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S1° AND BROUWERIAN AXIOMS

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Using the notation of [1] we discuss the effect of adding one or more Brouwerian axioms B_n : $\mathfrak{C}pL^nMp$ ($n \geq 1$) to S1°. We call a set of one or more such axioms *sufficient* or *insufficient* according as its addition to S1° does or does not yield S5. The means of proof available in S1° will be everywhere pre-supposed.

Theorem I. No set of axioms of the form B_{2k+1} ($k \geq 0$) *is sufficient.*

Proof is by the matrix:

K	1 2 3 4	N	M
* 1	1 2 3 4	4	1
2	2 2 4 4	3	3
3	3 4 3 4	2	2
4	4 4 4 4	1	4

which satisfies S1° and all B_{2k+1} but rejects $\mathfrak{C}pMp$.

Theorem II. Any pair B_1, B_{2k} ($k \geq 1$) *is sufficient.*

From B_1 we have (1) $\mathfrak{C}L^2pp$; from B_{2k} we have (2) $\mathfrak{C}LpL^{2(k+1)}p$; $k + 1$ applications of syllogism to (1), (2) give $\mathfrak{C}Lpp$ and so S1. But, as is known, $\{S1, B_{2k}\} = S5$.

Theorem III. For all m, n greater than 2, if m and n are co-prime, then B_{m-1}, B_{n-1} *are sufficient.*

If m and n are co-prime, there are positive integral r and s such that $rm = sn \pm 1$ and so $rm - 1 = sn - 1 \pm 1$. But B_{m-1} yields $\mathfrak{C}LpL^{m+1}p$ if $m > 2$, and so, by replacement, B_{rm-1} for all r not less than unity. Similarly from B_{n-1} we can obtain all B_{sn-1}, and proceed as follows:

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Case 1, $rm - 1 = sn$;

(1)	$\mathfrak{C}M^{rm-1}Lpp$	$[B_{rm-1}]$
(2)	$\mathfrak{C}M^{sn}Lpp$	[(1), hyp.]
(3)	$\mathfrak{C}pM^{sn}p$	$[B_{sn-1}]$
(4)	$\mathfrak{C}LpM^{sn}Lp$	[(3) p/Lp]
(5)	$\mathfrak{C}Lpp$	[(4), (2)]

Case 2, $rm = sn - 1$, proceeds similarly. Since B_{m-1}, B_{n-1} together yield (5), they are sufficient, as is clear from [1].

If m, n are greater than 2 but not co-prime, B_{m-1}, B_{n-1} are representable as B_{rp-1}, B_{sp-1} for some r, s, p, (r, $s \geq 1$, $p > 2$), and so are derivable from B_{p-1}. If $p - 1$ is odd, this axiom is insufficient by Theorem I. If $p - 1$ is even and the axiom is insufficient, the antecedent of *Theorem III* is a necessary condition. But I know of no proof that B_n is insufficient for arbitrary even n.

REFERENCES

[1] B. Sobociński: On the generalized Brouwerian axioms. *Notre Dame Journal of Formal Logic*, v. III (1962), pp. 123-8.

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