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The Borel Complexity of Isomorphism for Theories with Many Types

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Abstract During the Notre Dame workshop on Vaught's Conjecture, Hjorth and Kechris asked which Borel equivalence relations can arise as the isomorphism relation for countable models of a first-order theory. In particular, they asked if the isomorphism relation can be essentially countable but not tame. We show this is not possible if the theory has uncountably many types.

1 Preliminaries

We begin by recalling the basic definitions and background material. Suppose E_i is an equivalence relation on a standard Borel space X_i for $i = 1, 2$. We say that E_1 is *Borel reducible* to E_2 if there is a Borel measurable $f : X_1 \to X_2$ such that $x E_1 y$ if and only if $f(x) E_2 f(y)$ for all $x, y \in X_1$.

An equivalence relation is *countable* if every equivalence class is countable and *essentially countable* if it is Borel reducible to a countable equivalence relation. If $E_1 \leq_B E_2$ and $E_2 \leq_B E_1$, we write $E_1 \sim_B E_2$. A Borel equivalence relation E on X is *tame* if there is a Polish space Y and a Borel measurable $f : X \to Y$ such that $x E y$ if and only if $f(x) = f(y)$.

If $\mathcal{L}$ is a countable first-order language we let $X_{\mathcal{L}}$ be the Polish space of $\mathcal{L}$-structures with universe $\mathbb{N}$. For $\sigma \in \mathcal{L}_{\omega_1,\omega}$ let $\mathrm{Mod}(\sigma)$ be the Borel set of $\mathcal{M} \in X_{\mathcal{L}}$ with $\mathcal{M} \models \sigma$ and let $\cong_\sigma$ be the equivalence relation of isomorphism on $\mathrm{Mod}(\sigma)$. In general, $\cong_\sigma$ is Σ^1_1 but need not be Borel.

The following well-known theorem shows that $\cong_\sigma$ can be $\sim_B$ to any countable Borel equivalence relation.

Theorem 1.1 *Let $\hat{E}$ be a countable Borel equivalence relation on a Polish space X. There is $\tau \in \mathcal{L}_{\omega_1,\omega}$ such that $\hat{E} \sim_B \cong_\tau$.*

We give a sketch of the proof.[1]

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Sketch of proof Since $\hat{E}$ is a countable Borel equivalence relation, by the Feldman-Moore Theorem [2], $\hat{E}$ is the orbit equivalence relation of a Borel action of a countable discrete group G on a Polish space X. Let $E(G, 2^\omega)$ be the orbit equivalence relation of the natural shift action of G on $(2^\omega)^G$. There is a Borel reduction of $\hat{E}$ to $E(G, 2^\omega)$ (see [1], 1.2).

Let $\mathcal{L} = \{\hat{g} : g \in G\} \cup \{U_n : n \in \omega\}$ where each $\hat{g}$ is a unary function symbol and U_n is a unary predicate. Let σ be an $\mathcal{L}_{\omega_1,\omega}$-sentence such that $\mathcal{M} \models \sigma$ if and only if $\alpha(g, x) = \hat{g}(x)$ is a faithful and transitive action of G on $\mathcal{M}$.

For $\mathcal{M} \models \sigma$, we associate $f_\mathcal{M} \in (2^\omega)^G$ where $f_\mathcal{M}(g)(n) = 1$ if and only if $\mathcal{M} \models U_n(\hat{g}(0))$. This is a Borel map from $\mathrm{Mod}(\sigma)$ to $(2^\omega)^G$ and $\mathcal{M} \cong \mathcal{N}$ if and only if $f_\mathcal{M} E(G, 2^\omega) f_\mathcal{N}$. Thus $\cong_\sigma \leq_B E(G, 2^\omega)$.

Fix $g_0, g_1, \ldots$ an enumeration of G with $g_0 = 1$. For $h \in (2^\omega)^G$, construct $\mathcal{M}_h \models \sigma$ such that $\hat{g}_i(0) = i$ and $\mathcal{M}_h \models U_n(i)$ if and only if $h(i)(n) = 1$. Then $f_{\mathcal{M}_h} = h$ and $\mathcal{M}_{h_1} \cong \mathcal{M}_{h_2}$ if and only if $h_1 E(G, 2^\omega) h_2$. Thus $\cong_\sigma \sim_B E(2^\omega, G)$.

We need one lemma to complete the proof. Recall that for E an equivalence relation on X, and $A \subseteq X$, the *saturation* of A is $[A]_E = \{x : \exists y \in A\ xEy\}$.

Lemma 1.2 *Suppose E, F, G are Borel equivalence relations on Polish spaces X, Y, Z; E and G are countable; and $f : X \to Y$ and $g : Y \to Z$ are Borel reductions of E to F and F to G, respectively. Then $[f(X)]_F$ is Borel and $E \sim_B F|[f(X)]_F$.*

Proof We make repeated use of the Lusin-Novikov Uniformization Theorem (see [4], 18.10) for Borel sets with countable sections. The map $g \circ f : X \to Z$ is countable-to-one; thus $g(f(X))$ is Borel. Since G is countable, $[g(f(X))]_G$ is Borel. Since $[(f(X))]_F = g^{-1}([g(f(X))]_G)$, $[f(X)]_F$ is Borel.

