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Types in Abstract Elementary Classes

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Abstract We suggest a method of finding a notion of type to abstract elementary classes and determine under what assumption on these types the class has a well-behaved homogeneous and universal "monster" model, where homogeneous and universal are defined relative to our notion of type.

1 Introduction

Let $(\mathbf{K}, \subseteq)$ be an abstract elementary class (see below) such that $\mathbf{K} = \overline{\mathbf{K}}_0^{\mu}$ is a class of bicolored (rank 2) fields (see Baldwin and Holland [1]; in Example 4.1 below, for technical reasons, a bit simplified case is studied) and $\subseteq$ is the submodel relation. Then $(\mathbf{K}, \subseteq)$ behaves very badly; it is almost impossible to analyze $\mathbf{K}$ via the abstract elementary class $(\mathbf{K}, \subseteq)$. However, if $\subseteq$ is replaced by 'strong submodel relation' $\leq$ (see [1]), we get a very nice abstract elementary class $(\mathbf{K}, \leq)$ and by working in it, one can see that $\mathbf{K}$ is a relatively simple class of structures: it is a class of all strong submodels of a well-behaved homogeneous (in fact "$\leq$-homogeneous", see below) monster model (not always saturated) and a lot is known about such classes of models.

It is easy to find examples like the one above from the recent studies in model theory: Zilber's weak Schanuel structures, Banach spaces, and so on. In fact, often finding the right submodel relation is the key problem in getting a good model theoretic analysis for a given class of structures. Notice also that the considerations like the one above tie in closely with the classical Robinson school studies in model theory.

In this paper we study the question of finding right submodel relations. Abstract elementary classes $(\mathbf{K}, \leq)$ provide a natural context to carry out such studies. Our criteria for a good submodel relation is that it should give rise to a well-behaved homogeneous and universal "monster" model M for the abstract elementary class. To give a meaning for the notions homogeneous and universal, we need a reasonable

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notion of type for the class (types can be seen as equivalence classes of some meaningful equivalence relation on pairs $(a, \mathcal{A})$, where $\mathcal{A} \in \mathbf{K}$ and a is a sequence of elements of $\mathcal{A}$).

Definition 1.1 Suppose we have a notion of type for sequences of elements of models from $\mathbf{K}$. Then we say that a model $\mathbf{M} \in \mathbf{K}$ is homogeneous if any two sequences of elements of $\mathbf{M}$ of length $< |\mathbf{M}|$ which have the same type are mapped to each other by an automorphism of $\mathbf{M}$ and we say that $\mathbf{M}$ is universal if for all $\mathcal{B} \in \mathbf{K}$ of power $\leq |\mathbf{M}|$, there is a type-preserving embedding of $\mathcal{B}$ into $\mathbf{M}$.

Now suppose that we have defined types for our class and that the class has a well-behaved homogeneous and universal monster model relative to these types. Then the types must satisfy the following:

1. A complete type of an infinite sequence is determined by its restrictions to finite subsequences.
2. There is only one type of the empty sequence.
3. There are only set many types of finite sequences.

This is for the following reason: of course, types must be preserved under isomorphisms and so (1) follows from type compactness which is the key model theoretic property of the first-order homogeneous structures and which we want our homogeneous structures to share (so we aim to have really homogeneous monster models, not just model homogeneous).

Fact 1.2 Assume $\mathcal{A}$ is a model and E is the following equivalence relation on ${}^{<|\mathcal{A}|}\mathcal{A}$: sequences a and b are E-equivalent if and only if there is an automorphism f of $\mathcal{A}$ such that $f(a) = b$. Then the following are equivalent:

(i) The E-equivalence class of a sequence a of elements of $\mathcal{A}$ is determined by the E-equivalence classes of the finite subsequences of a.

(ii) Suppose $(a_i)_{i<\gamma} \in {}^{<|\mathcal{A}|}\mathcal{A}$ and that for all finite $X \subseteq \gamma$, there is a finite sequence a_X of elements of $\mathcal{A}$ such that if $X \subseteq Y$, then $(a_X, a_i)_{i\in X}$ is E-equivalent with $(a_Y, a_i)_{i\in X}$ (in particular, the length of a_X does not depend on X). Then there is a finite sequence a of elements of $\mathcal{A}$ such that for all finite $X \subseteq \gamma$, $(a, a_i)_{i\in X}$ is E-equivalent with $(a_X, a_i)_{i\in X}$.

Notice that if $\mathcal{A}$ is a monster model of a complete first-order theory, then a and b are E-equivalent if they have the same first-order type (over $\emptyset$) and so (i) holds. Also then (ii) says that if p is a complete type over A $(= \{a_i |\ i < \gamma\})$ and for all finite $B \subseteq A$, $p \restriction B$ is realized in $\mathcal{A}$, then p is realized in $\mathcal{A}$. Thus also (ii) holds.

Proof The implication from (i) to (ii) can be proved as the analogous result was proved, for example, in Hyttinen [2]. We prove the other direction: Assume that $(a_i)_{i<\gamma}, (b_i)_{i<\gamma} \in {}^{<|\mathcal{A}|}\mathcal{A}$ are such that for all finite $X \subseteq \gamma$, $(a_i)_{i\in X}$ and $(b_i)_{i\in X}$ are E-equivalent. We need to find an automorphism f such that for all $i < \gamma$, $f(a_i) = b_i$. For this it is enough to show the following: for all $a \in \mathcal{A}$, there is $b \in \mathcal{A}$ such that for all finite $X \subseteq \gamma$, $(a, a_i)_{i\in X}$ and $(b, b_i)_{i\in X}$ are E-equivalent. But this is immediate by (ii). □

The properties (2) and (3) follow from universality and our wish that the number of types of finite sequences of elements of the monster model $\mathbf{M}$ is $< |\mathbf{M}|$ (without this the monster model is rather useless).

In this paper we will introduce a new notion of type to abstract elementary classes and we will show that if the types (as we will define them) satisfy (1) – (3), then the class has a well-behaved homogeneous and universal monster model.

