

(7.2) $$\sigma_n^{(1)} = \frac{1}{n!} \sum_{k=1}^{n} p_{nk} s_{k-1}.$$

Multiplying the recursion formula (2.2) by $s_{k-1}/n!$ and summing over $1 \leq k \leq n+1$ gives

(7.3) $$(n+1)\sum_{k=1}^{n+1}\frac{p_{n+1,k}}{(n+1)!}s_{k-1} = n\sum_{k=1}^{n+1}\frac{p_{nk}}{n!}s_{k-1} + \sum_{k=1}^{n+1}\frac{p_{n,k-1}}{n!}s_{k-1}$$

and hence

(7.31) $$(n+1)\,\sigma_{n+1}^{(1)} = n\sigma_n^{(1)} + \sigma_n.$$

Replacing n by k in (7.31), summing over $1 \le k \le n$, and using the fact that $\sigma_1^{(1)} = 0$, gives the simple formula

(7.4) $$\left(1+\frac{1}{n}\right)\sigma_{n+1}^{(1)} = \frac{\sigma_1+\sigma_2+\cdots+\sigma_n}{n},$$

which is equivalent to a formula proved in [5] by induction. Since the arithmetic mean transformation C_1 is regular, the hypothesis that $\sigma_n \to \sigma$ implies that $\sigma_n^{(1)} \to \sigma$. Since there exist divergent sequences evaluable C_1, it follows that there exist sequences for which σ_n diverges but $\sigma_n^{(1)}$ is convergent. Thus *the formula*

(7.5) $$L\{s_1, s_2, \cdots\} = L\{0, s_1, s_2, \cdots\}$$

is correct whenever the left side exists; but the right side can exist when the left side fails to exist. For infinite series, the analogous conclusion is that *the formula*

(7.6) $$u_0 + L\{u_1 + u_2 + \cdots\} = L\{u_0 + u_1 + u_2 + \cdots\}$$

is correct whenever the left side exists, but the right side can exist when the left side fails to exist. Taking the case in which $u_0 = 0$, we see that the method L_1 defined in Section 6 is in fact stronger than L, and the conclusion that a given series is evaluable L_1 is therefore weaker than the conclusion that the series is evaluable L.

8. RELATION BETWEEN L AND THE EULER-KNOPP METHODS E_r

In this section we show that *if a sequence (or series) is evaluable to* S *by the Euler-Knopp method* E_r *of order* r, *and if* r *is real and positive, then the sequence is also evaluable* L *to* S. *Moreover the conclusion fails to hold when* r *is not real and positive.* Thus $L \supset E_r$ if and only if $r > 0$. For each complex r, the Euler-Knopp transform of order r of a sequence $s_0, s_1, s_2, \cdots$ is defined by

(8.1) $$S_n = \sum_{k=0}^{n}\binom{n}{k}r^k(1-r)^{n-k}s_k \qquad (n = 0, 1, 2, \cdots),$$

and the sequence is evaluable E_r to S if $S_n \to S$ as $n \to \infty$. In case $r = 0$, the transformation is a trivial one for which $S_n = s_0$ for each n, and each sequence is evaluable to its first element. We suppose henceforth that $r \neq 0$. The transformation inverse to (8.1) is

(8.11) $$s_j = \sum_{k=0}^{j}\binom{j}{k}\left(\frac{1}{r}\right)^k\left(1-\frac{1}{r}\right)^{j-k}S_k.$$

The transformation E_r is regular if and only if r is real and $0 < r \leq 1$; and, when $0 < r \leq 1$, E_r belongs to the class of regular Hausdorff transformations. A treatment of these matters and references to the literature of the subject may be found in Agnew [1].

