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A Negation-free Proof of Cantor's Theorem

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Abstract We construct a novel proof of Cantor's theorem in set theory.

1 Introduction

It has been an important endeavor in logic and mathematics to determine whether the proofs of basic theorems can be reformulated without invoking certain kinds of logical primitives. A striking instance of such a reformulation was provided by Yablo's paradox ([1], [2], [3]) which demonstrated that it was possible to construct paradoxical sentences in logic without the need to invoke either direct or indirect self-reference. We carve another such path in this paper by constructing a new proof of Cantor's theorem in set theory without explicitly invoking the negation operation.

Every proof of Cantor's theorem—that for no set there is a function mapping its members onto all its subsets—constructs a subset which is *leftover* by any onto mapping from any set to its powerset. The traditional diagonalization proof involves an explicit invocation of the negation operation in order to define the *leftover* subset. Our new proof of Cantor's theorem, though it uses diagonalization at a certain level, constructs the *leftover* subset without explicitly invoking the negation operation. Further, our proof can also be rewritten in a form which uses negation explicitly.

2 Yablo's Paradox

Yablo's paradox ([1], [2], [3]) is a non-self-referential Liar's paradox. Before the formulation of Yablo's paradox, all known paradoxes in logic seemed to require circularity in an unavoidable way. Each of them used either direct self-reference or indirect looplike self-reference. Yablo's paradox demonstrated that self-reference was not a necessary condition for the construction of paradoxical sentences. It can be stated as follows.

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Consider the following infinite sequence of sentences S_i where the indices 'i, j, k' range over natural numbers:

$$(S_i): \text{ For all } j > i,\ S_j \text{ is untrue.}$$

Note that, in the above sequence of statements, each statement quantifies only over statements which occur later in the sequence. Now suppose S_k is true for some k. Then S_{k+1} is false, and so are all subsequent statements. As all subsequent statements are false, S_{k+1} is true, which is a contradiction. So S_k is false for all k. Looking at any particular i, this in turn means that S_i in fact holds, which is a contradiction.

3 New Proof of Cantor's Theorem

Theorem 3.1 (Cantor's Theorem) *The cardinality of the power set of a set X exceeds the cardinality of X, and in particular the continuum is uncountable.*

Proof Let X be any set, and $P(X)$ denote the power set of X. Assume that it is possible to define a one-to-one mapping $M : X \leftrightarrow P(X)$.

Define $s_0, s_1, s_2, \ldots$ to be a trace, where the first element of the trace is any arbitrary $s_0 \in X$, and all further elements s_j, where $j > 0$, of the trace are such that $s_j \in M(s_{j-1})$. Define $t \in X$ to be a simple element, if all possible traces beginning with t terminate. Note that a trace $s_0, s_1, s_2, \ldots, s_f$ terminates at s_f if $M(s_f)$ is the empty set. Define $N = \{t \in X \mid t \text{ is a simple element}\}$.

The set N, which is a subset of X, cannot lie in the range of M. Suppose there exists an $n \in X$ such that $M(n) = N$, then n should be a simple element since all traces beginning with element n also terminate. Thus $n \in N$, but then n is no longer a simple element, since not all traces beginning with n are terminating traces (e.g., "n, n, n, ..." is one such nonterminating trace). Thus the set N is out of the range of mapping M. □

In the above novel proof of Cantor's theorem, the construction of the set N does not require explicit negation. This is unlike the standard diagonalization proof which invokes the operation of negation in order to construct the *leftover* subset. Of course, one could say the same of the usual diagonal argument showing that the reals are uncountable, because the process of swapping 0s and 1s in the binary expansion of a real number need not be thought of as negation. Also, it is possible to rewrite the above proof in a slightly different way and bring out negation explicitly. This can be done by changing the definition of a simple element as one whose traces do not continue indefinitely.

References

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