Notre Dame Journal of Formal Logic
Volume VI, Number 2, April 1965

ON LOGIC AND EXISTENCE

KAREL LAMBERT

I. A device often used to help explain the intended (conventional) sense of the quantifiers is that of *quantifier expansion* in a finite, non-empty universe of objects. As Quine puts it:[1]

> If we think of the universe as limited to a finite set of objects $a, b, \ldots, h$, we can expand existential quantifications into alternations and universal quantifications into conjunctions; '$(\exists x)Fx$' and '$(x)Fx$' become respectively: $Fa \vee Fb \ldots \vee Fh$, $Fa . Fb \ldots . Fh$

The concept of a *universe,* or more conventionally a *universe* of *discourse,* has to do with the *values* of the argument variables. The present characterization of the quantifiers thus is an *ontological* characterization.

Alternatively, the conventional sense of the quantifiers can be conveyed by a characterization in terms of the *substituends* of the argument variables. This method has been utilized recently by Lejewski.[2] Simply replace 'universe' by 'language' and 'set of objects' by 'list of non-empty argument constants' and '$a, b \ldots, h$' by ''a', 'b'..., 'h'' in the preceding quotation. This characterization is a *semantical* characterization.

The ontological and semantical characterizations of the quantifiers are parallel in the sense that they both suggest, unequivocally, an *existential* interpretation of the quantifier '$(\exists x)$', and thus prompt the idiomatic reading: 'There exists (timelessly) an x such that ...'.

Those standard versions of the lower predicate calculus with identity which include an argument substitution rule face the following difficulties: (a) If *empty* arguments constants ('Pegasus', 'the round square of Phineas') are allowed among the primitive signs, then, without further restrictions, one can infer

(1) $(\exists x)(x$ does not exist)

from

(2) Pegasus does not exist

by the principle of *Particularization,* or in symbols

(3) $fy \supset (\exists x) fx$

Received December 13, 1964

This inference is called the *singular existence anomaly*. (b) The formulae '$(x)Fx \supset (\exists x)Fx$', '$(\exists x)(Fx \vee \sim Fx)$', '$(\exists x)(x = x)$' and so on, though theorems in standard versions of the lower predicate calculus with identity, have been criticized on the grounds that their truth depends upon *facts* (that the universe of discourse is non empty). But this clashes with the ideal that truths of logic are not dependent on the facts, that, to use an expression, the truths of logic are "analytic". The objection cannot be vitiated by appeal to the vagueness of the analytic-synthetic distinction. For in the present case the issue is very clear; a logical formula is *analytic* if and only if it holds in all domains *including* the empty one. Nor is the appeal to the relative inutility of the empty universe in scientific discourse very strong support for inclusion of the formulae in question as theorems in elementary logic. Those who object to them on philosophical grounds, on grounds that they fail to meet the ideal conditions demanded for theoremhood in a logical system, will hardly be turned by appeal to non-philosophical motives and interests. (c) The predicate of *singular existence,* that is, 'exists' as in 'Pegasus exists' as opposed to 'exists' as in 'men exist', is singularly difficult to explicate within the context of quantification theory with identity. Attempts to explicate this notion often have recourse to other materials, for example, modal concepts as in H. S. Leonard's recent treatment, or to Russell's theory of descriptions.[4] Even ignoring the highly controversial character of these further devices, their introduction into elementary logic produce often intolerable complications in that relatively well understood discipline and therefore ought to be resisted. On the other hand, explications of singular existence within purely quantificational materials usually are either trivially true or contradictory.[5] But the analysis of singular existence is philosophically important if for no other reason than to make the tacit in conventional quantification theory explicit.[6] Further, it appears to leave certain important philosophical theses, e.g., Quine's 'to be is to be the value of a variable', formally inexpressible.[7]