Using (1.4), we put the L transform $\sigma_1, \sigma_2, \cdots$ of the sequence $s_0, s_1, \cdots$ in the form

$$\sigma_{n+1} = \frac{1}{(n+1)!}\sum_{j=0}^{n} p_{n+1,j+1} s_j \qquad (n = 0, 1, 2, \cdots). \tag{8.2}$$

Substituting (8.11) in (8.2) and changing the order of summation gives

$$\sigma_{n+1} = \frac{1}{(n+1)!}\sum_{k=0}^{n} b_{nk} S_k, \tag{8.21}$$

where

$$b_{nk} = \left(\frac{1}{r}\right)^k \sum_{j=k}^{n} p_{n+1,j+1}\binom{j}{k}\left(1 - \frac{1}{r}\right)^{j-k} \tag{8.22}$$

when $0 \leq k \leq n$. We simplify some of our formulas by writing $b_{nk} = 0$ when $k < 0$ and when $k > n$. We see that $L \supset E_r$ if and only if the transformation (8.21) is regular. We can therefore prove that $L \supset E_r$ when $r > 0$, by proving that the three conditions

$$\frac{1}{(n+1)!}\sum_{k=0}^{n} |b_{nk}| \leq M \qquad (n = 0, 1, 2, \cdots), \tag{8.23}$$

$$\frac{1}{(n+1)!}\sum_{k=0}^{n} b_{nk} = 1 \qquad (n = 0, 1, 2, \cdots), \tag{8.24}$$

$$\lim_{n \to \infty} \frac{1}{(n+1)!} b_{nk} = 0 \qquad (k = 0, 1, 2, \cdots) \tag{8.25}$$

hold when $r > 0$.

The fact that (8.24) is valid whenever r is complex follows at once from (8.21) and the fact that if $s_k = 1$ for each k, then both $S_k = 1$ for each k and $\sigma_{n+1} = 1$ for each n. As we shall see at the end of this section, when $r = 1/2$ the coefficients b_{nk} are all nonnegative and (8.23) follows from (8.24); but in the case when $r = 1/3$, for example, many of the coefficients b_{nk} are negative and (8.23) becomes the critical one of the three conditions.

We now derive the recursion formula (8.4) below and use it to show that (8.23) holds when $r > 0$. Using the fact that the binomial coefficient in (8.22) is 0 when $j < k$, we can put (8.22) in the form

(8.3) $$b_{nk} = \left(\frac{1}{r}\right)^k \sum_{j=0}^{\infty} p_{n+1,j+1}\binom{j}{k}\left(1 - \frac{1}{r}\right)^{j-k}.$$

Using the result of replacing k by (j + 1) in (2.2) gives

(8.31) $$b_{nk} = nb_{n-1,k} + \left(\frac{1}{r}\right)^k \sum_{j=0}^{\infty} p_{n,j}\binom{j}{k}\left(1 - \frac{1}{r}\right)^{j-k}$$

and hence

(8.32) $$b_{nk} = nb_{n-1,k} + \left(\frac{1}{r}\right)^k \sum_{j=0}^{\infty} p_{n,j+1}\binom{j+1}{j}\left(1 - \frac{1}{r}\right)^{j-k+1}.$$

Replacing the binomial coefficient in (8.32) by the right member of the elementary formula

(8.33) $$\binom{j+1}{k} = \binom{j}{k-1} + \binom{j}{k}$$

and using (8.31) gives the recursion formula

(8.4) $$b_{nk} = \frac{1}{r}b_{n-1,k-1} + \left(n + 1 - \frac{1}{r}\right)b_{n-1,k}$$

for the coefficients b_{nk}; it is valid when $n \geq 1$ and $-\infty < k < \infty$.

Letting

(8.41) $$A_n = \frac{1}{(n+1)!}\sum_{k=0}^{\infty} |b_{nk}|,$$

we see from (8.4) that

(8.42) $$A_n \leq \frac{|r^{-1}| + |n + 1 - r^{-1}|}{n+1} A_{n-1}.$$

Supposing now that r is real and $r > 0$, we see from (8.42) that if $n + 1 > r^{-1}$, then $A_n \leq A_{n-1}$. This implies that (8.23) holds when $r > 0$.