Our method of finding types will be intuitive but we are not able to show that it works whenever there is some method that works. In the last section we give two examples in which our type-construction works. The second example shows also that besides our construction, there may be other methods that work and that there need not be a weakest notion of type that gives rise to a well-behaved homogeneous and universal monster model.

We want to point out that the word "elementary" is used in this paper in an abstract sense, that is, in the sense of abstract elementary classes. It has nothing to do with the usual meaning of the word.

2 Construction of Types

In this section we give our type construction. In order to simplify the notation, unless otherwise stated, $a \in A$ means that a is a sequence of elements of A.

Shelah has defined the concept of an abstract elementary class: We let $\mathbf{K}$ be a class of structures in a fixed similarity type τ such that it is closed under isomorphism. By a $\mathbf{K}$-model we mean a member of $\mathbf{K}$ (and similarly for other classes of models). We let $\leq$ be a partial order on $\mathbf{K}$ and assume the following.

Assumptions on $(\mathbf{K}, \leq)$ For all $\mathbf{K}$-models $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, the following hold:

(a) $\mathcal{A} \leq \mathcal{A}$;
(b) $\mathcal{A} \leq \mathcal{B}$ implies that $\mathcal{A}$ is a substructure of $\mathcal{B}$;
(c) if $\mathcal{A}$ is a substructure of $\mathcal{B}$, $\mathcal{A} \leq \mathcal{C}$ and $\mathcal{B} \leq \mathcal{C}$, then $\mathcal{A} \leq \mathcal{B}$;
(d) $\leq$ is preserved under isomorphisms;
(e) if $(\mathcal{A}_i)_{i<\gamma}$ is a continuous $\leq$-increasing sequence of $\mathbf{K}$-models, then for all $i < \gamma$, $\mathcal{A}_i \leq \cup_{j<\gamma} \mathcal{A}_j \in \mathbf{K}$;
(f) if $(\mathcal{A}_i)_{i<\gamma}$ is a continuous $\leq$-increasing sequence of $\mathbf{K}$-models and for all $i < \gamma$, $\mathcal{A}_i \leq \mathcal{A}$, then $\cup_{j<\gamma} \mathcal{A}_j \leq \mathcal{A}$;
(g) there is a cardinal $LS(\mathbf{K})$ such that $LS(\mathbf{K}) \geq |\tau|$ and for all $\mathbf{K}$-models $\mathcal{A}$ and subsets $A \subseteq \mathcal{A}$ of power $\leq LS(\mathbf{K})$, there is a $\mathbf{K}$-model $A \subseteq \mathcal{B} \leq \mathcal{A}$ of power $\leq LS(\mathbf{K})$.

If in addition $(\mathbf{K}, \leq)$ has the property (h) below, we say that $(\mathbf{K}, \leq)$ is an abstract elementary class with the amalgamation property;

(h) for all $\mathbf{K}$-models $\mathcal{A}_i$, $i < 3$, the following holds: if $\mathcal{A}_0 \leq \mathcal{A}_1$ and $\mathcal{A}_0 \leq \mathcal{A}_2$ then there is a $\mathbf{K}$-model $\mathcal{A}_3$ and embeddings $f : \mathcal{A}_1 \to \mathcal{A}_3$ and $g : \mathcal{A}_2 \to \mathcal{A}_3$ such that $f \restriction \mathcal{A}_0 = g \restriction \mathcal{A}_0$, $f(\mathcal{A}_1) \leq \mathcal{A}_3$ and $g(\mathcal{A}_2) \leq \mathcal{A}_3$.

The assumption (g) is not needed in the main theorems (2.8 and 3.2), but notice that it gives the following.

Fact 2.1 For all cardinals $\lambda \geq LS(\mathbf{K})$, $\mathbf{K}$-models $\mathcal{A}$ and subsets $A \subseteq \mathcal{A}$ of power $\leq \lambda$, there is a $\mathbf{K}$-model $A \subseteq \mathcal{B} \leq \mathcal{A}$ of power $\leq \lambda$.

Proof Easy induction on λ using (c), (f), and (g). □

Prior to us Shelah has pointed out that if $(\mathbf{K}, \leq)$ has the amalgamation property, then a meaningful notion of a type over $\mathbf{K}$-models can be defined (strictly speaking, Shelah's definition of equality of types does not assume the amalgamation property, but it gives rise to a meaningful notion of type only if the relevant models are amalgamation bases). We push Shelah's idea a bit further. We define a notion of type over arbitrary sets and our notion is meaningful also without the amalgamation property. And we will determine under which conditions our notion of type makes **K** a homogeneous class of structures.

Definition 2.2

(i) By induction on ordinals α we define an α-elementary submodel relation $\prec_\alpha \subseteq \mathbf{K} \times \mathbf{K}$ and the notion α-elementary embedding as follows:
  (a) We say that **K**-model $\mathcal{A}$ is a 0-elementary submodel of **K**-model $\mathcal{B}$, if $\mathcal{A} \leq \mathcal{B}$ and for limit α, we say that $\mathcal{A}$ is an α-elementary submodel of $\mathcal{B}$, if $\mathcal{A} \prec_\beta \mathcal{B}$ for all $\beta < \alpha$.
  (b) We say that $f : \mathcal{A} \to \mathcal{B}$ is an α-elementary embedding if f is an isomorphism between $\mathcal{A}$ and some $\mathcal{C} \prec_\alpha \mathcal{B}$.
  (c) We say that **K**-model $\mathcal{A}$ is an $\alpha + 1$-elementary submodel of **K**-model $\mathcal{B}$ if the following holds: $\mathcal{A} \prec_\alpha \mathcal{B}$ and if $\mathcal{C}$ is a **K**-model and $\mathcal{A} \prec_\alpha \mathcal{C}$, then there are a **K**-model $\mathcal{D}$ and α-elementary embeddings $f : \mathcal{B} \to \mathcal{D}$ and $g : \mathcal{C} \to \mathcal{D}$ such that $f \restriction \mathcal{A} = g \restriction \mathcal{A}$.

