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S1° AND GENERALIZED S5-AXIOMS

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We call axioms $A_{j,k}$, $\mathfrak{C}M^{j}pL^{k}Mp$ ($1 \leq j$, $1 \leq k$) "generalized S5-axioms" since $A_{1,1}$ is commonly called "the characteristic axiom of S5." Some results of adding such an axiom to Feys's system S1° are investigated. For B_n, the generalized Brouwer axioms, see [1] and [2]. Proofs of the theorems depend largely on the rule:

$\mathcal{R}$ *In* S1° *if* $\vdash \mathfrak{C}M\alpha L\beta$ then $\vdash \mathfrak{C}\alpha\beta$

which [3] 4.2 clearly shows to be derivable.

Theorem I. If $j + k$ is odd, the matrix used in [2] shows that $A_{j,k}$ is insufficient to yield S5.

Theorem II. If $j = k$, $\{$S1°, $A_{j,k}\} =$ S5.

Proof: from $A_{k,k}$ we obtain by $\mathcal{R}$ $A_{1,1}$. The theorem follows by [3] 4.2.

Theorem III. If $j = k + 2$, $\{$S1°, $A_{j,k}\} =$ S5.

Proof: by $\mathcal{R}$ we obtain from $A_{k+2,k}$, $\mathfrak{C}M^2pMp$ and so $\mathfrak{C}LpL^2p$; hence we have $A_{k+2,k+2}$ and the theorem follows by theorem II.

Theorem IV. If $j = k + 2n$ $(n > 1)$, then $\{$S1°, $A_{j,k}\} = \{$S1°, $B_{2n-2}\}$.

Proof: from left to right we proceed:

(1)	$\mathfrak{C}M^{k+2n}pL^{k}Mp$	[by hyp.]
(2)	$\mathfrak{C}M^{2n}pMp$	[*(1)*, $\mathcal{R}$]
(3)	$\mathfrak{C}LpL^{2n}p$	[*(2)*, S1°]
(4)	$\mathfrak{C}M^{k+2n}pL^{k+4n-2}p$	[*(1)*, *(3)*, S1°]
(5)	$\mathfrak{C}pL^{2n-2}Mp$	[*(4)*, $\mathcal{R}\cdot$]

For the converse deduction it is enough to show that from B_{2n-2} we can prove $\mathfrak{C}M^2pLMp$, $\mathfrak{C}M^3pL^2Mp$, . . . , $\mathfrak{C}M^{2n-1}L^{2n-2}Mp$, since under B_{2n-2} all perpositive indices are strictly equivalent to one of *1*, *2*, . . . , $2n - 1$. This

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series of theses is obtainable by $B_{2n-2}\ p/M^{2n-1}p$ and $\mathcal{R}$, the resulting $M^{2n}p$ at the end of each being reducible to Mp.

Theorem V. If $k = j + 2n$ $(n \geq 1)$, then $\{S1°, A_{j,k}\} = \{S1°, B_{2n}\}$.

Proof: from left to right by $\mathcal{R}$. From right to left: express B_{2n} as B_{2m-2}. Then by the last theorem we have $\vdash \mathfrak{C}M^{k+2m}pL^kMp$ and so, since $M^{2m}p$ reduces to Mp, $\vdash \mathfrak{C}M^{k+1}pL^kMp$. With B_{2n} the Becker rule is obtainable, and thus $\mathfrak{C}L^{2n}M^{k+1}pL^{k+2n}Mp$, and further, by $B_{2n}\ p/M^kp$, $\vdash \mathfrak{C}M^kpL^{2n}M^{k+1}p$. From these last two theses we have by S1°, $\vdash \mathfrak{C}M^kpL^{k+2n}Mp$. *Q.E.D.*

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