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Sets without Subsets of Higher Many-One Degree

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Abstract Previously, both Soare and Simpson considered sets without subsets of higher $\leq_T$-degree. Cintioli and Silvestri, for a reducibility $\leq_r$, define the concept of a $\leq_r$-introimmune set. For the most common reducibilities $\leq_r$, a set does not contain subsets of higher $\leq_r$-degree if and only if it is $\leq_r$-introimmune. In this paper we consider $\leq_m$-introimmune and $\leq_m^P$-introimmune sets and examine how structurally easy such sets can be. In other words we ask, What is the smallest class of the Kleene's Hierarchy containing $\leq_r$-introimmune sets for $\leq_r \in \{\leq_m, \leq_m^P\}$? We answer the question by proving the existence of $\leq_m$-introimmune sets in the class Π_1^0, bi-$\leq_m$-introimmune sets in Δ_2^0, and bi-$\leq_m^P$-introimmune sets in Δ_1^0.

1 Introduction

Cintioli and Silvestri [6] considered infinite sets of words which do not contain subsets of higher polynomial-time Turing degree. Let us briefly look at the motivation for considering such sets. (The reader who is not familiar with the basic concepts of Structural Complexity Theory and Recursion Theory might want to refer to the following: Balcázar et al. [4]; Bovet and Crescenzi [5]; Garey and Johnson [8]; Rogers [13]; Odifreddi [12].)

In the fifties and sixties, Recursion Theory studied the notion of introreducibility, a notion arising from questions surrounding retraceable sets. A set $\{a_0 < a_1 < \cdots\}$ is *retraceable* if there exists a partial recursive function φ such that $\varphi(a_0) = a_0$ and $\varphi(a_{n+1}) = a_n$, for every $n \geq 0$. A set is said to be *introreducible* if it is Turing reducible to every infinite subset (see Jockush [9]). In Dekker and Myhill [7] it was proven that any retraceable set is introreducible. At this point, Jockush [9] asked just which properties of retraceable sets extend to the introreducible ones. For example, it was proven that for every set A, if both A and its complement are retraceable, then A is recursive (Mansfield, see Odifreddi [12]).

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Question 1.1 *Does this property hold for introreducible sets too? That is, is it true that if a set and its complement are introreducible, then the set is recursive?*

This question was answered in the affirmative by Seetapun and Slaman [15].

In [6] the authors take under consideration the above question about the introreducible sets and formulate its counterpart in Structural Complexity Theory. This is based on the notion of polynomial-time Turing reducibility. A set X is polynomial-time Turing reducible to the set Y, in short $X \leq_T^P Y$, if and only if there exists a deterministic oracle Turing machine M that, with oracle Y, runs in polynomial time and accepts X (equivalently, $X \in \mathbf{P}^Y$; Garey and Johnson [8] and Ladner et al. [11]). Likewise, a set X is nondeterministically polynomial-time reducible to the set Y, in short $X \leq_T^{NP} Y$, if and only if there exists a nondeterministic oracle Turing machine M that, with oracle Y, runs in polynomial time and accepts X (equivalently, $X \in \mathbf{NP}^Y$; [8] and [11]).

Definition 1.2 ([6]) A set is polynomial-time Turing introreducible, in short $\leq_T^P$-introreducible, if it is polynomial-time Turing reducible to every infinite subset.

Question 1.3 *If we assume that a language and its complement are $\leq_T^P$-introreducible, is the language in* $\mathbf{P}$*?*

In contrast with the corresponding problem of Recursion Theory, this problem is easily solvable. Indeed a stronger fact holds.

Theorem 1.4 ([6]) *A language is $\leq_T^P$-introreducible if and only if it is in* $\mathbf{P}$.

