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A Question about Suslin Trees and the Weak Square Hierarchy

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Abstract We present a question about Suslin trees and the weak square hierarchy which was contributed to the list of open problems of the BIRS workshop.

The topic of Magidor's lectures at the BIRS workshop was the following hierarchy of weak square principles which was introduced in Schimmerling [5]. Let κ and λ be cardinals. We say that a sequence $\langle \mathcal{F}_\nu \mid \kappa < \nu < \kappa^+ \rangle$ is a $\square_\kappa^{<\lambda}$ sequence if and only if, whenever ν is a limit ordinal and $\kappa < \nu < \kappa^+$,

1. $1 \leq |\mathcal{F}_\nu| < \lambda$ and
2. if $C \in \mathcal{F}_\nu$, then
   (a) C is a closed unbounded subset of ν,
   (b) C has order type $\leq \kappa$, and
   (c) if μ is a limit point of C, then $C \cap \mu \in \mathcal{F}_\mu$.

One says that $\square_\kappa^{<\lambda}$ *holds* if and only if there exists a $\square_\kappa^{<\lambda}$ sequence. By '$\square_\kappa^{\lambda}$' we mean $\square_\kappa^{<\lambda^+}$. The relations to Jensen's principles are $\square_\kappa \equiv \square_\kappa^1$ and $\square_\kappa^* \equiv \square_\kappa^\kappa$. As was clear from Magidor's talks, it has been a fruitful project to search for the least λ such that $\square_\kappa^{<\lambda}$ does not suffice in the traditional applications of $\square_\kappa$. The following problem is motivated by Jensen's theorem which says that if κ is a singular cardinal, then $\mathrm{GCH} + \square_\kappa$ implies the existence of a κ^+-Suslin tree. (See Devlin's textbook [3].) A simple modification of Jensen's proof uses only $\square_\kappa^{<\omega}$.

Problem 1 Let κ be a singular cardinal and assume GCH. Find the least λ such that $\square_\kappa^{<\lambda}$ does not imply the existence of a κ^+-Suslin tree. In particular, is the theory

$$\mathrm{ZFC} + \mathrm{GCH} + \mathrm{SH}_{\aleph_\omega+1} + \square_{\aleph_\omega}^*$$

consistent relative to large cardinals? Is

$$\mathrm{ZFC} + \mathrm{GCH} + \mathrm{SH}_{\aleph_\omega+1} + \square_{\aleph_\omega}^{\omega}$$

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consistent relative to large cardinals?

Here are some models that might be relevant to this problem. Magidor and Shelah [4] showed that if there is a 2-huge cardinal, then there is a forcing extension in which there is no $\aleph_{\omega+1}$-Aronszajn tree. Cummings, Foreman, and Magidor [1] show that if κ is the limit of ω supercompact cardinals, then, in a forcing extension, $\kappa = \aleph_\omega$, $\square^\omega_\kappa$ holds and every stationary subset of κ^+ reflects. A different model of Cummings, Foreman, and Magidor [1] shows that $\square^*_{\aleph_\omega}$ is consistent with the strongest possible simultaneous reflection principle for stationary subsets of $\aleph_{\omega+1}$. The result of Cummings and Schimmerling [2]—that Prikry forcing at a measurable cardinal κ adds $\square^\omega_\kappa$—might also be relevant.

References

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