We now derive some formulas which show that (8.25) holds if and only if $\Re r > 0$. Letting

(8.5) $$x = 1 - r^{-1},$$

we start with (8.22) and obtain

$$(8.51)\qquad b_{nk} = \left(\frac{1}{r}\right)^k \frac{1}{k!}\frac{d^k}{dx^k}\sum_{j=k}^{n} p_{n+1,j+1}\, x^j$$

$$= \left(\frac{1}{r}\right)^k \frac{1}{k!}\frac{d^k}{dx^k}\sum_{j=0}^{n} p_{n+1,j+1}\, x^j = \left(\frac{1}{r}\right)^k \frac{1}{k!}\frac{d^k}{dx^k}\frac{1}{x}\sum_{j=1}^{n+1} p_{n+1,j}\, x^j.$$

Use of (1.2) and (1.3) then gives

$$(8.52)\qquad b_{nk} = \left(\frac{1}{r}\right)^k \frac{1}{k!}\phi_n^{(k)}(x),$$

where

$$(8.53)\qquad \phi_n(x) = (x+1)(x+2)\cdots(x+n).$$

Thus, as $n \to \infty$,

$$(8.54)\qquad b_{n,0} = \phi_n(x) = \frac{(n+1)!}{x!}\,\frac{(n+1+x-1)!}{(n+1)!} = \frac{(n+1)!}{x!}\, n^{x-1}[1+o(1)],$$

and we see that $b_{n,0} = o[(n+1)!]$ if and only if $\Re(x-1) < 0$ and hence $\Re r > 0$. Let

$$(8.6)\qquad \psi_n(x) = \frac{1}{x+1} + \frac{1}{x+2} + \cdots + \frac{1}{x+n}.$$

Then

$$(8.61)\qquad \phi_n'(x) = \phi_n(x)\,\psi_n(x),$$

$$(8.62)\qquad \phi_n''(x) = \phi_n(x)\{[\psi_n(x)]^2 + \psi_n'(x)\},$$

$$(8.63)\qquad \phi_n'''(x) = \phi_n(x)\{[\psi_n(x)]^3 + 3\psi_n(x)\,\psi_n'(x) + \psi_n'(x)\,\psi_n''(x)\},$$

and it follows easily by induction that, for each $k = 1, 2, 3, \cdots$, the derivative $\phi_n^{(k)}(x)$ is the product of $\phi_n(x)$ and a polynomial of degree k in $\psi_n(x)$ and its derivatives. Since

$$(8.64)\qquad \phi_n(x) = O[(n+1)!\, n^{x-1}], \qquad \psi_n(x) = O(\log n),$$

and $\psi^{(j)}(x) = O(1)$ when $j \geq 1$, it follows with the aid of (8.52) that

$$(8.7)\qquad b_{nk} = O\{(n+1)!\, n^{x-1}(\log n)^k\}.$$

This implies the conclusion (8.25) when $\Re(x-1) < 0$ and hence when $\Re r > 0$. Thus we have completed the proof that $L \supset E_r$ when $r > 0$.

It remains to prove that L does not include E_r when r is not both real and positive. In case $\Re r \leq 0$ we already have the conclusion because, as we saw in connection with (8.54), the condition $b_{n,0} = o[(n+1)!]$ fails to hold in this case. To complete the proof, it would be sufficient to prove the fact that if r is not real but has a positive real part, then (8.23) fails to hold. It is, however, more informative to give an indirect proof of this fact by proving the following. If r is a complex number which is not real, then there exist some values of z for which the geometric series $1 + z + z^2 + \cdots$ is evaluable E_r but is not evaluable L. As we see from (3.1) and (8.1), the series $1 + z + z^2 + \cdots$ is evaluable E_r to the value $1/(1 - z)$ if and only if the sequence $1, z, z^2, z^3, \cdots$ is evaluable E_r to 0. But from (8.1) we see that the E_r transform $S_n(z)$ of this sequence is

$$S_n(z) = (1 - r + rz)^n. \tag{8.8}$$

Thus the set Z_r of values of z for which the series $1 + z + z^2 + \cdots$ is evaluable E_r to $1/(1 - z)$ is the set for which

$$|1 - r + rz| < 1, \tag{8.81}$$

that is,

$$\left|z - \left(1 - \frac{1}{r}\right)\right| < \left|\frac{1}{r}\right|. \tag{8.82}$$

The set Z_r is therefore the set of points z in the complex plane which lie inside the circle which has its center at the point $(1 - r^{-1})$ and which passes through the point 1. When r is not real, the center of this circle does not lie on the real axis, and the interior of the circle therefore contains some points z for which $\Re z > 1$. For these values of z the series $1 + z + z^2 + \cdots$ is evaluable E_r but, as we see from Section 3, the series is not evaluable L. This establishes the conclusions.

