Notre Dame Journal of Formal Logic
Volume VI, Number 3, July 1965

A GENERALIZATION OF SIERPIŃSKI'S THEOREM ON STEINER TRIPLES AND THE AXIOM OF CHOICE

WILLIAM J. FRASCELLA

In the language of combinatorial analysis, a finite set F is said to possess a *Steiner triple system* if and only if there exists a family $\mathfrak{I}$ of subsets of F such that 1) each element of $\mathfrak{I}$ contains exactly three elements of F and 2) every subset of F, containing exactly two elements, is contained in exactly one of the elements of $\mathfrak{I}$. It has been long established that a necessary and sufficient condition for the existence of such a system for a finite set F is that $\overline{\overline{F}} \equiv 1$ or $3 \pmod 6$.

In [1], W. Sierpiński has showed that a Steiner triple system always exists for any set which is not finite. The proof of this result depends upon the axiom of choice. In [2], B. Sobociński has proved that the assumption that every non-finite set possesses a Steiner triple system is, in fact, equivalent to the axiom of choice.

The aim of the present paper is to further generalize these two results. We begin by making a

Definition 1: *An arbitrary set E is said to possess a Steiner system of order k (where k is a natural number >1) if there exists a family $\mathfrak{I}_k$ of subsets of E such that 1) each element of $\mathfrak{I}_k$ contains exactly k elements of E and 2) every subset of E, containing exactly k-1 elements, is contained in exactly one member of the family $\mathfrak{I}_k$.*

§1. With the aid of the axiom of choice we shall show that every set which is not finite possesses a Steiner system of order n for $n = 2,3,4,\ldots$. In addition, we shall establish that the assumption that every set which is not finite possesses a Steiner system of order n, for $n = 3,4,\ldots$, is equivalent to the axiom of choice. We are not able to demonstrate the necessity of the axiom of choice to establish the existence of a Steiner system of order *2* for any set which is not finite.

To this end we first prove, with the aid of the axiom of choice,

Theorem 1: *Let E be any set which is not finite. Then E possesses a Steiner system of order n for $n = 3,4,\ldots$.*

Received November 27, 1964

Proof: As mentioned above, the theorem has been proved by Sierpiński for $n = 3$. In the manner of induction we will assume

(1) *Theorem 1 is true for* $n-1$, $n > 3$.

Now the axiom of choice tells us that the non-finite set E has as its cardinal number some aleph. That is,

(2) $\overline{\overline{E}} = \aleph_\lambda$.

Thus without loss of generality we may impose a well-ordering on E such that $\overline{\overline{E}} = \overline{\omega}_\lambda$, where ω_λ is the initial ordinal number of the class of all ordinals whose cardinality is $\aleph_\lambda$. Hence, we may take E to be the set of all ordinal numbers less than ω_λ.

In [1], Sierpiński remarks that the set $P_2 = \{<\alpha,\beta>: \alpha < \beta < \omega_\lambda\}$ can be given a well-ordering such that $\overline{P}_2 = \omega_\lambda$. (Here, as elsewhere in this paper, $<,>$ is to be taken as the symbol for an ordered pair. Similarly, $<,,>$ is to be taken as an ordered triple, etc..... Also, all small Greek letters are to be regarded as ordinal numbers.) The proof of Theorem 1 will depend upon a generalization of this remark. Its statement will be given the form of a lemma whose demonstration will follow the proof of the theorem.

Lemma 1: *The set* $P_k = \{<\alpha_1, \ldots, \alpha_k>: \alpha_1 < \alpha_2 < \ldots < \alpha_k < \omega_\lambda\}$ *can be given a well-ordering such that* $\overline{P}_k = \omega_\lambda$, *for* $k = 2,3,4, \ldots$.

Now, in virtue of this lemma, we are in a position to index the elements of P_{n-1} and express this set as follows:

(3) $P_{n-1} = \{<\alpha_\xi^{(1)}, \ldots, \alpha_\xi^{(n-1)}>: \xi < \omega_\lambda\}$.

By (1) we know E possesses a Steiner system of order $n-1$. Hence there exists a family $\mathfrak{F}_{n-1}$ satisfying the properties of Definition 1 for $k = n-1$.

Before proceeding it is necessary to make some definitions.

Definition 2: *Let* γ *be an ordinal number less than* ω_λ. *Then* $F_\gamma^{(n-1)}$ *is that unique member of the family* $\mathfrak{F}_{n-1}$ *which contains the set* $\{\alpha_\gamma^{(1)}, \ldots, \alpha_\gamma^{(n-2)}\}$.

In addition, suppose that

(4) $F_\gamma^{(n-1)} = \{\alpha_\gamma^{(1)}, \ldots, \alpha_\gamma^{(n-2)}, \beta\}$

and that

(5) $\alpha_\gamma^{(1)} < \ldots < \alpha_\gamma^{(i)} < \beta < \alpha_\gamma^{(i+1)} < \ldots < \alpha_\gamma^{(n-2)} < \omega_\lambda$.

We now formulate another

Definition 3:[1]

$$S(F_\gamma^{(n-1)}) = \begin{cases} \alpha_\gamma^{(n-1)} & \text{if } \alpha_\gamma^{(n-1)} \neq \beta \\ \left(\sum_{i=1}^{n-1} \alpha_\gamma^{(i)}\right) + 1 & \text{if } \alpha_\gamma^{(n-1)} = \beta \end{cases}$$

1. In this paper the symbol Σ will represent the standard addition of either ordinal or cardinal numbers. On the other hand, the symbol $\cup$, which later appears in (10), represents the standard concept of set-theoretical union.