Sketch of Proof The proof is practically as follows: given a language $L \notin \mathbf{P}$ over the alphabet $\{0, 1\}$, we construct by diagonalization an infinite subset B of L such that $L \not\leq_T^P B$. Furthermore, the constructed set B is not in $\mathbf{P}$. Besides this, the construction does not provide any further structural information on B. So it could be that $L \not\leq_T^P B$ because B is structurally lacking. This raises the problem of the existence of non-$\leq_T^P$-introreducible sets L witnessed by subsets structurally rich with respect to L. For example, by subsets B of L with $L \leq_T^{NP} B$. The existence of such sets easily follows from the existence of an oracle C such that $\mathbf{P}^C \neq \mathbf{NP}^C$ (see Baker et al. [2]). Then we ask whether *every* non-$\leq_T^P$-introreducible language has an infinite subset B with $L \not\leq_T^P B$ and $L \leq_T^{NP} B$. The answer to this question is no and can be deduced from the existence of sets without subsets of higher Turing degree. $\square$

Theorem 1.5 (Soare [17]) *There exists a set of natural numbers A such that, for every $B \subseteq A$ with $|A - B| = \infty$, $A \not\leq_T B$.*

Corollary 1.6 *There exists a non-$\leq_T^P$-introreducible language L such that for every $B \subseteq L$, $L \leq_T^P B \Leftrightarrow L \leq_T^{NP} B$.*

Proof Take the language L over the alphabet $\{0, 1\}$ whose words are the numerals encoding the numbers in A of Theorem 1.5, and let B be a subset of L. If $|L - B| = \infty$ then $L \not\leq_T^{NP} B$ as $L \not\leq_T B$, hence $L \not\leq_T^P B$. If $|L - B| < \infty$ then, of course, $L \leq_T^{NP} B$ and $L \leq_T^P B$. $\square$

In general, if $\leq_r$ is any kind of reducibility between sets or languages, we say that an infinite set L is $\leq_r$-*introimmune* if, for every $B \subseteq L$ with $|L-B| = \infty$, it is $L \not\leq_r B$.

If both L and $\overline{L}$ are $\leq_r$-introimmune, then L is bi-$\leq_r$-introimmune. Essentially, for the most common reducibilities $\leq_r$, an $\leq_r$-introimmune set does not contain sets of higher $\leq_r$-degree. This concludes our look at the motivation for considering infinite sets of words which do not contain subsets of higher polynomial-time Turing degree.

Now, any $\leq_m^P$-introimmune language is clearly not in **P**. Actually, it is even **P**-immune [6]. Likewise, any $\leq_m^{NP}$-introimmune language is **NP**-immune, and any $\leq_m$-introimmune language is immune, hence not recursively enumerable [6]. Here, $\leq_m^{NP}$ is the nondeterministic version of $\leq_m^P$.

However, given a generic reducibility $\leq_r$ we ask how structurally easy an $\leq_r$-introimmune or bi-$\leq_r$-introimmune set can be. With the locution "how structurally easy can a set be" we mean principally, but not exclusively, What is the smallest class of one of the main well-known hierarchies, such as the Polynomial Time Hierarchy **PH** or the Kleene's Hierarchy, containing such sets?

This question is not entirely new. For example, [17] proved that $\leq_T$-introimmune sets are Δ_0^1-hard with respect to every arithmetic reducibility, and Simpson [16] proved that $\leq_T$-introimmune sets are Δ_1^1-hard with respect to Turing reducibility $\leq_T$. Here Δ_0^1 and Δ_1^1 denote, respectively, the classes of arithmetical and hyperarithmetical sets.

In particular, our question includes the following question: Are there $\leq_m^P$-introimmune sets that belong to **PH**, or that are at least recursive? The existence of an $\leq_m^P$-introimmune set in **PH** implies $\mathbf{P} \neq \mathbf{PH}$. Hence, even if such set exists, its existence is hard to prove.

Coming back to the general question, we consider the many-one reducibilities $\leq_m$ and $\leq_m^P$. As previously stated, an $\leq_m$-introimmune set cannot be recursively enumerable. In this paper we prove the existence of an $\leq_m$-introimmune set in the class Π_1^0, the existence of a bi-$\leq_m$-introimmune set in the class Δ_2^0, and the existence of a bi-$\leq_m^P$-introimmune set in the class Δ_1^0 of the Kleene's Hierarchy.

These are the best possible results relative to the research of the smallest class of Kleene's Hierarchy containing such sets. Should the former result suggest that there exists an $\leq_m^P$-introimmune language in $\Pi_1^p = \text{co-}\mathbf{NP}$? We expect the answer to be no.