Lejewski[8] finds the source of these problems in the contemporary logician's interpretation of the quantifiers, in particular, in the definite existential cast associated with the quantifier '$(\exists x)$'. He recommends a non-existential interpretation of the quantifier '$(\exists x)$'. His characterization of the quantifiers is a semantical one. Briefly, he does not demand that the constants 'a', 'b' . . . , 'h' each be non-empty; rather each may be either empty or non-empty. For example, let 'a' be 'Pegasus'. Then, following the instructions in the method of quantifier expansion outlined above, '$(\exists x)$ (does not exist)' expands as 'Pegasus does not exist $\vee$ b does not exist $\vee$ . . . , h does not exist'. Accordingly, (1), under Lejewski's interpretation of '$(\exists x)$', in contrast to the usual interpretation, is true. Analogous remarks hold for the universal quantifier. Lejewski shows how his interpretation of '$(\exists x)$' dissolves the anomalous character of the inference from (2) to (1), by showing that (1) no longer is false. Further, he explains that under the new interpretation of '$(\exists x)$', formulae like '$(x)Fx \supset (\exists x)Fx$' ect. are valid in every doman including the empty one. Indeed, he proposes that '$(\exists x)$' be given the neutral reading 'for some x . . . ,' and be renamed the "particular quantifier" to divorce it from its traditional ontological associations.

Finally, he introduces an *inclusion* symbol into his quantification theory, provides a semantical interpretation of it, and, following Leśniewski, shows how it can be used to define both singular and general existence.[9]

This paper does not contest Lejewski's solutions to the three problems mentioned above. His solutions, indeed, are interesting and ingenious. However, his view on the source of these same problems is open to question. The difficulties in question do not necessarily come from merging quantification with existence.[10] To prove this, it will be sufficient to construct a quantification theory, under the conventional "existential" interpretation of the quantifiers, which provides solutions to the problems provoking Lejewski's study.

II. During the last decade several quantification theories have been constructed which are free of existence assumptions so far as their argument constants are concerned.[11] In most of these theories the law of *Particularization* (or the rule of *Existential Generalization*) has been modified to hold only under the condition that the argument constant(s) in the initial clause or premise designates. For example, the law of Particularization gets replaced by

(4) $(fy \cdot \mathrm{E!}y) \supset (\exists x)Fx$

where 'E!' means 'exists'. Elsewhere, I have called such systems *free logics*.[12] The quantifiers, however, retain their *conventional character*; note especially that '$(\exists x)$' is read: 'There exists (timelessly) as x....'.

During the same period, quantification theories have been constructed, under the conventional sense of the quantifiers, whose theorems are valid in every universe *including the empty* one.[13] Of special interest are the systems constructed by Hailperin and Quine, and Hintikka. In these systems, formulae like '$(x)Fx \supset (\exists x)Fx$' are no longer provable. Hintikka's system is of the greatest interest for the present purpose; for, contrary to Belnap's remarks,[14] his system is truly free of existence assumptions *both* in the sense that it does not require its argument constants to be non-empty and in the sense that its *provable* formulae are only those holding in every universe including the empty one.[15] His system thus is both a free logic and is analytic (in the sense described above).

Hintikka's system[16] may be described roughly as follows. First, he distinguishes between argument *placeholders* and *variables*, letting the former be symbolized by $a, b, c\ldots$, and the latter by $x, y, z\ldots$. He restricts formulae to expressions in which all variables are actually bound to some quantifier, and designates them, and expressions like them except for containing argument placeholders in the place of variables, by $f, g, h\ldots$; the expression '$f(a/x)$' indicates the expressions which results from replacing x by a in f. Secondly, he introduces a transitive two-place metalogical relation of equivalence '$\leftrightarrow$', and defines a consequence relation '$\rightarrow$' (read: 'therefore') in terms of it; '$f \rightarrow q$' for '$f \leftrightarrow (f \,\&\, q)$' where '&' is 'and'. Thirdly, he shows that an analytic, non-free first order quantification theory with identity can be represented by means of the following rules.

(1) Formulae which are tautologically equivalent by the propositional calculus are equivalent provided that they contain occurrences of exactly the same argument placeholders, and so are expressions obtained from them by replacing one or more argument placeholders by variables.
(2) (*a*) $f(y/x) \rightarrow (\exists x)f$.
(2) (*b*) $f(a/x) \rightarrow (\exists x)f$.
(3) If g does not contain x, then $(\exists x)(f \,\&\, g) \leftrightarrow ((\exists x)f \,\&\, g)$
(4) (*a*) If x occurs in f, $f \rightarrow x = x$.
(4) (*b*) If a occurs in f, $f \rightarrow a = a$.
(5) (*a*) $x = y \,\&\, f(y/x) \rightarrow f$.
(5) (*b*) $x = a \,\&\, f(a/x) \rightarrow f$.
(5) (*c*) $a = b \,\&\, f(b/a) \rightarrow f$.