We are now in a position to construct, after the manner of Sierpiński in [1], with certain modifications, a transfinite sequence of ordinal numbers indexed by all ordinals less than ω_λ. Let $\varphi_1 = 1$. Assume δ to be an arbitrary ordinal number such that $1 < \delta < \omega_\lambda$. Now suppose φ_ξ has been defined for all $\xi < \delta$. Then we let φ_δ be the smallest ordinal μ which satisfies the following condition:

(6) $\{\alpha_\mu^{(1)}, \ldots, \alpha_\mu^{(n-1)}\} \not\subset \{F_{\varphi_\xi}^{(n-1)} \cup S(F_{\varphi_\xi}^{(n-1)}): \xi < \delta\}$.

To establish that this construction is non-vacuous it is sufficient to exhibit a μ such that (6) holds. To accomplish this we construct the following sets:

(7) $R_i = \{f_i(\varphi_\xi): \xi < \delta\}$

where $f_i(\varphi_\xi) = \alpha_{\varphi_\xi}^{(i)}$ for $i = 1,2,\ldots,(n-1)$. It is clear that for each i we have

(8) $\overline{R}_i \leqq \delta$

where R_i has the order induced by the indices of the elements of the transfinite sequence already defined. Hence

(9) $\overline{\overline{R}}_i \leqq \overline{\delta}$ *for* $i = 1,2,\ldots,(n-1)$.

But clearly $\overline{\delta}$ is either a finite cardinal number or an aleph. If we now construct

(10) $R = \bigcup_{i=1}^{n-1} R_i$

it is clear that $\overline{\overline{R}} \leqq \sum_{i=1}^{n-1} \overline{\overline{R}}_i$. Now if $\overline{\delta}$ is a finite cardinal number it is immediate that $\overline{\overline{R}} < \aleph_\lambda = \overline{\overline{E}}$. On the other hand, however, if $\overline{\delta}$ is an aleph, say $\aleph_*$, we have, in virtue of the fact that $\aleph_* + \aleph_* = \aleph_*$,

(11) $\overline{\overline{R}} \leqq \aleph_*$.

But since $\delta < \omega_\lambda$ and ω_λ is an initial number

(12) $\aleph_* < \aleph_\lambda$

and therefore we again arrive at

(13) $\overline{\overline{R}} < \aleph_\lambda = \overline{\overline{E}}$.

It is clear, then, that there must exist n-2 elements of E which are not contained in R. That is, there exists $\alpha^{(1)}, \ldots, \alpha^{(n-2)}$ such that

(14) $\alpha^{(i)} \in E - R$ for $i = 1,2,\ldots,(n-2)$.

Hence by (7) and (14) no $\alpha^{(i)}$ can be considered an image point of the function f_j for all $\langle i,j \rangle \in \{1,2,\ldots,(n-2)\} \times \{1,2,\ldots,(n-1)\}$. Therefore

(15) $\alpha^{(i)} \notin \{F^{(n-1)}_{\varphi_\xi}: \ \xi < \delta\}$ for $i = 1,2,\ldots,(n\text{-}2)$.

If we suppose $\alpha^{(1)} < \ldots < \alpha^{(n-2)}$ we have

(16) $<\alpha^{(1)},\ldots,\alpha^{(n-2)}, \alpha^{(n-2)}+1> \in P_{n-1}$.

In virtue of (3) we may write

(17) $\alpha^{(1)} = \alpha^{(1)}_\mu; \ldots; \alpha^{(n-2)} = \alpha^{(n-2)}_\mu; \alpha^{(n-2)}+1 = \alpha^{(n-1)}_\mu$,

for some $\mu < \omega_\lambda$.

Therefore

(18) $\{\alpha^{(1)}_\mu,\ldots,\alpha^{(n-1)}_\mu\} \not\subset \{F^{(n-1)}_{\varphi_\xi} \cup S(F^{(n-1)}_{\varphi_\xi}): \xi < \delta\}$.

Thus the construction of the transfinite sequence is well formed. We now state and prove an important property of this transfinite sequence.

Lemma 2: *The transfinite sequence* $\{\varphi_\xi\}_{\xi<\omega_\lambda}$ *is strictly increasing.*

Proof: To the contrary suppose we have either of the following:

Case 1: $\eta_1 < \eta_2 < \omega_\lambda$ *and* $\varphi_{\eta_1} = \varphi_{\eta_2}$

Case 2: $\eta_1 < \eta_2 < \omega_\lambda$ *and* $\varphi_{\eta_1} > \varphi_{\eta_2}$.

If *Case 1* occurs we have by (6), $\varphi_{\eta_2} = \mu_2$ to be the smallest ordinal such that

(19) $\{\alpha^{(1)}_{\mu_2},\ldots,\alpha^{(n-1)}_{\mu_2}\} \not\subset \{F^{(n-1)}_{\varphi_\xi} \cup S(F^{(n-1)}_{\varphi_\xi}): \ \xi < \eta_2\}$.

But since $\eta_1 < \eta_2$, we must have

(20) $\{\alpha^{(1)}_{\mu_2},\ldots,\alpha^{(n-1)}_{\mu_2}\} \not\subset \{F^{(n-1)}_{\varphi_{\eta_1}} \cup S(F^{(n-1)}_{\varphi_{\eta_1}})\}$.