2 Sets without Subsets of Higher Many-One Degree

We are going to exibit an $\leq_m$-introimmune set in the class Π_1^0. This result is structurally optimal relative to the classification in the Kleene's Hierarchy, since any $\leq_m$-introimmune set is immune and, hence, not recursively enumerable. We start by proving that every $\leq_m$-introimmune set is immune.

Proposition 2.1 *Every $\leq_m$-introimmune set is immune.*

Proof Let $A \subseteq \mathbb{N}$ be an $\leq_m$-introimmune set and let us suppose that A is not immune: this means that there exists an infinite recursive subset of A. Let W be such an infinite recursive subset and let $W' := A - W$: it is $W' \subseteq A$ and $|A - W'| = |W| = \infty$. We show that $A \leq_m W'$, contradicting the assumption that A is $\leq_m$-introimmune. Let us define $f : \mathbb{N} \to \mathbb{N}$ as follows: for all x,

$$f(x) := \begin{cases} b & \text{if } x \in W, \\ x & \text{if } x \notin W, \end{cases}$$

where b is a fixed element in W'. Now, f is recursive and, for every x, $x \in A \Leftrightarrow f(x) \in W'$. $\square$

Observe that the converse of Proposition 2.1 is not true. In fact, take A immune. Then $A \oplus A = \{2x : x \in A\} \cup \{2x+1 : x \in A\}$ is an immune, not $\leq_m$-introimmune, set because the recursive function $f(2x) = 2x$, and $f(2x+1) = 2x$ $\leq_m$-reduces $A \oplus A$ to its co-infinite subset $2A = \{2x : x \in A\}$. Moreover, both A and $A \oplus A$ are in the same $\leq_m$-degree. However, it is easy to see that the $\leq_m$-introimmune property is recursively invariant. We will now prove the existence of an $\leq_m$-introimmune set in Π^0_1, first proving that every cohesive set is $\leq_m$-introimmune.

Lemma 2.2 *Every cohesive set is $\leq_m$-introimmune.*

Proof Let A be a cohesive set, let $B \subseteq A$ with $|A - B| = \infty$, and let us suppose that $A \leq_m B$ via a recursive function f. First of all, for every $x \in A$ the set $\{x, f(x), f(f(x)), \ldots\}$ is finite, otherwise A would have an infinite recursively enumerable subset, contradicting the cohesiveness of A. Let $W := \{x : x \neq f(x)\}$. W is recursively enumerable, so $A \cap W$ is finite or $A \cap \overline{W}$ is finite. The former is impossible as B is a co-infinite subset of A, hence $A \cap \overline{W}$ is finite. By the immunity of A, for any $x \in B$ there can be at most finitely many elements y such that $f(y) = x$. Therefore, A contains infinitely many finite disjoint orbits given by f. Moreover, such orbits are all contained in B. Let $W' := \{x : (\exists n \geq 2) f^{(n)}(x) = x\}$, where $f^{(n)}(x)$ denotes the n-iterate of $f(x)$. W' is recursively enumerable, with $A \cap W'$ infinite and $A \cap \overline{W'}$ infinite, since $A \cap \overline{W'} \supseteq A - B$. But this contradicts the cohesiveness of A. $\square$

We will see later that Lemma 2.2 cannot be reversed.

Corollary 2.3 *There exists an $\leq_m$-introimmune set in Π^0_1.*

Proof Take a maximal set A. By definition A is recursively enumerable and its complement is cohesive. So $\overline{A}$ is $\leq_m$-introimmune and is in Π^0_1. $\square$

We can be more precise on the position of an $\leq_m$-introimmune set inside Π^0_1. First of all, there do not exist $\leq_m$-introimmune sets Π^0_1-complete with respect to $\leq_m$. This is because every Π^0_1-complete set with respect to $\leq_m$ is productive. Every productive set is not immune and so, by Proposition 2.1, are not $\leq_m$-introimmune. Second, there are $\leq_m$-introimmune sets Π^0_1-complete with respect to Turing reducibility $\leq_T$. This follows from the existence of a maximal set which is Σ^0_1-complete with respect to $\leq_T$ (Yates [18]). Finally, not every $\leq_m$-introimmune set in Π^0_1 is complete with respect to $\leq_T$. This follows from the existence of a maximal set which is not complete with respect to $\leq_T$ (Sacks [14]).