To obtain an analytic, free logic Hintikka simply drops (2) (b)-which corresponds to existential generalization on argument constants. He observes that in the resulting system, one can infer $(\exists x)f$ from $f(a/x)$, only on the condition that $(\exists x)(x = a)$. Since '$(\exists x)(x = a)$' amounts to saying that a is non-empty (or is purely referential), no restriction on the kind of argument constants is needed in his system. Finally, a formula f is *provable* if and only if $f \leftrightarrow (f \vee \sim f)$[17].

This very interesting system, which by the way appears to be capable of even further reduction, if definite descriptions are introduced[18], offers alternative solutions to the problems provoking Lejewski's analysis *without changing the conventional sense of the quantifiers.*

Consider the existence anomaly licensed by '$fy \supset (\exists x)fx$'. The analogue of this principle is not deducible in Hintikka's system.[19] Instead, as has been pointed out above, the conclusion '$(\exists x)$(x does not exist)' is deducible from the premise 'Pegasus does not exist' only on the condition that '$(\exists x)(x = \text{Pegasus})$', a condition which in fact is not satisfied. Elsewhere, I have indicated that this treatment of the singular anomaly is consistent with the spirit of the modern logic's treatment of existential import.[20] As such, Hintikka's result ought to be seen as bringing into relief the tacit in traditional quantification theory. Rephrasing Lejewski's quotation of Quine on the idea behind Particularization,[21] we may now more accurately say: the idea behind such inference is that whatever is true of the object purported to be designated by a given substantive is true of something, *provided that there is such an object as that purported to be designated by that substantive.**

Concerning the question of the contingency of formulae like '$(x)Fx \supset (\exists x)Fx$', notice that such troublesome formulae are not provable in Hintikka's system; for they do not hold in the empty universe. Indeed, in Hintikka's system no such formulae are provable. (Such formulae are detectable by the following method: write '**T**' for every formula beginning with a universal quantifier, and '**F**' for anyone beginning with an existential quantifier. If, after such assignment, truth evaluation yields a '**T**', the

*I owe this way of expressing the matter to Professor George Goe.

formula holds in the empty universe; if 'F', the formula does not hold in the empty universe. The technique is Quine's.[22]) The philosophical importance of this result is far reaching. For example, one can now explicate Leibniz's "true in all possible worlds" as "true in all universes including the empty one". As suggested above this presents a fairly precise explication of "analytic". Therefore, elementary logic (truth function theory, quantification theory, and identity theory) can be formulated so that all of its provable propositions are true in all possible worlds, are analytic. Consider now set theory, viz. as in Quine's *Mathematical Logic*.[23]

From his reformulation of the principle of abstraction,

$$(\exists y)(x)(x \ \varepsilon \ y \equiv (x \ \varepsilon \ \text{V}.\ x = x))$$

is provable.[24] But, by Quine's test, this formula does not hold in the empty universe. For Kant, who did not think of set theory as a part of logic, Hintikka's system offers proof that the theorems of logic are analytic, but that mathematics includes as theorems synthetic propositions-under the explications of these terms proposed above.

Finally, we come to the question of the analysis of singular existence, The analysis is straightforward. For since '$(\exists x)(x = a)$' is no longer provable in Hintikka's system, it is not longer trivial (or contradictory) to express, for example, 'Pegasus does not exist' (or "Pegasus exists') as 'It is false that there exists something which is Pegasus' (or 'There exists something which is Pegasus'). Generalizing, we can define 'E(xists)! a' as '$(\exists x)(x = a)$'. Furthermore, as Hintikka has shown, '$(\exists x)(x = a)$' may be taken as the formal equivalent of 'a is a value of the bound variables'.[25] It follows therefore that Quine's thesis that 'to be is to be the value of a variable' is formally expressible by the forementioned definition of '$\text{E!}\,a$'. This completes the argument against the view that the difficulties facing traditional quantificational logic are the product of merging existence and quantification. For in Hintikka's system they are willfully merged; yet the system solves the problems giving rise to Lejewski's analysis.