But by assumption, $\varphi_{\eta_1} = \varphi_{\eta_2}$; hence

(21) $\{\alpha^{(1)}_{\mu_2},\ldots,\alpha^{(n-1)}_{\mu_2}\} \not\subset \{F^{(n-1)}_{\varphi_{\eta_2}} \cup S(F^{(n-1)}_{\varphi_{\eta_2}})\}$

which contradicts the very definitions of $F^{(n-1)}_{\varphi_{\eta_2}}$ and $S(F^{(n-1)}_{\varphi_{\eta_2}})$. Thus *Case 1* never obtains.

Suppose *Case 2* occurs. By (6) we have $\varphi_{\eta_1} = \mu_1$, the smallest ordinal such that

(22) $\{\alpha^{(1)}_{\mu_1},\ldots,\alpha^{(n-1)}_{\mu_1}\} \not\subset \{F^{(n-1)}_{\varphi_\xi} \cup S(F^{(n-1)}_{\varphi_\xi}): \xi < \eta_1\}$.

In the same manner we have $\varphi_{\eta_2} = \mu_2$ to be the smallest ordinal such that

(23) $\{\alpha^{(1)}_{\mu_2},\ldots,\alpha^{(n-1)}_{\mu_2}\} \not\subset \{F^{(n-1)}_{\varphi_\xi} \cup S(F^{(n-1)}_{\varphi_\xi}): \ \xi < \eta_2\}$.

But (23) and the fact that $\eta_1 < \eta_2$ implies

(24) $\{\alpha_{\mu_2}^{(1)}, \ldots, \alpha_{\mu_2}^{(n-1)}\} \not\subset \{F_{\varphi_\xi}^{(n-1)} \cup S(F_{\varphi_\xi}^{(n-1)}) : \xi < \eta_1\}$.

But (24) and the fact that $\mu_2 < \mu_1$ contradict the definition of φ_{η_1}. Thus *Case 2* never obtains and Lemma 2 is proved.

Finally we are in a position to define a family of subsets of E which will insure the existence of a Steiner system of order n.

Definition 4: $\mathcal{J}_n = \{F_{\varphi_\xi}^{(n-1)} \cup S(F_{\varphi_\xi}^{(n-1)}) : \xi < \omega_\lambda\}$.

To show this is the family in question, let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n-1)}$ be any n-1 distinct elements of E. We may assume that

(25) $\alpha^{(1)} < \alpha^{(2)} < \ldots < \alpha^{(n-1)} < \omega_\lambda$.

Clearly we have $< \alpha^{(1)}, \ldots, \alpha^{(n-1)} > \epsilon P_{n-1}$. Therefore by (6) there must exist an ordinal number $\mu < \omega_\lambda$ such that

(26) $\alpha_\mu^{(i)} = \alpha^{(i)}$ for $i = 1,2,\ldots,(n-1)$.

But since the sequence $\{\varphi_\xi\}_{\xi < \omega_\lambda}$ is strictly increasing, there exists an ordinal $\delta < \omega_\lambda$ such that

(27) $\varphi_\delta > \mu$.

But by the definition of φ_δ, there must exist an ordinal $\xi_0 < \delta$ such that

(28) $\{\alpha_\mu^{(1)}, \ldots, \alpha_\mu^{(n-1)}\} \subset \{F_{\varphi_{\xi_0}}^{(n-1)} \cup S(F_{\varphi_{\xi_0}}^{(n-1)})\}$

for otherwise (27) could not hold. Therefore (28) shows every n-1 distinct elements of E is contained in at least one member of the family $\mathcal{J}_n$.

On the other hand, suppose we have n-1 distinct elements of E contained in two distinct members of the family $\mathcal{J}_n$. That is, suppose we have $\eta < \xi < \omega_\lambda$ such that

(29) $\{\alpha^{(1)}, \ldots, \alpha^{(n-1)}\} \subset \{F_{\varphi_\eta}^{(n-1)} \cup S(F_{\varphi_\eta}^{(n-1)})\}$

(30) $\{\alpha^{(1)}, \ldots, \alpha^{(n-1)}\} \subset \{F_{\varphi_\xi}^{(n-1)} \cup S(F_{\varphi_\xi}^{(n-1)})\}$.

Again by (6) there must exist an ordinal number $\mu < \omega_\lambda$ such that $\alpha^{(i)} = \alpha_\mu^{(i)}$ for $i = 1, \ldots, n$-1. By Definition 3, $S(F_{\varphi_\eta}^{(n-1)})$ and $S(F_{\varphi_\xi}^{(n-1)})$ are the greatest elements (according to magnitude) of the sets $\{F_{\varphi_\eta}^{(n-1)} \cup S(F_{\varphi_\eta}^{(n-1)})\}$ and $\{F_{\varphi_\xi}^{(n-1)} \cup S(F_{\varphi_\xi}^{(n-1)})\}$, respectively. But since we assume $\alpha^{(1)} < \ldots < \alpha^{(n-1)}$, we must have

(31) $\{\alpha^{(1)}, \ldots, \alpha^{(n-2)}\} \subset F_{\varphi_\eta}^{(n-1)}$

and

(32) $\{\alpha^{(1)}, \ldots, \alpha^{(n-2)}\} \subset F^{(n-1)}_{\varphi_\xi}$.

But since

(33) $F^{(n-1)}_{\varphi_\eta} \in \mathcal{J}_{n-1}$ and $F^{(n-1)}_{\varphi_\xi} \in \mathcal{J}_{n-1}$,

we must have

(34) $F^{(n-1)}_{\varphi_\eta} = F^{(n-1)}_{\varphi_\xi}$

since $\mathcal{J}_{n-1}$ is the Steiner family of order $n-1$.