We conclude this section showing the existence of bi-$\leq_m$-introimmune sets in Δ^0_2. This is the best possible result, since the class Δ^0_2 is the smallest class of the Kleene's Hierarchy that can contain such sets. In fact, by Proposition 2.1, a bi-$\leq_m$-introimmune set cannot be in $\Sigma^0_1 \cup \Pi^0_1$. By the fact that there cannot be bicohesive sets we conclude that Lemma 2.2 cannot be reversed.

Theorem 2.4 *There exists a bi-$\leq_m$-introimmune set in Δ^0_2.*

Proof Let $\varphi_0, \varphi_1, \ldots$ be a fixed effective enumeration of all the unary computable functions. Let $f : \mathbb{N} \to \mathbb{N}$ be the function defined by $f(0) = 1$ and

$f(n+1) := \max_{u,v\leq n+1}\{f(n), \varphi_u(v)|\varphi_u(v)\downarrow\}+1$, for every $n \in \mathbb{N}$. Then f is an increasing function that satisfies both of the following properties:

i. it is K-recursive, where $K = \{x \in \mathbb{N} : \varphi_x(x)\downarrow\}$, and
ii. for every recursive function φ_n there exists a natural number m such that for every $m' \geq m$, for every $s \leq m'$, $\varphi_n(s) < f(m')$.

We recall here two technical definitions, the former from Balcázar and Schöning [3] and the latter from Kämper [10]. We say that a function f is *1-1 a.e.* if the set $\{(x, y) : x \neq y \wedge f(x) = f(y)\}$ is finite.

Definition 2.5 A set $X \subseteq \mathbb{N}$ is *strongly bi-$\leq_m$-immune* if and only if every $\leq_m$-reduction from X to any set $Y \subseteq \mathbb{N}$ is 1-1 a.e.

Observe that X is strongly bi-$\leq_m$-immune if and only if its complement is also strongly bi-$\leq_m$-immune.

Definition 2.6 Let $g : \mathbb{N} \to \mathbb{N}$ be a function such that for every $n \in \mathbb{N}$, $g(n) > n$, and let $X \subseteq \mathbb{N}$ be an infinite set. Then X has *g-gaps* if and only if for every natural number n there exists a natural number m such that $X \cap \{x \in \mathbb{N} : n+m < x \leq g(n+m)\} = \varnothing$. For every natural number n, we call the set $\{x \in \mathbb{N} : n < x \leq g(n)\}$ a *g-gap*.

We are going to construct, by a so-called finite-extension argument, a set A with the following two properties:

1. A is strongly bi-$\leq_m$-immune, and
2. both A and $\overline{A}$ have f-gaps, where f is the function defined above.

To guarantee the first property (1) we construct A in such a way to satisfy the following requirements R_i, for $i = 0, 1, \ldots$:

> R_i: If $\{(x, y) : x \neq y \wedge \varphi_i(x) = \varphi_i(y)\}$ is infinite, then there exist $u \neq v$ with $\varphi_i(u) = \varphi_i(v)$, $u \in A$, and $v \notin A$.

To guarantee the second property (2), whenever the procedure finds an index that can be diagonalized at some stage n, we insert an f-gap $\{x \in \mathbb{N} : z < x \leq f(z)\}$ into $\overline{A}$ and an f-gap $\{x \in \mathbb{N} : f(z) < x \leq f(f(z))\}$ into A, for an appropriate natural number z. That is, we make sure that $\{x \in \mathbb{N} : z < x \leq f(z)\} \cap \overline{A} = \varnothing$ and $\{x \in \mathbb{N} : f(z) < x \leq f(f(z))\} \cap A = \varnothing$. The construction is by stages. The set D contains the indices that could be diagonalized. For every $z \in \mathbb{N}$ let $\mathrm{GAP}(z, f) := \{x \in \mathbb{N} : z < x \leq f(f(z))\}$. At every stage, if the procedure finds an index that can be diagonalized, then this index is removed from D, the process stops for a moment and tries to insert $\{x \in \mathbb{N} : z < x \leq f(z)\}$ and $\{x \in \mathbb{N} : f(z) < x \leq f(f(z))\}$ into $\overline{A}$ and A, respectively, for some $z \in \mathbb{N}$. The only problem here is that the two f-gaps above could contain a word x for which an index in D can be diagonalized. For this reason, the procedure enters the while loop and looks for a natural number z for which $\mathrm{GAP}(z, f)$ does not have this conflict. Note that the while loop always ends, with D empty if necessary.