(ii) We say that (possibly infinite) sequences $a \in \mathcal{A}$ and $b \in \mathcal{B}$ have the same α-type, $t^\alpha_{\mathcal{A}}(a, \emptyset) = t^\alpha_{\mathcal{B}}(b, \emptyset)$, if there are a **K**-model $\mathcal{D}$ and α-elementary embeddings $f : \mathcal{A} \to \mathcal{D}$ and $g : \mathcal{B} \to \mathcal{D}$ such that $f(a) = g(b)$.

(iii) For **K**-models $\mathcal{A}$ and $\mathcal{B}$, we say that (possibly infinite) sequences $a \in \mathcal{A}$ and $b \in \mathcal{B}$ have the same type, $t_{\mathcal{A}}(a, \emptyset) = t_{\mathcal{B}}(b, \emptyset)$, if for all α they have the same α-type. We say that a function is elementary if it preserves the types and $\mathcal{A}$ is an elementary submodel of $\mathcal{B}$ ($\mathcal{A} \prec \mathcal{B}$) if $\mathrm{id}_{\mathcal{A}}$ is elementary.

(iv) If $a, b \in \mathcal{A}$ are sequences and $A \subseteq \mathcal{A}$, then we write $t_{\mathcal{A}}(a, A) = t_{\mathcal{A}}(b, A)$ if for some enumeration of A, $a \frown A$ and $b \frown A$ have the same type.

So $\prec$ is some kind of an inflationary fixed point of $\leq$ relative to amalgamation. In addition to the assumptions (a)–(g) above, we define the following properties.

Definition 2.3 *Ax*1: For all α and infinite cardinals γ, if $(a_i)_{i<\gamma}$ and $(b_i)_{i<\gamma}$ are sequences of elements of **K**-models $\mathcal{A}$ and $\mathcal{B}$, respectively, and for all $i < \gamma$, $(a_j)_{j<i}$ and $(b_j)_{j<i}$ have the same α-type, then $(a_i)_{i<\gamma}$ and $(b_i)_{i<\gamma}$ have the same α-type.

*Ax*2: For all **K**-models $\mathcal{A}$ and $\mathcal{B}$ there exist a **K**-model $\mathcal{D}$ and elementary embeddings $f : \mathcal{A} \to \mathcal{D}$ and $g : \mathcal{B} \to \mathcal{D}$.

*Ax*3: There is a cardinal κ such that for all α, if a_i, $i < \kappa$ are finite sequences of elements of **K**-models $\mathcal{A}_i$, then there are $i < j < \kappa$ such that $t^\alpha_{\mathcal{A}_i}(a_i, \emptyset) = t^\alpha_{\mathcal{A}_j}(a_j, \emptyset)$.

Notice that *Ax*2 says that all **K**-models are "elementarily equivalent," that is, there is only one type of the empty sequence. Notice also that *Ax*1 can be seen as a strong form of tameness for the class $(\mathbf{K}, \prec_\alpha)$.

We will show that *Ax*1, *Ax*2, and *Ax*3 imply the existence of a nice monster model and that excluding (g), $(\mathbf{K}, \prec)$ satisfies all the requirements of an abstract elementary class with the amalgamation property (a weak version of (g) holds, see Corollary 3.3). For notational reasons, we define also the following properties.

$Ax1^-$: For all α, Ax1^-_α holds, where $Ax1^-_\alpha$ is the following property: for all infinite cardinals γ, if $(\mathcal{A}_i)_{i<\gamma}$ is a continuous $\prec_\alpha$-increasing sequence of **K**-models and for all $i < \gamma$, $\mathcal{A}_i \prec_\alpha \mathcal{A}$, then $\cup_{j<\gamma}\mathcal{A}_j \prec_\alpha \mathcal{A}$.

$Ax3^-$: For some α, $\prec_{\alpha+1}=\prec_\alpha$.

We will show that $Ax1$ and $Ax3$ together imply $Ax3^-$ and that $Ax1$ implies $Ax1^-$.

Lemma 2.4

(i) *For all $\alpha < \beta$, $\mathcal{A} \prec_\beta \mathcal{B}$ implies $\mathcal{A} \prec_\alpha \mathcal{B}$ and $t^\beta_{\mathcal{A}}(a, \emptyset) = t^\beta_{\mathcal{B}}(b, \emptyset)$ implies $t^\alpha_{\mathcal{A}}(a, \emptyset) = t^\alpha_{\mathcal{B}}(b, \emptyset)$.*

(ii) *For all α, $\prec_\alpha$ is transitive and if $\mathcal{A}$ is a substructure of $\mathcal{B}$, $\mathcal{A} \prec_\alpha \mathcal{C}$ and $\mathcal{B} \prec_\alpha \mathcal{C}$, then $\mathcal{A} \prec_\alpha \mathcal{B}$.*

(iii) *Suppose that $Ax1^-_\beta$ holds for all $\beta < \alpha$. If $(\mathcal{A}_i)_{i<\gamma}$ is a continuous $\prec_\alpha$-increasing sequence of* **K***-models, then for all $i < \gamma$, $\mathcal{A}_i \prec_\alpha \cup_{j<\gamma}\mathcal{A}_j \in \mathbf{K}_\alpha$.*

(iv) *Suppose $Ax3^-$ holds. There is α such that*
   (a) *if $t^\alpha_{\mathcal{A}}(a, \emptyset) = t^\alpha_{\mathcal{B}}(b, \emptyset)$, then $t_{\mathcal{A}}(a, \emptyset) = t_{\mathcal{B}}(b, \emptyset)$,*
   (b) *$\prec_\alpha$ implies $\prec$,*
   (c) *α-elementary implies elementary.*

(v) *If $Ax3^-$ holds, then $t_{\mathcal{A}}(a, \emptyset) = t_{\mathcal{B}}(b, \emptyset)$ is a transitive relation.*

Proof (i) Immediate.