In conclusion, I should like to make a remark on the analysis of singular existence offered above and to prove an interesting remark of Lejewski on the interpretation of the quantifiers. First, since the above analysis of singular existence does not require materials beyond elementary logic, there is some theoretical advantage in the above proposal over that of Lejewski. For Lejewski requires the introduction of a primitive sign of inclusion '$\subset$' in his analysis of existence and thus enlarges the primitive frame of elementary logic. Secondly, Lejewski remarks that under his interpretation of the quantifiers '$(x)Fx$' and '$(\exists x)Fx$', the traditional interpretation of the quantifiers may be equated with, respectively, '$(x)(\text{E!}x \supset Fx)$' and '$(\exists x)(\text{E!}x.\ Fx)$'.[26] This result is provable in Hintikka's system. For since '$(x)(\exists x)(x = y)$' is provable, which by definition is $(x)\text{E!}x$, it is easy to prove '$(x)(\text{E!}x \supset Fx) \supset (x)Fx$'. But the converse is trivial. Hence, we get '$(x)(\text{E!}x \supset Fx) \equiv (x)Fx$'. But from the foregoing principle, '$(\exists x)(\text{E!}x.\ Fx) \equiv (\exists x)Fx$' is easy to prove. The traditional interpretation of the quantifiers thus is shown to be equivalent to certain *restricted interpretations* of the quantifiers.

NOTES

1. W. V. Quine, *Methods of Logic,* Holt (Revised) 1959, p. 88.
2. C. Lejewski, 'Logic and Existence', *British Journal for the Philosophy of Science',* Vol. V: No. 18, 1954, pp. 1-16.
3. H. S. Leonard, 'The Logic of Existence', Philosophical Studies, June: 1956, pp. 49-64.
4. Op. cit, Methods of Logic, pp. 196-224.
5. Jaakko Hintikka, 'Existential Presuppositions and Existential Commitments' *Journal of Philosophy,* January: 1959, p. 133.
6. W. V. Quine, *Word and Object,* Wiley: 1960, p. 187.
7. Op. cit., 'Existential Presuppositions and Existential Commitments', p. 133.
8. Op. cit., 'Logic and Existence', p. 6.
9. Ibid, pp. 12-15.
10. Ibid, p. 11.
11. See, for example, the papers by H. S. Leonard and Jaakko Hintokka. Also, see the paper by T. Hailperin and H. Leblanc, 'Nondesignating Singular Terms', *Philosophical Review,* April: 1959, pp. 239-244. For description theories based on such systems, see K. J. J. Hintikka's 'Towards a Theory of Definite Descriptions', *Analysis* 19: 79-85, (1959), and Karel Lambert 'Notes on E! III: A Theory of Descriptions', *Philosophical Studies,* June: 1962, pp. 51-59.
12. Op. cit., 'Notes on E! III: A Theory of Descriptions', p. 52.
13. Op. cit., 'Existential Presuppositions and Existential Commitments'. Also, see W. V. Quine, 'Quantification and the Empty Domain' *Journal of Symbolic Logic,* Vol. 19 (1954), pp. 177-179. Quine's paper also contains references to the systems developed by Hailperin and Mostowski.
14. See N. Belnap's review of Hintikka, in the *Journal of Symbolic Logic,* Vol. 25 (1960), p. 88.
15. In 'Existential import revisited', *Notre Dame Journal of Formal Logic,* v. IV (1963) pp. 222-292 I have suggested another such system. The chief difference between Hintikka's system and my version is that the latter is an axiomatic system. My friend, Mr. B. Van Fraassen, has recently proved (essentially) my system to be complete. His proof is being readied for publication.
16. Op. cit., 'Existential Presuppositions and Existential Commitments', pp. 129-130.

17. The notion of provability mentioned on page 130 of Hintikka's paper 'Existential Presuppositions and Existential Commitments' is explained more fully in his 'Distributive Normal Forms in the Calculus of Predicates', *Acta Philosophica Fennica,* Fasc. VI, pp. 13-14.

18. Karel Lambert, 'A Reduction in Free Quantification Theory with Identity and Descriptions', forthcoming.

19. Op. cit., 'Existential Presuppositions and Existential Commitments', p. 132.

20. Op. cit., forthcoming in the *Notre Dame Journal of Formal Logic.*

21. Op. cit., 'Logic and Existence', p. 4.

22. Op. cit., *Methods of Logic,* p. 97.

23. W. V. Quine, *Mathematical Logic,* (Revised), Harvard: 1958.

24. Ibid, p. 162.

25. Op. cit., 'Existential Presuppositions and Existential Commitments', p. 134-135.

26. Op. cit., 'Logic and Existence', p. 8.

West Virginia University
Morgantown, West Virginia