In order to be able to point out the connection between [5] and this section, we derive an additional formula. Starting with (8.52) and using Taylor's formula, (8.5), and (1.2) we obtain, when $n > 1$,

$$\sum_{k=0}^{n} b_{nk} t^k = \sum_{k=0}^{\infty} \frac{\phi_n^{(k)}(x)}{k!}\left(\frac{t}{r}\right)^k \tag{8.9}$$

$$= \phi_n\left(x + \frac{t}{r}\right) = \phi_n\left(\frac{t - 1}{r} + 1\right) = p_n\left(\frac{t - 1 + 2r}{r}\right)$$

and hence

$$\sum_{k=0}^{n} b_{nk} t^k = \sum_{k=1}^{n} p_{nk}\left(\frac{t - 1 + 2r}{r}\right)^k. \tag{8.91}$$

This formula does not seem to be particularly useful when $r \neq 1/2$, but when $r = 1/2$ and $n \geq 1$ it shows that $b_{nk} = 2^k p_{nk}$ and hence that (8.21) reduces to the formula

$$\sigma_{n+1} = \frac{1}{(n + 1)!} \sum_{k=1}^{n} 2^k p_{nk} S_k. \tag{8.92}$$

In [5], attention is confined to the case in which $r = 1/2$ and E_r reduces to the particular one of the methods E_r used by Euler. The formula (8.92) is derived and is used to show that $L \supset E_{1/2}$.

9. INTEGRAL FORMULAS FOR $L\{u_0 + u_1 + \cdots\}$

Let $u_0 + u_1 + \cdots$ be a given series and suppose there exists a positive constant R_1 such that the series in

$$f(t) = \sum_{k=0}^{\infty} \frac{u_k}{k!} t^k \tag{9.1}$$

converges when $|t| < R_1$. In case $u_0 + u_1 + \cdots$ is evaluable L, this hypothesis holds because (5.41) holds and the series in (9.1) converges when $|t| < \log 2$. Let the right member of the formula

$$f(t) = \left\{ \sum_{k=0}^{\infty} \frac{u_k}{k!} t^k \right\}^* \tag{9.11}$$

denote the analytic extension, along radial lines from the origin, of the function $f(t)$ defined by (9.1) when $|t| < R_1$. Thus the statement that the right member of (9.11) exists when $t > 0$ is equivalent to the statement that the function $f(t)$ defined by (9.1) when $|t| < R_1$ has an extension which is analytic over the real half-line $t \geq 0$ and which is therefore analytic over an open set containing this real half-line. The hypothesis that the series in (9.1) converges when $|t| < R_1$ implies the existence of a positive constant R such that the series in

$$F(z) = \sum_{k=0}^{\infty} \frac{u_k}{k!} [-\log(1 - z)]^k \tag{9.2}$$

converges when $|z| < R$. (The logarithm in (9.2) is determined unambiguously by the elementary formula $-\log(1 - z) = z + z^2/2 + \cdots$.) Let the right member of the formula

$$F(z) = \left\{ \sum_{k=0}^{\infty} \frac{u_k}{k!} [-\log(1 - z)]^k \right\}^* \tag{9.21}$$

denote the analytic extension, along radial lines from the origin, of the function $F(z)$ defined by (9.2) when $|z| < R$. The function $F(z)$ defined by (9.2) when $|z| < R$ is analytic when $|z| < R$, and hence

$$F(z) = \sum_{n=0}^{\infty} \frac{F^{(n)}(0)}{n!} z^n \tag{9.3}$$

when $|z| < R$. Differentiating (9.2) shows that the formula

$$F^{(n)}(z) = \frac{1}{(1 - z)^n} \sum_{j=1}^{n} p_{nj} \sum_{k=j}^{\infty} \frac{u_k}{(k - j)!} [-\log(1 - z)]^{k-j} \tag{9.31}$$

is valid when $n = 1$. Suppose that (9.31) holds for a given n; considering the right side of (9.31) as the product of $(1 - z)^{-n}$ and another function, we differentiate this product and use the recursion formula (2.2) to obtain the result of replacing n by

$n + 1$ in (9.31). Thus it is proved by induction that (9.31) holds when $n \geq 1$. From (9.2), (9.31), and (9.3) we obtain

$$F(z) = u_0 + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{p_{nj} u_j}{n!} \right) z^n \tag{9.32}$$

when $|z| < R$. It follows that

$$\int_0^r F(z)\,dz = r \left\{ u_0 + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{p_{nj} u_j}{(n+1)!} \right) r^n \right\} \tag{9.33}$$

when $0 < r < R$.