There now follows two cases:

Case 1: $\alpha^{(n-1)} \in F^{(n-1)}_{\varphi_\eta} = F^{(n-1)}_{\varphi_\xi}$

Case 2: $\alpha^{(n-1)} \notin F^{(n-1)}_{\varphi_\eta} = F^{(n-1)}_{\varphi_\xi}$.

If *Case 1* occurs, by Definition 3 we must have

(35) $S(F^{(n-1)}_{\varphi_\eta}) = S(F^{(n-1)}_{\varphi_\xi}) = \alpha^{(1)} + \alpha^{(2)} + \ldots + \alpha^{(n-1)} + 1$.

But this contradicts our assumption that the members of $\mathcal{J}_n$ are distinct.

If *Case 2* occurs we must have

(36) $S(F^{(n-1)}_{\varphi_\eta}) = \alpha^{(n-1)} = S(F^{(n-1)}_{\varphi_\xi})$.

But this, too, leads to the same contradiction.

Thus the family $\mathcal{J}_n$ has the properties of Definition 1 and the existence of a Steiner system of order n for the set E is assured. The induction being completed, Theorem 1 is proved.

We now return to the unfinished business of proving Lemma 1.

Proof of Lemma 1: Since the lemma is true for $n = 2$ it will be sufficient to proceed by induction. Hence we assume

(37) *Lemma 1 to be true for* $k = n-1$ $(n \geqq 3)$.

Since $\overline{\overline{E}} = \aleph_\lambda$ and the fact that $\aleph \cdot \aleph = \aleph$ for any aleph, $\aleph$, we have

(38) $\overline{\overline{(\underbrace{E \times \ldots \times E}_{n\text{-times}})}} = \aleph_\lambda$.

But according to the definition of P_n we have

(39) $P_n \subset (\underbrace{E \times \ldots \times E}_{n\text{-times}})$

and hence

(40) $\overline{\overline{P}}_n \leqq \aleph_\lambda$.

We now consider the following subsets of P_{n-1}:

(41) $P^{\#}_{n-1} = \{<0,\alpha^{(2)},\ldots,\alpha^{(n-1)}>: 0 < \alpha^{(2)} < \ldots < \alpha^{(n-1)} < \omega_\lambda\}$

(42) $P^*_{n-1} = P_{n-1} - P^{\#}_{n-1}$.

By (37) we have $\overline{\overline{P}}_{n-1} = \aleph_\lambda$. Also since, by (42), $P^*_{n-1} \subset P_{n-1}$:

(43) $\overline{\overline{P^*_{n-1}}} \leqq \aleph_\lambda$.

But it is possible to map the set E in an one-one manner onto a certain subset of P^*_{n-1}. Let

(44) $f{:}E \to P^*_{n-1}$

where,

(45) $f(\alpha) = \begin{cases} <1,2,\ldots,(n-2),\alpha> & \text{if } \alpha > n-2 \\ <\alpha+1, \alpha+2,\ldots,\alpha+n-1> & \text{if } \alpha \leqq n-2. \end{cases}$

We remark that f is well constructed, since by the definition of P^*_{n-1}, no n-1 tuple in this set has 0 as its first coordinate. Thus the set E is equinumerous to some subset of P^*_{n-1}. Hence

(46) $\overline{\overline{P^*_{n-1}}} \geqq \aleph_\lambda = \overline{\overline{E}}$.

Thus (43) and (46) yield

(47) $\overline{\overline{P^*_{n-1}}} = \aleph_\lambda$.

We now construct the following subset of P_n:

(48) $P^*_n = \{<0,\alpha_2,\ldots,\alpha_n>: 0 < \alpha_2 < \ldots < \alpha_n < \omega_\lambda\}$.

An one-one correspondence naturally arises between the sets P^*_n and P^*_{n-1}. Namely,

(49) $g{:}P^*_{n-1} \to P^*_n$

where

(50) $g(<\alpha_1,\ldots,\alpha_{n-1}>) = <0,\alpha_1,\ldots,\alpha_{n-1}>$.

The very definitions of the sets P^*_{n-1} and P^*_n insure that the map g is well defined. Hence,

(51) $\overline{\overline{P^*_{n-1}}} = \overline{\overline{P^*_n}}$.

Together with (47) we have

(52) $\overline{\overline{P^*_n}} = \aleph_\lambda$.

But since $P^*_n \subset P_n$ we conclude

(53) $\bar{\bar{P}}_n \geqq \aleph_\lambda$.

Therefore (40) and (53) establish

(54) $\bar{\bar{P}}_n = \aleph_\lambda$.

Thus we may impose a well-ordering on P_n such that $\overline{P}_n = \omega_\lambda$. The induction being completed, Lemma 1 is proved.

§2. In order to further our results, we will establish a functional characterization for a Steiner system of arbitrary order.

Theorem 2: Let E be any set which is not finite. Then E possesses a Steiner system of order n, for $n = 2,3,\ldots$ if and only if there exists a set function f such that

1° *The domain of f is the family of all subsets of E which contain exactly n-1 elements.*

2° *The range of f is some subset of E.*

3° $f(\{a_1,a_2,\ldots,a_{n-1}\}) \notin \{a_1,a_2,\ldots,a_{n-1}\}$.

4° *If* $f(\{a_1,\ldots,a_{n-1}\}) = b \in E$ *then* $f(\{a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_{n-1}\}) = a_i$ *for* $i = 1,2,\ldots,n-1$.