Since E is countable, the set $A = \{(x, y) : x \in X, f(x)Fy\}$ is Borel with countable sections in X and the projection of A to Y is $[f(X)]_F$. Thus there is a Borel $h : [f(X)]_F \to X$ such that $(h(y), y) \in A$ for all $y \in [f(X)]_F$. Let $y_1, y_2 \in [f(X)]_F$. Since $f(h(y_i))Fy_i$ and f is a reduction of E to F,

$$y_1 F y_2 \Leftrightarrow f(h(y_1)) F f(h(y_2)) \Leftrightarrow h(y_1) E h(y_2).$$

Thus h is a reduction of $F|[f(X)]_F$ to E. □

We can now finish the proof of the theorem. Since $\hat{E} \leq_B E(2^\omega, G) \sim_B \cong_\sigma$, we can apply Lemma 1.2 to find a Borel $C \subseteq \mathrm{Mod}(\sigma)$ that is isomorphism invariant and $\hat{E} \sim_B \cong_\sigma |C$. By Lopez-Escobar (see [4], 16.8) C is $\mathrm{Mod}(\tau)$ for some $\mathcal{L}_{\omega_1,\omega}$-sentence τ and $\hat{E} \sim_B \cong_\tau$. □

Hjorth and Kechris asked if the same result is true for first-order theories. It is easy to give examples of theories T with continuum many countable models where $\cong_T$ is tame. For example, let T be the theory of an equivalence relation with infinitely many classes where each class contains an algebraically closed field. Then models are determined up to isomorphism by the set of transcendence degrees of the equivalence classes. Are there any first-order theories T with $\cong_T$ essentially countable but not tame? We show that any such theory must have few types.

Let $\mathcal{C}$ be the Cantor space 2^ω. Fix $\langle\ ,\ \rangle : \omega^2 \to \omega$ a pairing function. For $x \in \mathcal{C}$, let $A_x \subseteq \mathcal{C}$ be the set $\{x_1, x_2, \ldots\}$ where $x_i(j) = x(\langle i, j\rangle)$. We say $x F_2\, y$ if and only if $A_x = A_y$.

The equivalence relation F_2 is not essentially countable. See [3], Exercise 2.64.

Theorem 1.3 *Let T be a first-order theory in a countable language where the type space $S(T)$ is uncountable. Then $F_2 \leq_B \cong_T$. Thus $\cong_T$ is not essentially countable.*

This result is not surprising as the set of realized types is a natural invariant of a model.

2 Theories with Many Types

Suppose T is a first-order theory in a countable language with $S(T)$ uncountable. We can find $\mathcal{T}$ a perfect tree of types in $S(T)$. Choose $r_T \in \mathcal{C}$ such that $\mathcal{L}, T, \mathcal{T} \leq_T r_T$. Using $\mathcal{T}$ we can code elements of the Cantor space as types.

Lemma 2.1 *There is continuous one-to-one map $\tau : \mathcal{C} \to S(T)$ such that $\tau(x) \leq_T x \oplus r_T$ and $x \leq_T \tau(x) \oplus r_T$, where $x \oplus y$ is the join of x and y.*

2.1 Scott sets

Definition 2.2 We say that $S \subseteq \mathcal{C}$ is a *Scott set*

(i) if $x \in S$ and $y \leq_T x$, then $x \in S$;
(ii) if $x, y \in S$, then $x \oplus y \in S$;
(iii) if $x \in S$ codes an infinite subtree t of $2^{<\omega}$, then there is $y \in S$ an infinite path through t.

We need a refinement of recursively saturated models.

Definition 2.3 Let T be a complete first-order theory in a countable language and let S be a Scott set with $T \in S$. We say that $\mathcal{M} \models T$ is *S-saturated* if

(i) for all $x \in S$ if $a_1, \ldots, a_n \in M$ and $p(v, a_1, \ldots, a_n)$ is a partial type recursive in some $x \in S$, then p is realized in $\mathcal{M}$;
(ii) $\mathrm{tp}(a_1, \ldots, a_n) \in S$ for all $a_1, \ldots, a_n \in M$.

S-saturated models were studied in papers of Knight and Nadel ([5], [6]) and Wilmers [8]. The next result summarizes the facts that we will need.

Proposition 2.4 *Let T be a first-order theory in a countable language. Let S be a countable Scott set with $T \in S$.*

(i) *There is a countable S-saturated model of T.*
(ii) *S-saturated models of T are ω-homogeneous.*
(iii) *Any two countable S-saturated models of T are isomorphic.*

The proof of (i) is a Henkin argument where one alternates trying to realize types in S, witnessing existential sentences and making sure that for all Henkin constants $c_1, \ldots, c_n$, $\mathrm{tp}(c_1, \ldots, c_n) \in S$. The uniformity of this construction (and the uniqueness of S-saturated models) allows us to prove the following.