We now suppose that the given series $u_0 + u_1 + \cdots$ is evaluable L. Then, as we see from (6.41), the series in the formal statement

$$\int_0^1 F(z)\,dz = u_0 + \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{p_{nj} u_j}{(n+1)!} \tag{9.34}$$

must be convergent. This implies that the series in (9.32) must converge and define $F(z)$ when $|z| < 1$, that the series in (9.33) must converge when $0 < r < 1$, and that, because of Abel's theorem on power series, we can let $r \to 1$ in (9.33) to obtain (9.34). Thus, by use of (6.41), we see that the formula

$$L\left\{ \sum_{k=0}^{\infty} u_k \right\} = \int_0^1 F(z)\,dz \tag{9.4}$$

is valid whenever the series $u_0 + u_1 + \cdots$ is evaluable L, that is, whenever the left member exists. Using (9.21), we see that the formula

$$L\left\{ \sum_{k=0}^{\infty} u_k \right\} = \int_0^1 \left\{ \sum_{k=0}^{\infty} \frac{u_k}{k!} [-\log(1-z)]^k \right\}^* dz \tag{9.41}$$

is valid whenever the left member exists. Changing the variable of integration in (9.41) by setting $t = -\log(1 - z)$ shows that *the formula*

$$L\left\{ \sum_{k=0}^{\infty} u_k \right\} = \int_0^{\infty} e^{-t} \left\{ \sum_{k=0}^{\infty} \frac{u_k}{k!} t^k \right\}^* dt \tag{9.42}$$

is valid whenever the left member exists. It would not be correct to remove the star superscripts from the right members of (9.41) and (9.42); for while the hypothesis that the left members of (9.41) and (9.42) exist implies that the series in (9.32) converges when $|z| < 1$, it does not imply that the series in (9.2) converges when $|z| < 1$, and it does not imply that the series in (9.1) converges when $t > 0$.

With the aid of (9.33), we see that the formula

$$(9.5)\qquad \lim_{r\to 1}\left\{u_0+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\frac{p_{nj}u_j}{(n+1)!}\right)r^n\right\}=\int_0^{\infty}e^{-t}\left\{\sum_{k=0}^{\infty}\frac{u_k}{k!}t^k\right\}^* dt$$

is correct whenever the left member exists and whenever the right member exists.

10. THE SERIES $B_1/1 + B_2/2 + \cdots$

Letting $B_0, B_1, B_2, \cdots$ denote the Bernoulli numbers for which $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_3 = 0, \cdots$, *we show in this section that*

$$(10.1)\qquad L\left\{\frac{B_1}{1}+\frac{B_2}{2}+\frac{B_3}{3}+\cdots\right\}=\gamma-1,$$

where γ *is Euler's constant.* Starting with the familiar formula

$$(10.2)\qquad \frac{t}{e^t-1}=\sum_{k=0}^{\infty}\frac{B_k}{k!}t^k,$$

which is valid when $|t| < 2\pi$, and letting $u_k = B_{k+1}/(k+1)$ and

$$(10.21)\qquad f(t)=\frac{1}{e^t-1}-\frac{1}{t},\qquad F(z)=\frac{1}{z}+\frac{1}{\log(1-z)}-1,$$

we see that (9.1) is valid when $|t| < 2\pi$, that (9.32) is valid when $|z| < 1$, and that (9.33) is valid when $0 < r < 1$. If the given series is evaluable L, we can use (9.4) to obtain

$$(10.3)\qquad L\left\{\sum_{k=0}^{\infty}\frac{B_{k+1}}{k+1}\right\}=\int_0^1\left[\frac{1}{z}+\frac{1}{\log(1-z)}\right]dz-1,$$

and the desired conclusion (10.1) will follow from a known integral formula which appears even in short tables of definite integrals.