Begin Construction

Stage 0. $A := \varnothing$, $D := \{0\}$, $k(1) := 0$.

Stage $n > 0$.
$D := D \cup \{n\}$;
If $\exists i \in D$ and $\exists x < k(n)$ such that $\varphi_i(x) = \varphi_i(k(n))$ **then**
  begin let i_0 be the smallest such index and let x_0 be the smallest such number.
  if $x_0 \notin A$ **then** $A = A \cup \{k(n)\}$; **endif**
  $D := D - \{i_0\}$;
  $z := k(n)$;
  GAP := GAP(z, f);
  while $\exists i \in D$, $\exists x \in$ GAP and $\exists y < x$ such that $\varphi_i(x) = \varphi_i(y)$ **do**
   begin
    let x_0 be the smallest such number x;
    let i_0 be the smallest such index i;
    let $y_0 < x_0$ be the smallest number such that $\varphi_{i_0}(y_0) = \varphi_{i_0}(x_0)$;
    if $y_0 \notin A$ **then** $A := A \cup \{x_0\}$; **endif**
    $D := D - \{i_0\}$;
    $z := x_0$;
    GAP := GAP(z, f)
  endwhile
  $A := A \cup \{x : z < x \leq f(z)\}$; {insert an f-gap into $\overline{A}$};

  $k(n+1) := f(f(z)) + 1$; {insert an f-gap into A};
  end
else
$k(n+1) := k(n) + 1$
endif
Go to stage $n + 1$.
End stage n.

End Construction

Set A is K-recursive, and hence in Δ_2^0. First of all, we observe that there are infinitely many stages at which some index is diagonalized (e.g., all the constant functions occur in the enumeration $\varphi_0, \varphi_1, \ldots$). At every such stage n, the procedure finds a natural number $z \geq k(n)$ and extends both A and $\overline{A}$ in such a way that $\{x : f(z) < x \leq f(f(z))\} \cap A = \varnothing$ and $\{x : z < x \leq f(z)\} \cap \overline{A} = \varnothing$. As $k(0) < k(1) < \cdots$, it follows that both A and $\overline{A}$ have f-gaps.

We show only that A is $\leq_m$-introimmune, since the same argument with A replaced by $\overline{A}$ shows that $\overline{A}$ is also $\leq_m$-introimmune.

Let $B \subseteq A$ with $|A - B| = \infty$, and let φ be a recursive function. If φ is not 1-1 a.e. then φ does not $\leq_m$-reduce A to B, because sooner or later φ will be diagonalized. So let us suppose that φ is 1-1 a.e.

For every $x \in A - B$ it is $\varphi(x) \neq x$ and for every $n \geq 1$ it is $\varphi^{(n)}(x) \neq x$. For every $x \in A - B$ let $S_x = \{x, \varphi(x), \varphi(\varphi(x)), \ldots\}$. φ is 1-1 a.e., hence there are only finitely many $x \in A - B$ with S_x finite. Let $x_0 \in A - B$ with $S_{x_0} = \{x_0, \varphi(x_0), \varphi(\varphi(x_0)), \ldots\}$ infinite. From (ii), and from the fact that A has f-gaps, there exists $m \geq 0$ such that the two conditions below hold:

1. $A \cap \{x \in \mathbb{N} : x_0 + m < x \leq f(x_0 + m)\} = \emptyset$, and
2. $\forall s \leq x_0 + m, \varphi(s) < f(x_0 + m)$.

Then there exists $y \in S_{x_0}$, $y \leq x_0 + m$, with $y \in A$ and $\varphi(y) \notin A$, hence φ does not $\leq_m$-reduce A to B. □

3 Sets without Subsets of Higher Polynomial-Time Many-One Degree

In this section we consider the polynomial-time many-one reducibility $\leq_m^P$ and discover a bi-$\leq_m^P$-introimmune recursive set. Of course, this result is the best possible one inside the Kleene's Hierarchy.