Remark: It is important to observe that the proof of Theorem 2 will not employ the axiom of choice.

Proof:

Necessity: Suppose the non-finite set E possesses a Steiner system of order n. Let us now construct

(55) $\mathcal{A} = \{A \subset E: \bar{\bar{A}} = n\text{-}1\}$.

By Definition 1, we know that for every $A \in \mathcal{A}$ there exists a unique element of $\mathcal{J}_n$ which contains A. We represent such an element of $\mathcal{J}_n$ by the symbol F_A. Next, we construct a map

(56) $f: \mathcal{A} \to E$

where

(57) $f(A) = F_A - A$ for each $A \in \mathcal{A}$.

Since $F_A \in \mathcal{J}_n$, it follows that $\bar{\bar{F}}_A = n$. But also $\bar{\bar{A}} = n\text{-}1$ and $A \subset F_A$. Hence $f(A) \in E$. Thus f satisfies properties 1° and 2° of the theorem. Clearly $f(A) = (F_A - A) \notin A$, and property 3° is thereby satisfied.

Now suppose $A \in \mathcal{A}$. Thus we may write

(58) $A = \{a_1,\ldots,a_{n-1}\}$.

And suppose

(59) $F_A = \{a_1,\ldots,a_{n-1},b\}$.

Therefore by (57) we have

(60) $f(A) = b$.

Now construct the set $A' = \{a_1, \ldots, a_{i-1}, b, a_{i+1}, \ldots, a_{n-1}\}$. By (59) we see $A' \subset F_A$. Hence

(61) $F_{A'} = F_A$.

Therefore we have

(62) $f(A') = F_{A'} - A' = F_A - A = a_i$.

Property **4°** holding for f, necessity is established.

Sufficiency: Suppose we have given the function f with the stated properties **1°-4°**. We now define for each $A \in \mathcal{A}$,

(63) $F_A^* = A \cup f(A)$.

By properties **2°** and **3°** of f we see that F_A^* is a subset of E consisting of exactly n elements.

We are now in a position to define a family of subsets of E consisting of exactly n elements. Namely,

(64) $\mathcal{F}_n = \{F_A^* : A \in \mathcal{A}\}$.

It remains to show that $\mathcal{F}_n$ establishes a Steiner system of order n.

By property **1°** of f and (64) it is clear any subset of E, consisting of exactly n-1 elements, is contained in at least one member of the family $\mathcal{F}_n$. Specifically

(65) $A \subset F_A^*$ *for each* $A \in \mathcal{A}$.

It remains to show that every $A \in \mathcal{A}$ is contained in, at most, one member of the family $\mathcal{F}_n$. To the contrary, suppose

(66) $A \subset F_X^* \in \mathcal{F}_n$

and

(67) $A \subset F_Y^* \in \mathcal{F}_n$

where

(68) $F_X^* \neq F_Y^*$.

Suppose $A = \{a_1, \ldots, a_{n-1}\}$. Then we have

(69) $F_X^* = \{a_1, \ldots, a_{n-1}, x\}$

and

(70) $F_Y^* = \{a_1, \ldots, a_{n-1}, y\}$.

But (69) and (70) together with property **4°** of f yields

(71) $f(A) = x$ and $f(A) = y$.

Hence $x = y$, which contradicts (68).

Thus the family $\mathcal{F}_n$ has the desired properties and, therefore, E possesses a Steiner system of order n. Sufficiency established, Theorem 2 is proved.

§3. After constructing and characterizing the Steiner system of order n, one naturally raises the question as to whether the existence of a Steiner system of arbitrary order implies the axiom of choice. An answer is obtained in

Theorem 3: *The assumption that every non-finite set possesses a Steiner system of order n implies the axiom of choice for any $n \geq 3$.*

Remark: The proof of this theorem follows, in substance, the proof given in [2] by B. Sobociński, who has established the result for the case when $n = 3$.

Proof: Let $\mathfrak{m}$ be an arbitrary cardinal number which is not finite. As is well known, to $\mathfrak{m}$ we may associate a certain aleph, $\aleph(\mathfrak{m})$, called Hartogs' aleph for $\mathfrak{m}$, where $\aleph(\mathfrak{m})$ is the least aleph with the property:

(72) $\aleph(\mathfrak{m}) \nleq \mathfrak{m}$.

Since $\aleph(\mathfrak{m})$ is an aleph, there must exist an ordinal number λ such that

(73) $\aleph(\mathfrak{m}) = \aleph_\lambda$.

Let ω_λ represent the initial number of the class of all ordinals whose cardinality is $\aleph_\lambda$. Elementary results tell us there exists a cardinal number, $\mathfrak{m} + \aleph(\mathfrak{m})$, which is not finite.

Hence, there must exist non-finite sets E, R and P such that

(74) $\overline{\overline{P}} = \aleph(\mathfrak{m}) = \aleph_\lambda$

(75) $P = \{\alpha : \alpha \textit{ is an ordinal } < \omega_\lambda\}$

(76) $\overline{\overline{R}} = \mathfrak{m}$

(77) $R \cap P = \phi$

(78) $E = R \cup P$

(79) $\overline{\overline{E}} = \overline{\overline{R}} + \overline{\overline{P}} = \mathfrak{m} + \aleph(\mathfrak{m})$.