Lemma 2.5 *Let $\mathcal{S} = \{x \in \mathcal{C} : A_x$ is a Scott set$\}$. Then $\mathcal{S}$ is Borel and there is a Borel $\mu : \mathcal{S} \to \mathrm{Mod}(T)$ such that $\mu(x)$ is the A_x-saturated model of T.*

In fact, by the main result of [7], if $T \in A_x$, then an A_x-saturated model can be constructed recursively in x.

2.2 Borel closure systems Let $\mathcal{F} = \{f_1, f_2, \ldots\}$ be a countable set of Borel functions $f_i : \mathcal{C}^{m_i} \to \mathcal{C}$. For $A \subseteq \mathcal{C}$, let $\mathrm{cl}_{\mathcal{F}}(A)$ be the closure of A under the functions in $\mathcal{F}$.

Definition 2.6 We say that $I \subseteq \mathcal{C}$ is $\mathcal{F}$*-independent* if

$$\mathrm{cl}_{\mathcal{F}}(A) \cap I = A$$

for all $A \subseteq \mathcal{I}$.

Lemma 2.7 *For any countable set of Borel functions $\mathcal{F}$, there is a perfect $\mathcal{F}$-independent set.*

Proof If P is a perfect set of suitably generic Cohen reals, then P is $\mathcal{F}$-independent. □

Let $\mathcal{F}$ be the following collection of functions:

(i) $j(x, y) = x \oplus y$;

(ii)

$$f_e(x) = \begin{cases} \varphi_e^x & \text{if } \varphi_e^x \text{ is a total function in } \mathcal{C} \\ x & \text{otherwise} \end{cases} \quad \text{for } e = 0, 1, \ldots.$$

(iii) $t(x)$ = leftmost path in the tree coded by x if x codes a tree on $2^{<\omega}$ and $t(x) = x$ otherwise.

(iv) the constant function $x \mapsto r_T$.

If $A \subseteq \mathcal{C}$, then $\mathrm{cl}_{\mathcal{F}}(A)$ is a Scott set containing $A \cup \{r_T\}$. The construction of closures is uniform.

Lemma 2.8 *There is a Borel $\nu : \mathcal{C} \to \mathcal{C}$ such that $A_{\nu(x)}$ is the $\mathcal{F}$-closure of A_x for all $x \in \mathcal{C}$. In particular, $A_{\nu(x)}$ is a Scott set containing $A_x \cup \{r_T\}$.*

2.3 Proof of Theorem 1.3 Let P be a perfect $\mathcal{F}$-independent set with $\rho : \mathcal{C} \to P$ a homeomorphism. There is a Borel $\rho^* : \mathcal{C} \to \mathcal{C}$ such that $A_{\rho^*(x)} = \rho(A_x)$.

For $A \subseteq \mathcal{C}$ countable, let $S_A = \mathrm{cl}_{\mathcal{F}}(\rho(A))$ and let $\mathcal{M}_A$ be the unique countable S_A-saturated model of T.

Lemma 2.9 *If $A \neq B$, then $\mathcal{M}_A \not\cong \mathcal{M}_B$.*

Proof Suppose $x \in A \setminus B$. Then $\rho(x) \in S_A$, but, since P is $\mathcal{F}$-independent, $\rho(x) \notin S_B$. Since $r_T \in S_A \cap S_B$, it follows from Lemma 2.1, that $\tau(\rho(x)) \in S(T) \cap S_A$ and $\tau(\rho(x)) \notin S(T) \cap S_B$. The type $\tau(\rho(x))$ is realized in $\mathcal{M}_A$ but not $\mathcal{M}_B$. Thus $\mathcal{M}_A \not\cong \mathcal{M}_B$. □

We now build our reduction of F_2 to $\cong_T$. For $x \in \mathcal{C}$, let $g(x) = \mu(\nu(\rho^*(x))$. Unraveling the definition,

(i) $A_{\rho^*(x)} = \rho(A_x)$;

(ii) $A_{\nu(\rho^*(x))} = \mathrm{cl}_{\mathcal{F}}(\rho(A_x))$;

(iii) $g(x)$ is a code for a $\mathrm{cl}_{\mathcal{F}}(\rho(A_x))$-saturated model of T.

Since S-saturated models are unique, if $x F_2 y$, then $g(x) \cong g(y)$. By Lemma 2.9, if $x \not F_2 y$, then $g(x) \not\cong g(y)$. Thus $F_2 \leq_B \cong_T$.

Remarks Let $\mathrm{hMod}(T) \subseteq \mathrm{Mod}(T)$ be the codes for homogeneous models of T. Countable homogeneous models are determined by the types they realize over $\varnothing$.

Corollary 2.10 *Suppose $S(T)$ is uncountable; then $F_2 \sim_B \cong_T \mid \mathrm{hMod}(T)$.*

Problem Find a first-order theory T where $\cong_T$ is not tame and $F_2 \not\leq_B \cong_T$. Note that counterexamples to Vaught's conjecture have this property. Is there an ω-stable theory with this property?

Note

1. I am grateful to Kechris for showing me this argument and the referee for pointing out an error in my original presentation.

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