Theorem 3.1 *There exists a bi-$\leq_m^P$-introimmune recursive set.*

Proof Let $T_1, T_2, \ldots$ be a fixed effective enumeration of polynomial-time Turing transducers on the alphabet $\{0, 1\}$ computing every polynomial-time computable function. Without loss of generality, we assume that the time bound of every T_i is the polynomial $p_i(n) = n^i + i$, $i = 1, 2, \ldots$. For every symbol $a \in \{0, 1\}$ and for every positive $n \in \mathbb{N}$, a^n is the word $aa \cdots a$ with a occurring n times, $\{0, 1\}^* = \{\varepsilon, 0, 1, 00, 01, 10, 11, \ldots\}$ is the set of all the words on the alphabet $\{0, 1\}$, and for every $w \in \{0, 1\}^*$ $|w|$ denotes the number of symbols of w. With $\leq_{\text{lex}}$ we denote the usual lexicographic order $\varepsilon, 0, 1, 00, 01, 10, 11, \ldots$ on $\{0, 1\}^*$, while with $<_{\text{lex}}$ we denote the strict version of $\leq_{\text{lex}}$. For every word $w \in \{0, 1\}^*$, $\text{succ}(w)$ is the next word in $\{0, 1\}^*$ with respect to $\leq_{\text{lex}}$.

Let $f : \mathbb{N} \to \mathbb{N}$ be the function $f(0) = 1$ and $f(n+1) := \max_{i, j \leq n+1}\{f(n), p_i(j)\} + 1$, for every $n \in \mathbb{N}$. Then f is an increasing computable function that satisfies the following condition similar to condition (ii) of Theorem 2.4:

> For every Turing transducer T_n of the above enumeration there exists a natural number m such that for every word w with $|w| \geq m$, for every word v with $|v| \leq |w|$, it holds that $|T_n(v)| < f(|w|)$.

We introduce here the two previous technical definitions in the framework of words on $\{0, 1\}$ and polynomial-time computations.

Definition 3.2 A set $X \subseteq \{0, 1\}^*$ is *strongly bi-$\leq_m^P$-immune* if and only if every $\leq_m^P$-reduction from X to any set $Y \subseteq \{0, 1\}^*$ is 1-1 a.e.

Definition 3.3 Let $g : \mathbb{N} \to \mathbb{N}$ be a function such that for every $n \in \mathbb{N}$, $g(n) > n$, and let $X \subseteq \{0, 1\}^*$ be an infinite set. Then X has *g-gaps* if and only if for every natural number n there exists a natural number m such that $X \cap \{x \in \{0, 1\}^* : n + m < |x| \leq g(n + m)\} = \emptyset$. For every word z, and for every word w of length $g(|z|)$, we call the set $\{x \in \{0, 1\}^* : z <_{\text{lex}} x \leq_{\text{lex}} w\}$ a *g-gap*.

Similar to Theorem 2.4, we construct a set $A \subseteq \{0, 1\}^*$ with the following two properties:

1. A is strongly bi-$\leq_m^P$-immune, and
2. both A and $\overline{A}$ have f-gaps, where f is the function defined above.

The construction of set A is by stages and is very similar to that of Theorem 1.5. In particular, to guarantee the first property (1) we satisfy the following requirements R_i, for $i = 1, 2, \ldots$:

> R_i: If $\{(x, y) : x \neq y \wedge T_i(x) = T_i(y)\}$ is infinite, then there exist $u \neq v$ with $T_i(u) = T_i(v)$, $u \in A$ and $v \notin A$.

To guarantee the second property (2), whenever the procedure finds an index that can be diagonalized at some stage n, we insert an f-gap $\{x : z <_{\text{lex}} x \leq_{\text{lex}} 1^{f(|z|)}\}$ into $\overline{A}$ and an f-gap $\{x : 1^{f(|z|)} <_{\text{lex}} x \leq_{\text{lex}} 0^{f(f(|z|))}\}$ into A, for an appropriate word z.

For every $z \in \Sigma^*$ let $\text{GAP}(z, f) := \{x \in \Sigma^* : z <_{\text{lex}} x \leq_{\text{lex}} 1^{f(f(|z|))}\}$.