By the hypothesis of Theorem 3, the non-finite set E possesses a Steiner system of order n, where n is a natural number greater than 3. Thus, by Definition 1, there must exist a family $\mathcal{J}$ of subsets of E such that

(80) *every element of $\mathcal{J}$ is a subset of E containing exactly n elements*

and

(81) *every n-1 distinct elements of E is contained in one, and only one, member of the family $\mathcal{J}$.*

Remark: As in Lemma 1, P_k will represent the collection:

(82) $\{<\alpha_1, \ldots, \alpha_k>: \alpha_1 < \ldots < \alpha_k < \omega_\lambda\}$

where the α_i's are all ordinal numbers and ω_λ is the initial number referred to above. The conclusion of Lemma 1 was

(83) $\overline{P}_k = \omega_\lambda$ for $k = 1, 2, \ldots$.

An immediate corollary to this result would be

(84) $\overline{\overline{P}}_k = \overline{\omega}_\lambda = \aleph_\lambda$ for $k = 1,2,\ldots$.

We now introduce another

Definition 5: *For any natural number n, $P_n^\dagger$ will represent the family of all subsets of P which contain exactly n elements.*

A correspondence naturally arises between $P_n^\dagger$ and P_n. For, every element $p \in P_n^\dagger$ is of the form $p = \{\alpha_1, \ldots, \alpha_n\}$, where $\alpha_i \in P$ for $i = 1, \ldots, n$. If we assume,

(85) $\alpha_1 < \ldots < \alpha_n$

we may associate to this element $p \in P_n^\dagger$ the element $<\alpha_1, \ldots, \alpha_n> \in P_n$. Such an association is clearly an one-one onto correspondence of the sets $P_n^\dagger$ and P_n. Hence,

(86) $\overline{\overline{P_n^\dagger}} = \overline{\overline{P_n}}$.

Together with (84) we have

(87) $\overline{\overline{P_n^\dagger}} = \overline{\omega}_\lambda = \aleph_\lambda$ for $n = 1,2,\ldots$.

As a matter of fact we have,

(88) $\overline{\overline{P_n^\dagger}} = \aleph(\mathfrak{m})$ for $n = 1,2,\ldots$.

This concludes our remark.

Returning to the proof of Theorem 3 we make a

Definition 6: *For any $r \in R$ we define a family of sets F_r as follows: $x \in F_r$ if and only if 1) $x \in \mathfrak{I}$ and 2) there exists n-2 distinct elements of P, say $\alpha_1, \ldots, \alpha_{n-2}$, such that a) the ordinal numbers 1,2,...,n-3 are contained in the set $\{\alpha_1, \ldots, \alpha_{n-2}\}$ and b) $\{r, \alpha_1, \ldots, \alpha_{n-2}\} \subset x$.*

Definition 6 immediately implies

(89) $F_r \subset \mathfrak{I}$ for any $r \in R$.

Lemma 3: *The family F_r is not empty for every $r \in R$.*

Proof: Let $r \in R$. Then by (78), $r \in E - P$. Certainly the set P contains the ordinals $1, \ldots, n\text{-}3$, and, at least, one additional ordinal α. Thus $\{r, 1, 2, \ldots, n\text{-}3, \alpha\}$ is a subset of E consisting of exactly $n\text{-}1$ elements. By (81), there exists a unique $x \in \mathfrak{I}$ such that

(90) $\{r, 1, 2, \ldots, n\text{-}3, \alpha\} \subset x$.

Clearly this x satisfies the requirements of Definition 6, and hence

(91) $x \in F_r$.

Thus for each $r \in R$, F_r is not empty. Lemma 3 is proved.

We now wish to exhibit certain distinguished members of the family F_r. To this end we state

Lemma 4: *Let $r \in R$. Then there exists an $x \in F_r$ such that $x = \{r,\alpha_1, \ldots, \alpha_{n-1}\}$ where all the α_i's are elements of the set P.*

Remark: Since $x \in F_r$, we are guaranteed that at least n-2 of the α_i's are elements of P. In fact, we know that the ordinals $1, 2, \ldots, (n\text{-}3)$ must be among them.

Proof: To the contrary, we assume

(**A**) $r \in R$

and

(**B**) *if* $\{r,1,2, \ldots, (n\text{-}3), \alpha, \ell\} \in F_r$, *where* $\{1,2, \ldots, (n\text{-}3), \alpha\} \subset P$, *then* $\ell \in R$.

It is now possible to construct a mapping

(92) $f_1 : P - \{1,2, \ldots, n\text{-}3\} \to F_r$

where

(93) *for* $\alpha \in P - \{1,2, \ldots, n\text{-}3\}$, $f_1(\alpha)$ *represents the unique element of* $\mathfrak{I}$ *which contains the* n-1 *distinct elements* $\{r,1,2, \ldots, n\text{-}3, \alpha\}$.

It is clear from (93) and Definition 6, that $f_1(\alpha) \in F_r$ and thus f_1 is well-defined. For each $z \in F_r$ we must have $z = \{r,1,2, \ldots, n\text{-}3, x, y\}$. But by Definition 6, *at least* one of the elements x, y must be an element of $P - \{1,2, \ldots, n\text{-}3\}$. But by (**B**), *at most* one of the elements x,y can belong to P. Hence, z contains a unique element $\alpha \in P - \{1, 2, \ldots, n\text{-}3\}$, such that $f_1(\alpha) = z$. Thus f_1 is onto.

