Notre Dame Journal of Formal Logic
Volume 46, Number 4, 2005

Automorphisms of Homogeneous Structures

A. Ivanov

Abstract We give an example of a simple ω-categorical theory such that for any finite set of parameters the corresponding constant expansion does not satisfy the PAPA. We describe a wide class of homogeneous structures with generic automorphisms and show that some natural reducts of our example belong to this class.

1 Introduction

Let T be a first-order theory over a countable language. It is assumed that models of T are elementary substructures of a sufficiently saturated monster model $\mathbb{C}$. We use A, B, C to denote subsets of $\mathbb{C}$, assumed to be much smaller than $\mathbb{C}$.

Property PAPA is defined as follows. Whenever $(A_1, \sigma_1) \subseteq (A_2, \sigma_2), (A_3, \sigma_3)$, where A_1, A_2, A_3 are algebraically closed (in T^{eq}) substructures of $\mathbb{C}^{\mathrm{eq}}$ and $\sigma_i \in \mathrm{Aut}(A_i)$, there exists an eq-algebraically closed substructure B of $\mathbb{C}^{\mathrm{eq}}$, $\sigma \in \mathrm{Aut}(B)$, and automorphism-preserving embeddings $(A_2, \sigma_2) \to (B, \sigma)$ and $(A_3, \sigma_3) \to (B, \sigma)$ which agree on A_1. We say that the PAPA holds for finite structures if it holds under the additional assumption that A_1, A_2, A_3 are acl-generated by finite sets.

The PAPA is assumed in a construction from Chatzidakis and Pillay [1] which assigns a model companion T_A (if it exists) to the theory of all structures (M, σ) ($\sigma \in \mathrm{Aut}(M)$) for models M of T. The theory ACFA of algebraically closed fields with a generic automorphism (Chatzidakis and Hrushovski [2]) is an example of such T_A.

Below we give an example of a simple ω-categorical theory such that for any finite set of parameters A, the corresponding constant expansion does not satisfy the PAPA. The question if such an example exists was formulated by Kikyo at Simplton 2002 (Lumini).

Received July 25, 2004; accepted January 3, 2005; printed December 8, 2005
2000 Mathematics Subject Classification: Primary, 03C45; Secondary, 03C10
Keywords: simple theories, homogeneous structures, generic automorphisms

Our example has some additional interesting properties. We will see that for any tuple $\bar{a}$ the stabilizer of $\bar{a}$ in Aut(M) does not have generic automorphisms. On the other hand, the example is a reduct of a structure constructed by the Fraissé method. The corresponding class K of finite structures satisfies property FAP defined as follows.

Let $\mathcal{L}$ be a countable relational language, and K a class of finite $\mathcal{L}$-structures. We say that K has the *free amalgamation property* (FAP), if given $A, B_1, B_2 \in K$ and embeddings $f_i : A \to B_i$, there is $C \in K$ containing B_1 and an embedding $h : B_2 \to C$, such that $h(f_2(x)) = f_1(x)$ for all $x \in A$, $h(B_2) \cup B_1 = C$, $h(B_2) \cap B_1 = f_1(A)$ and no tuple of $B_1 \cup h(B_2)$ which satisfies a relation of $\mathcal{L}$ meets both $h(B_2) \setminus B_1$ and $B_1 \setminus h(B_2)$. (It is clear that the embeddings f_i define C uniquely.)

The second result of the paper states that if the class of finite substructures of a countable homogeneous structure M has the FAP, then M has generic automorphisms. As a consequence we obtain that all finite reducts (= reducts to finite languages) of the theory without the PAPA presented in the paper have local generics.

Below we use the following notation. If $\bar{a}$ is a tuple from a model M, we often abuse notation by writing $\bar{a} \in M$. If $r(\bar{x})$ is a type, we denote by $r(M)$ the set of tuples from M which realize r. For any structure M and $A \subseteq M$, define $\mathrm{Aut}(M/A)$ to be the group of automorphisms of M which fix A pointwise.

2 Example

The example is based on some reducts of the random graph (Thomas [4]). This idea is not new; it was applied in examples of theories without the PAPA (in their basic language) found by Tsuboi and anounced at Simplton 2002.