(ii) The case $\alpha = 0$ follows from the assumptions on $\leq$ ($\leq$ is a partial order and (c)) and the rest can be proved by an easy induction on α.

(iii) We prove these by induction on α. The case $\alpha = 0$ follows from the assumption (e) on $\leq$ and limit cases are immediate. So assume $\alpha = \beta + 1$. For simplicity we assume that $\gamma = \omega$. Now $\cup_{j<\gamma}\mathcal{A}_j \in \mathbf{K}$ follows from (ii). So is enough to show that if $\mathcal{A}_k \prec_\beta \mathcal{C}$, $k < \gamma$, then there are a **K**-model $\mathcal{D}$ and β-elementary embeddings $f : \cup_{i<\gamma}\mathcal{A}_i \to \mathcal{D}$ and $g : \mathcal{C} \to \mathcal{D}$ such that $f \restriction \mathcal{A}_k = g \restriction \mathcal{A}_k$. By induction on $i < j < \gamma$ we can find a **K**-model $\mathcal{D}_j$ and β-elementary embeddings $f_j : \cup_{i<j}\mathcal{A}_i \to \mathcal{D}_j$ and $g_j : \mathcal{C} \to \mathcal{D}_j$ such that ($f_j \restriction \mathcal{A}_k = g_j \restriction \mathcal{A}_k$ and) for $j < j'$, $f \subseteq f_j \subseteq f_{j'}$ $g_j = g_{j'} = g$ and $\mathcal{D}_j \prec_\beta \mathcal{D}_{j'}$ (amalgamate over $\mathcal{A}_{j-1}$). By the induction assumption (ii) and $Ax1^-_\beta$, $\cup_{i<j<\gamma} f_j$, g and $\cup_{i<j<\gamma} D_j$ are as wanted.

(iv) Let α be as in $Ax3^-$. By an easy induction, one can see that $\mathcal{A} \prec_\alpha \mathcal{B}$ implies $\mathcal{A} \prec_\beta \mathcal{B}$ for all β. Thus (a) follows. For (b) and (c) it is now enough to notice that if $\mathcal{A} \prec_\alpha \mathcal{B}$, then by the definition of α-type, for all sequences a of elements of $\mathcal{A}$, $t^\alpha_{\mathcal{A}}(a, \emptyset) = t^\alpha_{\mathcal{B}}(a, \emptyset)$.

(v) Amalgamate twice and use (iv). □

Notice that by Lemma 2.4(v), $t_{\mathcal{A}}(a, \emptyset)$ can be defined as an equivalence class of the relation $t_{\mathcal{B}}(b, \emptyset) = t_{\mathcal{C}}(c, \emptyset)$. Since in general $t^\alpha_{\mathcal{B}}(b, \emptyset) = t^\alpha_{\mathcal{C}}(c, \emptyset)$ is not a transitive relation, (ii) in Lemma 2.5 below is not quite as strong as it seems. However, notice also that if α is a limit ordinal, then the relation $t^\beta_{\mathcal{B}}(b, \emptyset) = t^\beta_{\mathcal{C}}(c, \emptyset)$, for all $\beta < \alpha$, is transitive and so an equivalence relation.

Lemma 2.5 *Assume Ax1 holds.*

(i) *Let $a = (a_i)_{i<\gamma}$ and $b = (b_i)_{i<\gamma}$ be sequences of elements of* **K***-models $\mathcal{A}$ and $\mathcal{B}$, respectively. If for all finite $X \subseteq \gamma$, $t^\alpha_{\mathcal{A}}((a_i)_{i\in X}, \emptyset) = t^\alpha_{\mathcal{B}}((b_i)_{i\in X}, \emptyset)$, then $t^\alpha_{\mathcal{A}}(a, \emptyset) = t^\alpha_{\mathcal{B}}(b, \emptyset)$.*

(ii) *$\mathcal{A} \prec_\alpha \mathcal{B}$ if and only if for all finite sequences a of elements of $\mathcal{A}$, $t^\alpha_\mathcal{A}(a,\emptyset) = t^\alpha_\mathcal{B}(a,\emptyset)$.*
(iii) *$Ax1^-$ holds.*

Proof (i) Easy induction on $|\gamma|$.

(ii) From left to right the claim is clear. For the other direction, assume that for all finite sequences a of elements of $\mathcal{A}$, $t^\alpha_\mathcal{A}(a,\emptyset) = t^\alpha_\mathcal{B}(a,\emptyset)$. By (i), $t^\alpha_\mathcal{A}(\mathcal{A},\emptyset) = t^\alpha_\mathcal{B}(\mathcal{A},\emptyset)$ and so there is $\mathcal{C}$ such that $\mathcal{A} \prec_\alpha \mathcal{C}$ and $\mathcal{B} \prec_\alpha \mathcal{C}$. By Lemma 2.4(ii) (and assumption (b)), $\mathcal{A} \prec_\alpha \mathcal{B}$.

(iii) Assume not. Let α be the least ordinal in which $Ax1^-_\alpha$ fails and let $\mathcal{A}$ and $\mathcal{A}_i$, $i < \gamma$, witness the failure. Then $\alpha = \beta + 1$ for some β and there is $\mathcal{C}$ such that $\cup_{i<\gamma}\mathcal{A}_i \prec_\beta \mathcal{C}$ but $t^\beta_\mathcal{A}(\cup_{i<\gamma}\mathcal{A}_i,\emptyset) \neq t^\beta_\mathcal{C}(\cup_{i<\gamma}\mathcal{A}_i,\emptyset)$. By Ax1, there is finite sequence a from $\cup_{i<\gamma}\mathcal{A}_i$ such that $t^\beta_\mathcal{A}(a,\emptyset) \neq t^\beta_\mathcal{C}(a,\emptyset)$. Let $i < \gamma$ be such that $a \in \mathcal{A}_i$. By the choice of α, $\mathcal{A}_i \prec_\beta \cup_{i<\gamma}\mathcal{A}_i \prec_\beta \mathcal{C}$. This means that $t^\beta_{\mathcal{A}_i}(a,\emptyset) = t^\beta_\mathcal{C}(a,\emptyset)$. Since $\mathcal{A}_i \prec_\alpha \mathcal{A}$, $t^\beta_\mathcal{A}(a,\emptyset) = t^\beta_\mathcal{C}(a,\emptyset)$, a contradiction. □