Begin Construction

Stage 0. $A := \emptyset$, $D := \emptyset$, $w(1) := \varepsilon$.

Stage $n > 0$.
$D := D \cup \{n\}$;
If $\exists i \in D$ and $\exists x <_{\text{lex}} w(n)$ such that $T_i(x) = T_i(w(n))$ **then**
  begin let i_0 be the smallest such index and let x_0 be the smallest such word.
  if $x_0 \notin A$ **then** $A = A \cup \{w(n)\}$; **endif**
  $D := D - \{i_0\}$;
  $z := w(n)$;
  $\text{GAP} := \text{GAP}(z, f)$;
  while $\exists i \in D$, $\exists x \in \text{GAP}$ and $\exists y <_{\text{lex}} x$ such that $T_i(y) = T_i(x)$ **do**
    let x_0 be the smallest such word x;
    let i_0 be the smallest such index i;
    let $y_0 < x_0$ be the smallest word such that $T_{i_0}(y_0) = T_{i_0}(x_0)$;
    if $y_0 \notin A$ **then** $A := A \cup \{x_0\}$; **endif**
    $D := D - \{i_0\}$;
    $z := x_0$;
    $\text{GAP} := \text{GAP}(z, f)$
  endwhile
  $A := A \cup \{x : z <_{\text{lex}} x \leq_{\text{lex}} 1^{f(|z|)}\}$; {insert an f-gap into $\overline{A}$};
  $w(n+1) := 0^{f(f(|z|))+1}$; {insert an f-gap into A};
  end
else
$w(n+1) := \text{succ}(w(n))$
endif
Go to stage $n+1$.
End stage n.

End Construction

Set A is recursive. With a very similar argument to that of Theorem 2.4 it is possible to show that A is bi-$\leq_m^P$-introimmune. □

We conclude observing that with a little change to the construction of Theorem 3.1 it is possible to show the existence of a sparse $\leq_m^P$-introimmune recursive set. Recall that a set $X \subseteq \{0,1\}^*$ is sparse if and only if there exists a polynomial p such that for every $n \in \mathbb{N}$, $|\{x \in X : |x| \leq n\}| \leq p(n)$.

Theorem 3.4 *There exists a sparse $\leq_m^P$-introimmune recursive set.*

Actually, Ambos-Spies recently proved in [1] a much stronger result, showing the existence of a sparse $\leq_T^P$-introimmune set in **EXPTIME**.[1]

Theorems 3.1 and 3.4 produce optimal results inside Kleene's Hierarchy. As remarked in the introduction, $\leq_m^P$-introimmune sets cannot be members of the class **P**.

In fact, a very similar proof to that of Proposition 2.1 shows that $\leq_m^P$-introimmune sets are **P**-immune. The converse is not true: take an $\leq_m^P$-introimmune (and sparse) set $A \subseteq \{0,1\}^*$ and consider $A \oplus A := \{x0 : x \in A\} \cup \{x1 : x \in A\}$. Then $A \oplus A$ is **P**-immune (and sparse) but not $\leq_m^P$-introimmune. Likewise to the sequential observation of Proposition 2.1, both A and $A \oplus A$ are in the same $\leq_m^P$-degree, and it is easy to see that the $\leq_m^P$-introimmune property is p-isomorphic invariant. That is, if $L \subseteq \{0,1\}^*$ is a $\leq_m^P$-introimmune language and $f : \{0,1\}^* \to \{0,1\}^*$ is a polynomial-time computable permutation with f^{-1} polynomial-time computable, then $f(L)$ is also $\leq_m^P$-introimmune.

As a further line of research, we propose, given a reducibility $\leq_r$, the discovery of an $\leq_r$-introimmune set as down as possible in one of the resource bounded hierarchies, such as the Polynomial Time Hierarchy, the Exponential Time Hierarchy, or others. The work of Ambos-Spies [1] in this direction is excellent. However, the existence of $\leq_m^p$-introimmune sets in **PH** is very hard to prove, if they exist, since it implies the solution of the open problem **P=?PH**.

Note

1. **EXPTIME** $= \bigcup_{c>0}$DTIME(2^{cn}).

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