Let $\mathcal{L}_0 = \{R_1, R_2, \ldots, R_n, \ldots\}$ be a relational language, where each R_i has arity $2i$. The structure M_0 is built by a Fraissé construction, so we first specify a class K of finite $\mathcal{L}_0$-structures. In each $C \in K$ each relation R_n determines a symmetric graph on the set (denoted by $\binom{C}{n}$) of unordered n-element subsets of C. It is easy to see that K is a free amalgamation class: given $A, B_1, B_2 \in K$ with $B_1 \cap B_2 = A$, define $C \in K$ as $B_1 \cup B_2$, such that no tuple $\bar{c}_1\bar{c}_2 \in C$ which satisfies R_n meets both $B_2 \setminus B_1$ and $B_1 \setminus B_2$. Let M_0 be the corresponding universal homogeneous structure. Note that $\mathrm{Th}(M_0)$ is ω-categorical and admits elimination of quantifiers.

Claim 2.1 *The theory of M_0 is supersimple of SU-rank 1.*

Proof of Claim 2.1 Let $\varphi(\bar{x}, \bar{b})$, $|\bar{x}| = l$, be a quantifier-free formula and $(\bar{b}_i : i < \omega)$ be an indiscernible sequence of $tp(\bar{b})$. We may assume that $\varphi(\bar{x}, \bar{b})$ implies $\bar{x} \cap \bar{b} = \varnothing$. Then any set $B_n = \bigcup\{\bar{b}_i : i \leq n\}$ can be extended by a tuple $c_1, \ldots, c_l$ satisfying all $\varphi(\bar{x}, \bar{b}_i)$, $i \leq n$. Since M_0 is universal homogeneous, the tuple $\bar{c}$ can be found in M_0. We now see that any nonalgebraic type does not divide over $\varnothing$; thus M_0 is simple of SU-rank 1. $\square$

Let M be the reduct of M_0 to the language $\mathcal{L} = \{T_1, \ldots, T_n, \ldots\}$ of $3n$-relations where a triple of n-element sets C_1, C_2, and C_3 satisfies T_n if and only if it contains 0 or 3 edges with respect to R_n. By Thomas's classification of reducts of the random graph [4] any automorphism of the relation of T_n is an automorphism of R_n or maps R_n onto its complement.

Claim 2.2 *Let R'_n be the relation which is the complement of R_n on the set of all pairs $C \neq D$ with $C, D \in \binom{M}{n}$: $(C, D) \in R_n \leftrightarrow (C, D) \notin R'_n$. Then the structure M_0 is isomorphic with $M'_0 = (M, R_1, \ldots, R_{n-1}, R'_n, R_{n+1}, \ldots)$ and the structure M is the reduct of M'_0 obtained by the same definition as M is obtained from M_0.*

Proof of Claim 2.2 To prove the claim it suffices to note that any structure from K is embeddable into M'_0 and for every pair $A < A'$ from K with $A' \cap M'_0 = A$ there exists an A-embedding of A' into M'_0. Both conditions follow from the fact that M_0 is universal homogeneous. The second statement of the claim is obvious. □

By Claim 2.1 the structure M is supersimple. It is easy to see (by genericity) that for all $\bar{a}$ and A, $tp(\bar{a}/A) \vdash tp(\bar{a}/acl^{\mathrm{eq}}(A))$ with respect to both $\mathrm{Th}(M_0)$ and $\mathrm{Th}(M)$. Universality of M_0 also implies triviality of acl in $\mathrm{Th}(M)$ and that for every finite $A \subset M$ any automorphism of A uniquely determines its extension to $acl^{\mathrm{eq}}(A)$; this allows us to avoid acl in the PAPA.

Let $\bar{a} = (a_1, \ldots, a_n) \subset M$. Since M_0 is universal homogeneous, there are elements $b, c_1, d_1, \ldots, c_4, d_4 \in M_0 \setminus \bar{a}$ so that

$$M_0 \models \bigwedge_{i=3,4} (tp(c_i c_{7-i}/\bar{a}) = tp(bc_i/\bar{a}) = tp(bd_i/\bar{a}))\wedge$$

$$[tp(c_1c_3/\bar{a}) = tp(c_3c_4/\bar{a}) = tp(c_2c_4/\bar{a}) = tp(d_3d_4/\bar{a}) =$$

$$= tp(d_1d_3/\bar{a}) = tp(d_2d_4/\bar{a}) = tp(c_4d_4/\bar{a}) \neq tp(c_1c_2/\bar{a})]\wedge$$

$$\bigwedge_{i=3,4} \bigwedge_{j=3,4} (tp(c_4d_4/\bar{a}) = tp(c_id_j/\bar{a}) = tp(d_jc_i/\bar{a}) = tp(c_id_{5-j}/\bar{a}))\wedge$$