Remark 2.6

(i) Assume that $Ax1$ and $Ax3^-$ hold. If for all finite sequences c of elements of $\mathcal{A}\cap\mathcal{B}$, $t_\mathcal{A}(c,\emptyset) = t_\mathcal{B}(c,\emptyset)$, then there are a **K**-model $\mathcal{C}$ and $f : \mathcal{A}\cup\mathcal{B} \to \mathcal{C}$ such that $f \restriction \mathcal{A}$ and $f \restriction \mathcal{B}$ are elementary.
(ii) Assume that $Ax1$ and $Ax3$ hold. Then $\mathcal{A} \prec_{\alpha+1} \mathcal{B}$ if and only if $\mathcal{A} \prec_\alpha \mathcal{B}$ and the following holds: for all $A \subseteq \mathcal{A}$ and **K**-models $\mathcal{C} \supseteq A$, if $t^\alpha_\mathcal{A}(A,\emptyset) = t^\alpha_\mathcal{C}(A,\emptyset)$, then $t^\alpha_\mathcal{B}(A,\emptyset) = t^\alpha_\mathcal{C}(A,\emptyset)$.

Proof (ii) is immediate by the definitions and (i) follows from Lemma 2.4(iv) and Lemma 2.5(i). □

Lemma 2.7 *Assume $Ax1$ and $Ax3$ hold.*

(i) *$Ax3^-$ holds.*
(ii) *Let $\mathcal{A}$ and $\mathcal{B}$ be* **K***-models and $a = (a_i)_{i<\gamma}$ and $b = (b_i)_{i<\gamma}$ sequences of elements of $\mathcal{A}$ and $\mathcal{B}$, respectively. If for all finite $X \subseteq \gamma$, $t_\mathcal{A}((a_i)_{i\in X},\emptyset) = t_\mathcal{B}((b_i)_{i\in X},\emptyset)$, then $t_\mathcal{A}(a,\emptyset) = t_\mathcal{B}(b,\emptyset)$.*
(iii) *$\mathcal{A} \prec \mathcal{B}$ if and only if ($\mathcal{A} \leq \mathcal{B}$ and) for all finite sequences a of elements of $\mathcal{A}$, $t_\mathcal{A}(a,\emptyset) = t_\mathcal{B}(a,\emptyset)$.*
(iv) *If for all finite sequences c of elements of $\mathcal{A} \cap \mathcal{B}$, $t_\mathcal{A}(c,\emptyset) = t_\mathcal{B}(c,\emptyset)$, then there are a* **K***-model $\mathcal{C}$ and $f : \mathcal{A} \cup \mathcal{B} \to \mathcal{C}$ such that $f \restriction \mathcal{A}$ and $f \restriction \mathcal{B}$ are elementary.*

Proof We prove (i); the rest follows immediately from this and the lemmas above. Let $t_{\mathcal{A}_i}(a_i,\emptyset)$, $i < \xi$, list all types of finite sequences, that is, for all a, $\mathcal{A}$ and α, there is $i < \xi$ such that $t^\alpha_\mathcal{A}(a,\emptyset) = t^\alpha_{\mathcal{A}_i}(a_i,\emptyset)$ (this is possible by Lemma 2.4(i)). For all $i < j < \xi$, let α_{ij} be such that $t^{\alpha_{ij}}_{\mathcal{A}_i}(a_i,\emptyset) \neq t^{\alpha_{ij}}_{\mathcal{A}_j}(a_j,\emptyset)$ and let $\beta = \sup\{\alpha_{ij} \mid i < j < \xi\}$ and $\alpha = \beta + 1$. We claim that α is as wanted. For this, by Lemma 2.5(ii), it is enough to show that if $i < j < \xi$, $t_\mathcal{B}(b,\emptyset) = t_{\mathcal{A}_i}(a_i,\emptyset)$ and $t_\mathcal{C}(c,\emptyset) = t_{\mathcal{A}_j}(a_j,\emptyset)$, then $t^\alpha_\mathcal{B}(b,\emptyset) \neq t^\alpha_\mathcal{C}(c,\emptyset)$. Assume not. Clearly we may assume that $b = a_i$, $c = a_j$ and

(a) there is $\mathcal{D}_i$ such that $\mathcal{A}_i \prec_{\alpha+1} \mathcal{D}_i$ and $\mathcal{B} \prec_{\alpha+1} \mathcal{D}_i$,
(b) there is $\mathcal{D}_j$ such that $\mathcal{A}_j \prec_{\alpha+1} \mathcal{D}_j$ and $\mathcal{C} \prec_{\alpha+1} \mathcal{D}_j$.

Since $t^{\alpha}_{\mathcal{B}}(b, \emptyset) = t^{\alpha}_{\mathcal{C}}(c, \emptyset)$, by (a) and (b) and Remark 2.6(ii), we may assume that $b = c$ and that there is $\mathcal{D}$ such that $\mathcal{D}_i \prec_{\beta} \mathcal{D}$ and $\mathcal{D}_j \prec_{\beta} \mathcal{D}$ (amalgamate three times). But then $t^{\beta}_{\mathcal{A}_i}(a_i, \emptyset) = t^{\beta}_{\mathcal{A}_j}(a_j, \emptyset)$, a contradiction. $\square$

Theorem 2.8 *Assume $Ax1$ and $Ax3$. Then excluding property (g) (the Löwenheim-Skolem property), $(\mathbf{K}, \prec)$ satisfies all the requirements of an abstract elementary class with the amalgamation property.*

Proof The properties follow from the lemmas above. $\square$

In the next section we show that $Ax1$ and $Ax3$ imply a weak version of (g).