Suppose $\alpha, \beta \in P - \{1,2, \ldots, n\text{-}3\}$ such that $\alpha \neq \beta$. Then we have

(94) $f_1(\alpha) = \{r,1,2, \ldots, (n\text{-}3), \alpha, x\} \in F_r \subset \mathfrak{I}$

and

(95) $f_1(\beta) = \{r, 1,2, \ldots, (n\text{-}3), \beta, y\} \in F_r \subset \mathfrak{I}$.

By (**B**) we know

(96) $x,y \notin P$.

Thus, if we suppose $f_1(\alpha) = f_1(\beta)$ we must have by (94) and (95)

(97) $x = \beta$ and $y = \alpha$.

But this contradicts (96). Therefore,

(98) $f_1(\alpha) \neq f_1(\beta)$

which establishes the fact that f_1 is an one-one onto correspondence of the sets $P - \{1, \ldots, n\text{-}3\}$ and F_r. Hence,

(99) $\overline{\overline{P - \{1, \ldots, n\text{-}3\}}} = \overline{\overline{F}}_r$.

But it is clear, since P is not finite, that

(100) $\overline{\overline{P-\{1,\ldots,n-3\}}} = \overline{\overline{P}} = \aleph(\mathfrak{m})$.

Therefore (99) and (100) yield

(101) $\overline{\overline{F}}_r = \aleph(\mathfrak{m})$.

To complete the proof of Lemma 4 we shall need another

Definition 7: Let $r \in R$. *Then* R_r *will denote the set of all* $\ell \in R$ *such that 1)* $\ell \neq r$ *and 2) there exists an* $x \in F_r$ *such that* $\ell \in x$.

We note that Definition 7 implies

(102) $R_r \subset R$

while (**B**) insures that

(103) R_r is not empty.

Now we construct a mapping

(104) $f_2 : F_r \to R_r$

where

(105) *for each* $x \in F_r$, $f_2(x)$ *represents that element of* x, *which belongs to* R, *but different from* r.

Since $x \in F_r$, by Definition 6 we know x contains the element $r \in R$, the ordinals $1, 2, \ldots, n-3$ and, *at least*, one additional ordinal α. But (**B**) insures that x contains, *at most*, one additional ordinal α. Thus x, which contains r, must contain a unique element of R which is different from r. This shows f_2 to be well defined. Let $\ell \in R_r$. By Definition 7, we know

(106) $\ell \neq r$

and

(107) *there exists an* $x \in F_r$ *such that* $\ell \in x$.

By Definition 6, and using the same argument following (105), we see that x contains a unique element of R different from r. But (106) and (107) imply this element must be ℓ. Thus $f_2(x) = \ell$ and f_2 is shown to be onto. Let $x, y \in F_r$ such that $x \neq y$. And suppose

(108) $f_2(x) = f_2(y) = z$.

But (108) implies that both x and y have the following $n-1$ elements in common:

(109) $r, 1, 2, \ldots, (n-3), z$.

But since $x, y \in \mathcal{J}$, (81) gives

(110) $x = y$

contradicting our assumption. Hence we conclude that (108) is not true and the map f_2 is one-one. Thus

(111) $\bar{\bar{F}}_r = \bar{\bar{R}}_r$.

This, together with (101), gives

(112) $\bar{\bar{R}}_r = \aleph(\mathfrak{m})$.

But $R_r \subset R$. Therefore we obtain from (112)

(113) $\aleph(\mathfrak{m}) \leqq \bar{\bar{R}} = \mathfrak{m}$

which contradicts (72). Thus, the assumption that Lemma 4 is false leads to an absurdity. By the law of the excluded middle, Lemma 4 is proved.

In retrospect, we have been able to establish that for each $r \in R$, there exists an element $x \in F_r$ such that $x = \{r, 1, 2, \ldots, (n-3), \alpha, \beta\}$ where $1, 2, \ldots, (n-3)$, α and β are all elements of P. Continuing we introduce,

Definition 8: *Let $r \in R$. Then F_r^* denotes the set of all $x \in F_r$ such that x satisfies the conditions of Lemma 4.*

In a natural way, we may construct, for each $r \in R$, a map

(114) $f_3 : F_r^* \to P_{n-1}^{\dagger}$

where

(115) *for every $x \in F_r^*$, $f_3(x)$ represents the set of n-1 distinct elements of P, which by Definition 8 must be contained in x.*

It is clear that f_3 is well defined. Suppose $x, y \in F_r^*$, such that $x \neq y$. Since, $F_r^* \subset F_r$, we must have

(116) $r \in x$ and $r \in y$.

Thus the n-1 remaining elements of x (i.e. those different from r) cannot be identical with the n-1 remaining elements of y. But these sets of remaining elements for x and y are $f_3(x)$ and $f_3(y)$, respectively. Hence

(117) $f_3(x) \neq f_3(y)$,

and therefore f_3 is an one-one correspondence between F_r^* and some subset of $P_{n-1}^{\dagger}$.

Let $f_3(F_r^*)$ represent the range of f_3. Clearly,

(118) $f_3(F_r^*) \subset P_{n-1}^{\dagger}$.

Lemma 1 has showed that P_{n-1} is a well-ordered set. By (86) and (87) it is clear that $P_{n-1}^{\dagger}$ can also be considered a well-ordered set whose order is induced by P_{n-1}. Thus

(119) *$f_3(F_r^*)$ is a non-empty subset of the well-ordered set $P_{n-1}^{\dagger}$ for each $r \in R$*

and, therefore, $f_3(F_r^*)$ is, itself, well-ordered This enables us to make the following

Definition 9: *For each $r \in R$, $f^*[f_3(F_r^*)]$ is defined to be the initial element of the well-ordered set $f_3(F_r^*)$.*

Finally we are in a position to construct a mapping

(120) $f_4 : R \rightarrow P^{\dagger}_{n-1}$

where

(121) for each $r \in R, f_4(r) = f^*[f_3(F^*_r)]$.