$$\bigwedge\{tp(c_1c_2/\bar{a}) = tp(uv/\bar{a}) : \{u, v\} \text{ is a two-element subset of}$$

$$\{b, c_1, d_1, \ldots, c_4, d_4\} \text{ not arising in the equalities above }\}$$

$$\bigwedge\{tp(U/\bar{a}') = tp(V/\bar{a}') : U \text{ and } V \text{ are subsets of} \{b, c_1, d_1, \ldots, c_4, d_4\}$$

$$\text{of the same size and } \bar{a}' \text{ is a proper subtuple of } \bar{a}\}.$$

(We suggest that the reader draw a graph on $\{b, c_1, d_1, \ldots, c_4, d_4\}$ where c_3, c_4 forms an edge (corresponding to R_{n+1}).) It is clear that the pairs c_1c_2 and c_3c_4 have the same type over $\bar{a}$ with respect to the sublanguage $\{R_{n+2}, R_{n+3}, \ldots\}$. We also assume that for any pair C_1, C_2 with $C_1 \cup C_2 = c_3c_4\bar{a}$, the corresponding pair C'_1 and C'_2 (obtained by replacing c_i by c_{5-i}) satisfies R_{n+1} if and only if $(C_1, C_2) \notin R_{n+1}$. The same property is assumed for d_1, d_2, d_3, d_4.

Let R'_{n+1} be obtained from R_{n+1} as in Claim 2.2 (by complementing). Since the structure $M'_0 = (M, R_1, \ldots, R_n, R'_{n+1}, R_{n+2}, \ldots)$ is isomorphic with M_0, the type of c_3c_4 over $\bar{a}$ in M_0 is the same as the type of c_1c_2 over $\bar{a}$ in M'_0 (by our construction mutually corresponding subtuples from $c_3c_4\bar{a}$ and $c_1c_2\bar{a}$ satisfy the same relations). Applying the last statement of Claim 2.2 we see that the type of c_3c_4 over $\bar{a}$ in M is the same as the type of c_1c_2 over $\bar{a}$ in M.

Since M_0 is universal homogeneous the configuration above can be chosen so that there is an automorphism β of M fixing $\bar{a}b$ and taking $c_1c_2c_3c_4d_1d_2d_3d_4$ to $c_4c_3c_1c_2d_4d_3d_1d_2$ (then in our picture edges are replaced by non-edges). *We claim that there is no graph R on the set of $(n+1)$-element subsets of $\bar{a}bc_1d_1 \ldots c_4d_4$ which induces T_{n+1} and is preserved by β.* To see this suppose that R is such a relation and R coincides with R_{n+1} on $\bar{a}b, \bar{a}c_3, \bar{a}c_4$ (the opposite case is similar). Then any pair from $\bar{a}b, \bar{a}c_1, \bar{a}c_2$ forms an R-edge and there are no other edges in $\bar{a}b, \bar{a}c_1, \bar{a}c_2, \bar{a}c_3, \bar{a}c_4$ (by the T_{n+1}-structure on this set). Since the triple $\bar{a}b, \bar{a}c_1, \bar{a}d_1$ belongs to T_{n+1} and $\bar{a}b, \bar{a}c_1$ forms an R-edge, we see that $\bar{a}c_1, \bar{a}d_1$ and any pair from the triple $\bar{a}b, \bar{a}d_1, \bar{a}d_2$ (and from the triple $\bar{a}b, \bar{a}d_3, \bar{a}d_4$) forms an R-edge.

Since any triple of the form $\bar{a}b, \bar{a}c_i, \bar{a}d_j$, $i, j \in \{1, 2\}$, belongs to T_{n+1}, any pair of the form $\bar{a}c_i, \bar{a}d_j$, $i, j \in \{1, 2\}$, forms an R-edge. Since β preserves R we also have that any pair of the form $\bar{a}c_i, \bar{a}d_j$, $i, j \in \{3, 4\}$, forms an R-edge.

Since any triple of the form $\bar{a}d_i, \bar{a}d_{i+2}, \bar{a}c_j$, $i \in \{1, 2\}$, $j \in \{3, 4\}$, belongs to T_{n+1} and any pair of the form $\bar{a}c_i, \bar{a}d_j$, $i, j \in \{3, 4\}$, belongs to R, the pairs $\bar{a}d_1, \bar{a}d_3$ and $\bar{a}d_2, \bar{a}d_4$ form R-edges. This implies that the triples $\bar{a}b, \bar{a}d_1, \bar{a}d_3$ and $\bar{a}b, \bar{a}d_2, \bar{a}d_4$ belong to T_{n+1}. This is a contradiction with the definition of our configuration.