Remark 2.9 If $(\mathbf{K}, \leq)$ has the amalgamation property then $\prec = \leq$ and $Ax3$ holds.

Proof The claim $\prec = \leq$ is clear by the definitions and the number of types of finite sequences realized in $\mathbf{K}$-models is $\leq 2^{LS(\mathbf{K})}$ by Lemma 2.5(ii) and (g). $\square$

3 Homogeneous and Universal Models

In this section we look at the existence of a well-behaved monster model.

Definition 3.1

(i) We say that a $\mathbf{K}$-model $\mathcal{A}$ is λ-saturated if the following holds: For all $A \subseteq \mathcal{A}$ of power $< \lambda$, $\mathcal{B} \succ \mathcal{A}$ and $b \in \mathcal{B}$, there is $a \in \mathcal{A}$ such that $t_{\mathcal{A}}(a, A) = t_{\mathcal{B}}(b, A)$ (i.e., $t_{\mathcal{A}}(a \frown A, \emptyset) = t_{\mathcal{B}}(b \frown A, \emptyset)$).

(ii) We say that a $\mathbf{K}$-model $\mathcal{A}$ is strongly λ-homogeneous, if for all sequences a and b of elements of $\mathcal{A}$ of length $< \lambda$, $t_{\mathcal{A}}(a, \emptyset) = t_{\mathcal{A}}(b, \emptyset)$ implies that there is an automorphism f of $\mathcal{A}$ such that $f(a) = b$.

(iii) We say that a $\mathbf{K}$-model $\mathcal{A}$ is λ-universal if for all $\mathbf{K}$-models $\mathcal{B}$ of power $\leq \lambda$ there is an elementary embedding $f : \mathcal{B} \to \mathcal{A}$.

Theorem 3.2 *Assume that $Ax1$ and $Ax3$ hold. Let $\mathcal{A}$ be a $\mathbf{K}$-model and λ a cardinal.*

(i) *$\mathcal{A}$ is λ-saturated if and only if the following holds: For all $A \subseteq \mathcal{A}$ of power $< \lambda$, $\mathbf{K}$-models $\mathcal{B}$, $B \subseteq \mathcal{B}$ and $b \in \mathcal{B}$, if $t_{\mathcal{A}}(A, \emptyset) = t_{\mathcal{B}}(B, \emptyset)$, then there is $a \in \mathcal{A}$ such that $t_{\mathcal{A}}(a \frown A, \emptyset) = t_{\mathcal{B}}(b \frown B, \emptyset)$.*

(ii) *There is a λ-saturated strongly λ-homogeneous $\mathbf{K}$-model $\mathcal{B}$ such that $\mathcal{A} \prec \mathcal{B}$.*

(iii) *If in addition $Ax2$ holds, then every λ-saturated $\mathbf{K}$-model is λ-universal.*

Proof (i) Immediate by Lemma 2.7(iv) and the definitions.

(ii) and (iii) Usual argument (Fraïssé or Morley-Vaught) using Lemma 2.7(iv) and Theorem 2.8. $\square$

Corollary 3.3 *Assume that $Ax1$ and $Ax3$ hold. There is an infinite cardinal $LS^*(\mathbf{K})$ such that if $\mathcal{B}$ is an $LS^*(\mathbf{K})$-saturated $\mathbf{K}$-model, then the following holds: For all $A \subseteq \mathcal{B}$, there is a $\mathbf{K}$-model $\mathcal{A} \prec \mathcal{B}$ such that $A \subseteq \mathcal{A}$ and $|\mathcal{A}| = |A| + LS^*(\mathbf{K})$.*

Proof Let $\kappa_0 = \omega$ and κ_{i+1} be such that for all $A \subseteq \mathcal{A} \in \mathbf{K}$, if $|A| \leq \kappa_i$, then there is $\mathcal{C} \in \mathbf{K}$ such that $A \subseteq \mathcal{C}$, $t_{\mathcal{C}}(A, \emptyset) = t_{\mathcal{A}}(A, \emptyset)$ and $|\mathcal{C}| \leq \kappa_{i+1}$. Since there are no class many types $t_{\mathcal{A}}(A, \emptyset)$, κ_{i+1} exists. Let $LS^*(\mathbf{K}) = \cup_{i<\omega}\kappa_i$.

Suppose $\mathcal{B}$ is $LS^*(\mathbf{K})$-saturated. We show that for all $A \subseteq \mathcal{B}$ of power $\leq LS^*(\mathbf{K})$, there is $\mathcal{A} \prec \mathcal{B}$ of power $\leq LS^*(\mathbf{K})$ such that $A \subseteq \mathcal{A}$. As in Fact 2.1, an easy induction shows that this suffices.

Choose $A_i \subseteq A$, $i < \omega$, so that $\cup_{i<\omega} A_i = A$ and for all i, $|A_i| \leq \kappa_i$. For all $i < \omega$, by the choice of κ_{i+1} (and $LS^*(\mathbf{K})$-saturation of $\mathcal{B}$), we can choose $\mathcal{A}_i \prec \mathcal{B}$ of power $\leq \kappa_{i+1}$ such that $A_i \subseteq \mathcal{A}_i$ and if $j < i$, then also $\mathcal{A}_j \subseteq \mathcal{A}_i$. Let $\mathcal{A} = \cup_{i<\omega} \mathcal{A}_i$. By Theorem 2.8, $\mathcal{A} \prec \mathcal{B}$. □

Assume now that $Ax1$, $Ax2$ and $Ax3$ hold. Let $\mathbf{M}$ be a monster model for $\mathbf{K}$ given by Theorem 2.2. We extend τ to τ^* by adding new predicate symbols in the obvious way: For each $k < \omega$ and each equivalence class of pairs $(a, \mathcal{A})$, $a \in \mathcal{A}^k$, under the equivalence relation $t_{\mathcal{A}}(a, \emptyset) = t_{\mathcal{B}}(b, \emptyset)$, we add a new k-ary relation symbol $R_{[a,\mathcal{A}]}$, where $[a, \mathcal{A}]$ denotes the equivalence class of $(a, \mathcal{A})$. For all $\mathbf{K}$-models $\mathcal{B}$ we define a τ^*-expansion $\mathcal{B}^*$ so that $\mathcal{B}^* \models R_{[a,\mathcal{A}]}(b)$ if and only if $t_{\mathcal{B}}(b, \emptyset) = t_{\mathcal{A}}(a, \emptyset)$.