It remains to be shown that the given series is evaluable L, and we do this by means of some formulas obtained in [5]. Starting with the elementary formula

$$(10.4)\qquad \frac{1-e^{-t}}{t}=1-\frac{t}{2!}+\frac{t^2}{3!}-\cdots,$$

putting $u = (-1)^k/(k+1)$ and $t = -\log(1-z)$, and using formulas from (9.1) to (9.32) gives

$$(10.41)\qquad \frac{-z}{\log(1-z)}=1+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\frac{p_{nj}(-1)^j}{j+1}\right)\frac{z^n}{n!}=1+\sum_{n=1}^{\infty}\left(\int_0^1\sum_{j=1}^{n}p_{nj}(-y)^j\,dy\right)\frac{z^n}{n!}.$$

Using (1.2) and (1.3) then gives

(10.42)
$$\frac{z}{\log(1-z)} = -1 + \sum_{n=1}^{\infty} \left(\int_0^1 y(1-y)(2-y)\cdots(n-1-y)\,dy \right) \frac{z^n}{n!}.$$

Since (9.32) is valid when $u_j = B_{j+1}/(j+1)$, $|z| < 1$, and $F(z)$ is defined by (10.21), we have also

(10.43)
$$\frac{z}{\log(1-z)} = -1 + \frac{z}{2} + \sum_{n=1}^{\infty} \left(\sum_{j=1}^{n} \frac{B_{j+1}}{j+1} p_{nj} \right) \frac{z^{n+1}}{n!}$$

when $|z| < 1$. Comparing coefficients in (10.42) and (10.43), we obtain

(10.44)
$$\sum_{j=1}^{n} \frac{B_{j+1}}{j+1} p_{n,j} = \frac{1}{n+1} \int_0^1 y(1-y)(2-y)\cdots(n-y)\,dy.$$

Hence

(10.45)
$$0 < \sum_{j=1}^{n} \frac{B_{j+1}}{j+1} p_{nj} < \frac{n!}{n+1},$$

and

(10.46)
$$\sum_{n=1}^{\infty} \frac{1}{(n+1)!} \sum_{j=1}^{n} \frac{B_{j+1}}{j+1} p_{nj} < \sum_{n=1}^{\infty} \frac{1}{(n+1)^2}.$$

As we see from (6.41), this implies that the given series is evaluable L, and (10.1) is proved.

Starting with (10.42), transposing the first term on the right side, dividing by z, and integrating over $0 < z < 1$ gives the interesting formula

(10.5)
$$\gamma = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} \frac{1}{n!} \int_0^1 y(1-y)(2-y)\cdots(n-y)\,dy$$

for Euler's constant.

11. RELATIONS BETWEEN L AND THE BOREL METHODS

A series $u_0 + u_1 + \cdots$ and its sequence $s_0, s_1, \cdots$ of partial sums are said to be evaluable to $B\{s_0, s_1, \cdots\}$ by the Borel (or Borel exponential) method B if the series in

$$(11.1) \qquad B\{s_0, s_1, \cdots\} = \lim_{t\to\infty} e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} s_k$$

converges for each $t > 0$ and (11.1) holds. A series $u_0 + u_1 + \cdots$ and its sequence $s_0, s_1, \cdots$ of partial sums are said to be evaluable to $BI\{u_0 + u_1 + \cdots\}$ by the Borel integral method BI if, in the right member of the statement

$$(11.2) \qquad BI\{u_0 + u_1 + \cdots\} = \int_0^\infty e^{-t} \sum_{k=0}^{\infty} \frac{u_k}{k!} t^k dt,$$

the series converges for each $t > 0$ and the integral exists in the sense of $\lim_{x\to\infty} \int_0^x$.