Let α be the identity on some $bb'\bar{a}$ and β be defined on $\bar{a}bc_1d_1 \dots c_4d_4$ as above. Let (C, γ), $\gamma \in \mathrm{Aut}(C)$, be an amalgamation of α and β and C be embeddable into M over $\bar{a}b$. As we noted above any automorphism of (C, T_{n+1}) extending α must preserve R_{n+1}. On the other hand, any automorphism of (C, T_{n+1}) extending β must map R_{n+1} onto R'_{n+1}. This shows that α and β cannot be amalgamated. Thus the PAPA does not hold.

3 Generic Automorphisms of Finitely Homogeneous Structures

For a countable structure M we study $\mathrm{Aut}(M)$ as a closed subgroup of $\mathrm{Sym}(\omega)$. Here we consider $\mathrm{Sym}(\omega)$ as a complete metric space by defining $d(g, h) = \Sigma\{2^{-n} : g(n) \neq h(n) \text{ or } g^{-1}(n) \neq h^{-1}(n)\}$. An automorphism $\alpha \in \mathrm{Aut}(M)$ is *generic* if its conjugacy class in $\mathrm{Aut}(M)$ is comeager. If the conjugacy class is comeager in some nonempty open set, then α is called locally generic. We will consider only countable universal homogeneous structures. There are a number of results stating the existence of generic automorphisms for such structures. We mention the papers Herwig and Lascar [3] and Truss [5].

It is easy to see that the example of Section 2 does not have local generics. Indeed, for any $\bar{a} \in M$ and sufficiently large n the subgroup of $\mathrm{Aut}(M/\bar{a})$ consisting of automorphisms preserving R_n is normal in $\mathrm{Aut}(M/\bar{a})$ of index 2. This shows that $\mathrm{Aut}(M/\bar{a})$ does not have generic automorphisms. Since cosets of such subgroups form a base of the space $\mathrm{Aut}(M)$, we see that $\mathrm{Aut}(M)$ does not have local generics.

Nevertheless, the following theorem implies that finite reducts of that structure have local generics (see the discussion after the proof).

Theorem 3.1 *Let M be a universal homogeneous structure over a countable relational language $\mathcal{L}$, and suppose that the class K of finite structures which embed into M has the* FAP. *Then M has generic automorphisms.*

Proof Truss has shown in [5] that if the set $\mathbf{P}$ of all finite partial maps in the structure M extendible to automorphisms of M contains a cofinal subset $\mathbf{P}'$ closed under conjugacy and having the amalgamation property and the joint embedding property then there is a generic automorphism.

Let K be the class of all finite structures embeddable into M. Let K_a be the class of all pairs (A, α) where $A \in K$ and α is an isomorphism between substructures of A extendible to an automorphism of M. Let $K_{\mathrm{per}} \subset K_a$ consist of pairs where α is an automorphism of A. We want to show that K_{per} is cofinal in K_a and satisfies the joint embedding and the amalgamation properties. Then we can apply the theorem of Truss formulated in the previous paragraph.

We start with cofinality. Let $(A_0, \alpha_0) \in K_a$ and $D_0 = \mathrm{Dom}(\alpha_0)$. Let (A_1, α_1, D_1) be a copy of (A_0, α_0, D_0). Identifying each $d' \in D_1$ with $\alpha_0(d)$ for the corresponding $d \in D_0$ (where the original isomorphism between A_0 and A_1 maps d to d') consider $A_0 \cup A_1$ as the result of free amalgamation. Then α_0 and α_1 agree on $D_0 \cap D_1$ (under the identification above α_1 acts on this intersection as $\alpha_0(d) \to \alpha_0^2(d)$). In $A_0 \cup A_1$ the map α_0 can be naturally extended to $\alpha_0' : A_0 \to A_1$ (by the isomorphism between A_0 and A_1) so that A_1 becomes the range of the map. Note that for any $a \in A_0 \setminus D_0$, $\alpha_0'(a) \in A_1 \setminus A_0$.