Then $\mathbf{M}^*$ is a monster model for the class $\mathbf{K}^* = \{\mathcal{A}^* | \ \mathcal{A} \in \mathbf{K}\}$ in the sense that it is first-order homogeneous and every $\mathbf{K}^*$-model is embeddable into $\mathbf{M}^*$. Further more (the finite diagram of) $\mathbf{M}^*$ has the semi-elimination of quantifiers, that is, for all first-order τ^*-formulas $\varphi(\overline{x})$ there are atomic or negated atomic τ^*-formulas $\theta_{ij}(\overline{x})$, $i \in I$ and $j \in J_i$, such that for all τ^*-models $\mathcal{A}$, if the finite diagram of $\mathcal{A}$ is a subset of that of $\mathbf{M}^*$, then

$$\mathcal{A} \models \forall \overline{x}(\varphi(\overline{x}) \leftrightarrow \vee_{i\in I} \wedge_{j\in J_i} \theta_{ij}(\overline{x})).$$

So if $\mathcal{A}$ is an ω-saturated $\mathbf{K}$-model, then $\mathcal{A}^*$ can be embedded to $\mathbf{M}^*$ first-order elementarily.

This reduction allows us to transfer results from the theory of good finite diagrams to $(\mathbf{K}, \leq)$ (see Shelah [4]). As an example we show the stability hierarchy theorem. We say that $\mathbf{K}$ is λ-stable if for all $\mathbf{K}$-models $\mathcal{A}$ and $A \subseteq \mathcal{A}$ of power $\leq \lambda$, the number of types (of finite sequences) over A realized in $\mathcal{A}$ is $\leq \lambda$.

Corollary 3.4 *Assume $Ax1$, $Ax2$, and $Ax3$ hold. If $\mathbf{K}$ is λ-stable for some λ, then there are cardinals $\lambda(\mathbf{K})$ and $\kappa(\mathbf{K})$ such that $\mathbf{K}$ is ξ-stable if and only if $\xi = \lambda(\mathbf{K}) + \xi^{<\kappa(\mathbf{K})}$.*

Proof By the reduction above, this follows immediately from [4]. □

Question 3.5 *Do $Ax1$, ($Ax2$), and $Ax3$ imply that $(\mathbf{K}, \prec)$ satisfies the property (g) from the definition of an abstract elementary class?*

4 Examples

As an example of our construction of types, we look at bicolored fields. This example is based on a construction due to Poizat (see [1]).

Example 4.1 Let $\mathbf{K}'$ be the class of all algebraically closed fields of char 0 with additional unary predicate P. For all finite (or finitely generated algebraically closed subfields) $X \subseteq \mathcal{A} \in \mathbf{K}'$, we define $\delta(X) = 2 \cdot d_f(X) - |P^{\mathcal{A}} \cap X|$, where $d_f(X)$ is the transcendence degree of X. We let $\mathbf{K}$ consist of those $\mathcal{A} \in \mathbf{K}'$ for which the following holds: For all finite $X \subseteq \mathcal{A}$, $\delta(X) \geq 0$. We let $\leq$ be the submodel relation.

Then it is easy to see that $(\mathbf{K}, \leq)$ is an abstract elementary class, $Ax1$, $Ax2$, and $Ax3$ hold, and $\prec = \prec_1 = \leq_s$, where $\mathcal{A} \leq_s \mathcal{B}$ if $\mathcal{A}$ is a submodel of $\mathcal{B}$ and for all finite $X \subseteq \mathcal{B}$, $\delta(X/X \cap \mathcal{A}) = \delta(X) - \delta(X \cap \mathcal{A}) \geq 0$.

Sketch of proof It is trivial to verify that $(\mathbf{K}, \leq)$ is an abstract elementary class. From [1] (with small additional work) it follows that $(\mathbf{K}, \leq_s)$ is a homogeneous class of structures, in particular $(\mathbf{K}, \leq_s)$ has the amalgamation property. Thus when we establish that $\prec_1 = \leq_s$ (see Definition 1.2), it follows that $Ax1$, $Ax2$, and $Ax3$ hold and $\prec = \leq_s$ (we do need to establish that $\prec_1 = \leq_s$, since it might happen, for example, that $\prec_1$ is strictly stronger than $\leq_s$ in which case the observation that $(\mathbf{K}, \leq_s)$ is a nice class of structures tells us nothing).

Suppose first that $\mathcal{A} \leq_s \mathcal{B}$. Let $\mathcal{C}$ be such that $\mathcal{A} \leq \mathcal{C}$. Clearly we may assume that $\mathcal{C} \cap \mathcal{B} = \mathcal{A}$. Let $\mathcal{D}$ be the algebraically closed field we get by amalgamating fields $\mathcal{B}$ and $\mathcal{C}$ over $\mathcal{A}$ freely (tensor product over $\mathcal{A}$). Let $P^{\mathcal{D}} = P^{\mathcal{B}} \cup P^{\mathcal{C}}$. An easy predimension calculation shows that $\mathcal{D} \in \mathbf{K}$. Thus $\mathcal{A} \prec_1 \mathcal{B}$.