In addition to these well-known methods which are treated in the book [4] of Hardy, we introduce a generalized Borel integral method BI* which is related to BI in the same way that a familiar generalized Abel method A* is related to the Abel method A. We shall say that the series $u_0 + u_1 + \cdots$ is evaluable to $BI^*\{u_0 + u_1 + \cdots\}$ by the method BI* if

$$(11.3) \qquad BI^*\{u_0 + u_1 + \cdots\} = \int_0^\infty e^{-t} \left\{ \sum_{k=0}^{\infty} \frac{u_k}{k!} t^k \right\}^* dt.$$

In (11.3), the star superscript has the same significance as in (9.11) and (9.5). For validity of (11.3), it is necessary that the power series in (11.3) have a positive radius of convergence, but it is not necessary that the radius of convergence be infinite. The formula (11.3) can be very convenient. If, for example, $B_0, B_1, \cdots$ denote the Bernoulli numbers, then (11.3), (10.2), and formulas found in tables of definite integrals give

$$(11.31) \qquad BI^*\left\{ \sum_{k=0}^{\infty} B_k \right\} = \int_0^\infty e^{-t} \frac{t}{e^t - 1} dt = \int_0^\infty \frac{t}{e^t - 1} dt - \int_0^\infty te^{-t} dt = \frac{\pi^2}{6} - 1.$$

The series $B_0 + B_1 + \cdots$ is, however, not evaluable BI, because the series in (10.2) diverges when $|t| > 2\pi$. For the exponential integral series (4.1), we see that the formula

$$\mathrm{BI}^*\left\{\sum_{k=0}^{\infty} \frac{(-1)^k k!}{z^k}\right\} = \int_0^\infty e^{-t}\left\{\sum_{k=0}^{\infty}\left(\frac{-t}{z}\right)^k\right\}^* dt = z\int_0^\infty \frac{e^{-t}}{z+t}dt \tag{11.32}$$

is valid for each z for which the last integral exists, and hence for each $z \neq 0$ if we allow Cauchy principal value integrals to occur in the definition (11.3) of BI*. There is no z for which this series is evaluable BI.

From (9.5) and (11.3) we see that the formula

$$\mathrm{BI}^*\{u_0 + u_1 + \cdots\} = \lim_{r\to 1}\left\{u_0 + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{n} \frac{p_{nj}u_j}{(n+1)!}\right)r^n\right\} \tag{11.4}$$

is correct whenever at least one of the two members exists. The quantity in braces in the right member of (11.4) is the Abel power series transform of the L series transform of the series $u_0 + u_1 + \cdots$. Hence, in standard terminology, if the right side of (11.4) exists then the series $u_0 + u_1 + \cdots$ is said to be evaluable AL to that value. Thus the statement involving (11.4) means that BI* and AL are equivalent methods for evaluation of series. Since A is regular, it follows that BI* $\supset$ L. Since BI* $\supset$ BI, *it follows from* (11.4) *that the formula*

$$\mathrm{BI}\{u_0 + u_1 + \cdots\} = \lim_{r\to 1}\left\{u_0 + \sum_{n=1}^{\infty}\left(\sum_{j=1}^{\infty} \frac{p_{nj}u_j}{(n+1)!}\right)r^n\right\} \tag{11.5}$$

is correct whenever the left side exists, and that BI $\subset$ AL. Since A is regular, it follows from (11.5) that the formula

$$\mathrm{BI}\{u_0 + u_1 + \cdots\} = \mathrm{L}\{u_0 + u_1 + \cdots\} \tag{11.6}$$

is valid whenever both members exist. This means that BI *and* L *are consistent*, and since BI $\supset$ B it follows that B *and* L *are consistent.*

Our work does not show whether the relations L $\supset$ B and L $\supset$ BI are valid: In [5], a formula equivalent to (11.5) is obtained and some conclusions are drawn from it. There is a lack of precision in the treatment, and it is erroneously asserted that B and BI are equivalent methods for evaluation of series. Correct relations between B and BI are given in [4].

REFERENCES

1. R. P. Agnew, *Euler transformations*, Amer. J. Math., 66 (1944), 313-338.
2. L. Euler, *De seriebus divergentibus*, Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae, Tomus 5 ad Annum 1754-1755 (1760), 205-237.
3. I. J. Good, *Note on the summation of a classical divergent series*, J. London Math. Soc. 16 (1941), 180-182.
4. G. H. Hardy, *Divergent series*, Oxford, 1949.
5. A. V. Lototsky (or Lotockii), *On a linear transformation of sequences and series*, Ivanov. Gos. Ped. Inst. Uc. Zap. Fiz.-Mat. Nauki 4 (1953), 61-91 (in Russian).

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