Taking the next copy (A_2, α_2, D_2) and naturally identifying D_2 with $\alpha_1(D_1)$ define the corresponding free amalgamation. In the obtained structure we can now extend the map α_0' to a map $A_0 \cup A_1 \to A_1 \cup A_2$ so that it agrees with α_1 on D_1 (and α_0 on D_0). Continuing this procedure we eventually find a number n, structure $C \in K$ ($C = A_0 \cup \cdots \cup A_n$) and a partial isomorphism $\gamma : A_0 \cup \cdots \cup A_{n-1} \to A_1 \cup \cdots \cup A_n$ such that A_0 is contained in $\mathrm{Dom}(\gamma^n)$ as a substructure, γ extends all α_i, $i \leq n$, and for any $d \in A_0 \cap A_n$, $\gamma^n(d) = d$ (then α_0 and α_n agree on $D_0 \cap D_n$). We can arrange that $A_0 \cap A_n$ and $A_1 \cap A_n$ are the same and consist of all $d \in D_0$ such that for some i, $\gamma^i(d) = d$. Let β be the isomorphism from A_0 onto A_n induced by γ^n.

Let $C' = A_0' \cup \cdots \cup A_n'$ be a copy of $C = A_0 \cup \cdots \cup A_n$ and γ' be the corresponding copy of γ. The isomorphism β naturally induces isomorphisms $\beta_1 : A_0' \to A_n$ and $\beta_2 : A_n' \to A_0$. Moreover, $\beta_1 \cup \beta_2$ is an isomorphism between substructures of C' and C. By free amalgamation we obtain a structure defined on $C' \cup C$. Note that the partial maps induced by γ and γ' on $A_0 \cup A_n$ and $A_0' \cup A_n'$, respectively, agree under the identification $\beta_1 \cup \beta_2$ (this follows from the property that α_0 and α_n agree on $D_0 \cap D_n$ and that $\gamma(A_0) \cap A_n = A_0 \cap A_n$). So γ and γ' define an automorphism δ on the obtained structure.

We now verify the amalgamation (the joint embedding) property in K_{per}. Let $(A, \alpha), (B, \beta), (C, \gamma) \in K_{\mathrm{per}}$, $A = B \cap C$ and α agree with β and γ on A. Then $\beta \cup \gamma$ is a permutation of the structure $B \cup C$ obtained by free amalgamation. Since the relations of the structure are just the unions of the corresponding relations from B and C, we see that $\beta \cup \gamma$ is an automorphism.

As a result K_{per} satisfies all the conditions of Theorem 2.1 from [5]. □

Let M be the structure from Section 2. If M' is the reduct of M to $\{T_1, \ldots, T_n\}$, then for any $2n$-element tuple $\bar{a}$ the automorphisms of $(M', \bar{a})$ coincide with automorphisms of $(M', R_1, \ldots, R_n, \bar{a})$ (they cannot map R_i to its complement). Since the latter structure has the FAP, by Theorem 3.1 the structure $(M', \bar{a})$ has generics.

References

[1] Chatzidakis, Z., and A. Pillay, "Generic structures and simple theories," *Annals of Pure and Applied Logic*, vol. 95 (1998), pp. 71–92. Zbl 0929.03043. MR 1650667. 419

[2] Chatzidakis, Z., and E. Hrushovski, "Model theory of difference fields," *Transactions of the American Mathematical Society*, vol. 351 (1999), pp. 2997–3071. Zbl 0922.03054. MR 1652269. 419

[3] Herwig, B., and D. Lascar, “Extending partial automorphisms and the profinite topology on free groups,” *Transactions of the American Mathematical Society*, vol. 352 (2000), pp. 1985–2021. Zbl 0947.20018. MR 1621745. 422

[4] Thomas, S., “Reducts of the random graph,” *The Journal of Symbolic Logic*, vol. 56 (1991), pp. 176–81. Zbl 0743.05049. MR 1131738. 420

[5] Truss, J. K., “Generic automorphisms of homogeneous structures,” *Proceedings of the London Mathematical Society. Third Series*, vol. 65 (1992), pp. 121–41. Zbl 0723.20001. MR 1162490. 422, 423

Acknowledgments

The research was supported by KBN grant 2 P03A 007 19. The research was finished when the author held a visiting position at Institute of Mathematics of Polish Academy of Sciences. The referee has informed the author that E. Hrushovski and H. Kikyo have found another example of a simple ω-categorical theory such that for any set of parameters the corresponding constant expansion does not satisfy the PAPA.

Institute of Mathematics
Wrocław University
pl Grunwaldzki 2/4
50-384 Wrocław
POLAND
ivanov@math.uni.wroc.pl