Suppose then that $\mathcal{A} \not\leq_s \mathcal{B}$. Let $X \subseteq \mathcal{B}$ witness this. We may assume that $X \subseteq P^{\mathcal{B}}$ and that for all $Z \subseteq X$, if $X \cap \mathcal{A} \subseteq Z$ and $Z \neq X$, then $\delta(X/Z) < 0$. Let $q < 0$ be such that for all such Z, $\delta(X/Z) < q$. Then one can find $\mathcal{C} \in \mathbf{K}$ and finite $Y \subseteq P^{\mathcal{C}}$ such that $\mathcal{A} \leq \mathcal{C}$, $X \cap \mathcal{A} \subseteq Y$ and $\delta(Y) < -q$. In addition we can choose these so that if F_Y is the subfield of $\mathcal{C}$ generated by Y and F_X is the subfield generated by $X \cup (Y \cap \mathcal{A})$, then there is no field embedding $h : F_X \to F_Y$ such that $h \restriction (Y \cap \mathcal{A}) = \mathrm{id}$ and $h(X) \subseteq Y$. But then there is no $\mathcal{D} \in \mathbf{K}$ and embeddings $f : \mathcal{B} \to \mathcal{D}$ and $g : \mathcal{C} \to \mathcal{D}$ such that $f \restriction \mathcal{A} = g \restriction \mathcal{A}$. Thus $\mathcal{A} \not\prec_1 \mathcal{B}$. □

Our second example shows that there need not exist a weakest notion of type that gives rise to a nice monster model for the class of structures. For the terminology used in the example, see Hyttinen [3].

Example 4.2 Let the similarity type consist of unary predicates P_i, $i < 3$, and unary functions F_i, $i < 2$. We let $\mathbf{K}$ be a class of those models $\mathcal{A}$, which satisfy

(i) $P_i^{\mathcal{A}}$, $i < 3$, is a partition of the universe of $\mathcal{A}$,
(ii) for all x, $F_0^{\mathcal{A}}(x) \in P_0^{\mathcal{A}}$ if $x \in P_1^{\mathcal{A}}$ and otherwise $F_0^{\mathcal{A}}(x) = x$,
(iii) for all x, $F_1^{\mathcal{A}}(x) \in P_0^{\mathcal{A}}$ if $x \in P_2^{\mathcal{A}}$ and otherwise $F_1^{\mathcal{A}}(x) = x$,
(iv) there are no elements $x \in P_0^{\mathcal{A}}$, $y \in P_1^{\mathcal{A}}$, and $z \in P_2^{\mathcal{A}}$ such that $F_0^{\mathcal{A}}(y) = x$ and $F_1^{\mathcal{A}}(z) = x$.

Notice that $\mathbf{K}$ is first-order axiomatizable by universal sentences. Let $\leq$ be the submodel relation. Then $(\mathbf{K}, \leq)$ is an abstract elementary class.

Let $\mathcal{L}_0$ be the least collection of first-order formulas, which is closed under negation and replacing free variables by terms and which contains all atomic formulas together with

(0) $\exists y(F_0(y) = x \wedge y \neq x)$.

$\mathcal{L}_1$ is defined similarly except that (0) is replaced by

(1) $\exists y(F_1(y) = x \wedge y \neq x)$.

For $i \in \{0, 1\}$, let $\leq_i$ $(= \preceq_{\mathcal{L}_i})$ be the $\mathcal{L}_i$-elementary submodel relation ($\mathcal{A} \leq_i \mathcal{B}$ if for all $\varphi \in \mathcal{L}_i$ and sequences a of elements of $\mathcal{A}$, $\mathcal{A} \models \varphi(a)$ if and only if $\mathcal{B} \models \varphi(a)$). Then $(\mathbf{K}, \leq_i)$ is an abstract elementary class which has a (first-order) saturated monster (= universal and homogeneous with respect to $\leq_i$) model $\mathbf{M}_i$ with $\mathcal{L}_i$-elimination of quantifiers (= elimination of quantifiers with atomic formulas replaced by $\mathcal{L}_i$-formulas). Note that $\mathbf{M}_0$ is isomorphic to $\mathbf{M}_1$ (if they have the same cardinality). In particular, a model $\mathcal{A} \in \mathbf{K}$ is $\mathcal{L}_0$-1-existentially closed if and only if it is $\mathcal{L}_1$-1-existentially closed and in such models $\mathcal{A}$ for all sequences a and b of elements of $\mathcal{A}$, a and b have the same $\mathcal{L}_0$-type if and only if they have the same $\mathcal{L}_1$-type ($\mathcal{L}_i$-1-existentially closed is defined as 1-existentially closed but with atomic formulas replaced by $\mathcal{L}_i$-formulas). Still these notions of type do not have a reasonable lower bound:

For a contradiction, assume that there is some notion of type such that relative to this notion of type, $\mathbf{K}$ has a homogeneous and universal model and such that the notion of type is weaker than both $\mathcal{L}_0$-type and $\mathcal{L}_1$-type. We define models $\mathcal{A}_i$, $i < 3$, so that they belong to $\mathbf{K}$ and, in addition,

1. the universe of $\mathcal{A}_i$ is $\{0, i\}$,
2. $0 \in P_0^{\mathcal{A}_i}$,
3. $F_0^{\mathcal{A}_1}(1) = 0$ and $F_1^{\mathcal{A}_2}(2) = 0$.

Then $\mathcal{A}_0 \leq_1 \mathcal{A}_1$ and $\mathcal{A}_0 \leq_0 \mathcal{A}_2$. Since the notion of type is weaker than both $\mathcal{L}_0$-type and $\mathcal{L}_1$-type and there is a homogeneous and universal model relative to it, we should be able to amalgamate $\mathcal{A}_1$ and $\mathcal{A}_2$ over $\mathcal{A}_0$, which is clearly impossible.

Finally, we point out that in Example 4.2, our construction of types will give the notion of type which is the same as $\mathcal{L}_0 \cup \mathcal{L}_1$-type. Thus the monster model that results is not even first-order elementarily equivalent with $\mathbf{M}_0$ (the model is the natural one though).

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