Definition 9 and (119) show that f_4 is well defined. Now suppose r, $\ell \in R$ such that

(122) $r \neq \ell$.

In order to show that $f_4(r) \neq f_4(\ell)$ it will be enough to show that the sets $f_3(F^*_r)$ and $f_3(F^*_\ell)$ have no elements in common. Since, if this were true, it would follow that their respective initial elements, $f_4(r)$ and $f_4(\ell)$, could not be identical. Therefore, suppose there exists a $p \in P^{\dagger}_{n-1}$ such that

(123) $p \in f_3(F^*_r) \cap f_3(F^*_\ell)$.

Since $p \in P^{\dagger}_{n-1}$ we may express $p = \{\alpha_1, \ldots, \alpha_{n-1}\}$, where $\alpha_i \in P$ for $i = 1, \ldots, n-1$. But (123) and the definition of the mapping f_3, given in (115), immediately imply

(124) $\{r, \alpha_1, \ldots, \alpha_{n-1}\} \in F^*_r \subset \mathfrak{J}$

and

(125) $\{\ell, \alpha_1, \ldots, \alpha_{n-1}\} \in F^*_\ell \subset \mathfrak{J}$.

Thus (81) shows $r = \ell$, contradicting (122). Therefore the sets $f_3(F^*_r)$ and $f_3(F^*_\ell)$ are disjoint and, thereby, the mapping f_4 is one-one.

Since f_4 is a well defined one-one map of the set R onto some subset of $P^{\dagger}_{n-1}$, it naturally follows

(126) $\overline{\overline{R}} \leqq \overline{\overline{P^{\dagger}_{n-1}}}$.

Thus, from (76) and (87), it follows that

(127) $\mathfrak{m} \leqq \aleph_\lambda = \aleph(\mathfrak{m})$.

But (72) restricts us further to

(128) $\mathfrak{m} < \aleph_\lambda = \aleph(\mathfrak{m})$.

We have thus shown that in assuming any non-finite set possesses a Steiner system of order n, for $n > 3$, one can establish the fact that any non-finite cardinal number $\mathfrak{m}$ is strictly less than some aleph, and, consequently, is itself an aleph. This is nothing other than the establishment of the axiom of choice. Theorem 3 is proved.

§4. With regard to the Steiner system of order *2*, we recognize at once that a non-finite set E possesses of Steiner system of order *2* if, and only if, there exists a decomposition of E into disjoint pairs. Thus we may prove, with the aid of the axiom of choice,

Theorem 4: Any non finite set E possesses a Steiner system of order 2.

Proof: It is well known that, with the aid of the axiom of choice, we can establish, for any non-finite cardinal $\mathfrak{m}$, the relation:

(129) $\mathfrak{m} + \mathfrak{m} = \mathfrak{m}$.

Thus, if E is any non-finite set, there must exist a non-finite cardinal $\mathfrak{m}$ such that

(130) $\overline{\overline{E}} = \mathfrak{m}$.

Therefore, there must also exist non-finite sets S and T such that

(131) $\overline{\overline{S}} = \overline{\overline{T}} = \mathfrak{m}$

(132) $S \cap T = \phi$

(133) $E = S \cup T$.

By (131) there must exist an one-one onto correspondence

(134) $g:S \to T$.

We construct a family of pairs of E as follows:

(135) $\mathfrak{I} = \{\{s,g(s)\}: s \in S\}$.

Clearly, $\mathfrak{I}$ represents a collection of disjoint pairs of E which exhausts E. Hence, by Definition 1, E possesses a Steiner system of order 2. This proves Theorem 4.

Final Remarks: In virtue of Theorem 1, we have shown that the axiom of choice is sufficient to establish the existence of a Steiner system of order n for $n = 3, 4, \ldots$, for any non-finite set E. By Theorem 4 we extended this result to the case where $n = 2$.

Moreover, since Theorem 3 was established without the aid of the axiom of choice, the existence of a Steiner system of order n for $n = 3,4, \ldots$, always implies the axiom of choice. Hence, the axiom of choice is necessary to establish the existence of a Steiner system of order n for $n = 3,4, \ldots$, for any non-finite set E.

It therefore follows that the existence of a Steiner system of order n for $n = 3,4, \ldots$, for any non-finite set E, is equivalent to the axiom of choice.

We conclude, on the basis of the above discussion, with a simple corollary to Theorem 2:

Corollary: If we designate the function f in Theorem 2 as f_n, where n refers to the order of the Steiner system f_n establishes for E, we then have, for $n = 3,4, \ldots$, the following equivalent to the axiom of choice:

For every non-finite set E, there exists a function f_n with properties 1°-4° *as stated in Theorem 2.*

BIBLIOGRAPHY

[1] W. Sierpiński: Sur un probleme de triades. *Comptes Rendus des Séances de la Société des Sciences et des Lettres de Varsovie* Classe III. Vol. 33-38(1940-1945), pp. 13-16.

[2] B. Sobociński: A theorem of Sierpiński on triads and the axiom of choice. *Notre Dame Journal of Formal Logic*, v. 5(1964), pp. 51-58.

Seminar in Symbolic Logic
University of Notre Dame
Notre Dame, Indiana