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PROOF ROUTINES FOR THE PROPOSITIONAL CALCULUS

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I prove in the pages that follow a conjecture of mine, to wit:

Any metastatement of the form

$$A_1, A_2, \ldots, A_n \vdash B,$$

where $A_1, A_2, \ldots, A_n (n \geq 0)$, *and* B *are wffs of* PC *and* '$\vdash$' *is the customary yields sign, is provable, when valid, by means of the three structural rules in Table I and the intelim rules in Table I for such of the connectives* '$\sim$', '$\supset$', '&', 'v', *and* '$\equiv$' *as occur in* $A_1, A_2, \ldots, A_n \vdash B$,

and sketch a routine for proving $A_1, A_2, \ldots, A_n \vdash B$, when valid, for each one of the 32 cases covered by the conjecture.[1] I also discuss a related conjecture of mine concerning the intuitionist fragment of PC.

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I

Let all five of '$\sim$', '$\supset$', '&', 'v', and '$\equiv$' be elected to serve as the primitive connectives of PC; let 'A', 'B', 'C', and 'D' be elected to range over the well-formed formulas (wffs) of PC; let a metastatement of the form $A_1, A_2, \ldots, A_n \vdash B$, called for short a T-statement, be rated valid if, in case $n = 0$, B is satisfied by any assignment of truth-values to the propositional variables occurring in B, or, in case $n > 0$, B is satisfied by any assignment of truth-values to the propositional variables occurring in A_1, $A_2, \ldots, A_n$, and B which simultaneously satisfies $A_1, A_2, \ldots,$ and A_n; let a T-statement be rated provable if it is the last entry in a finite column of T-statements each one of which is of the form **R** in Table I or follows from one or more previous T-statements in the column by application of one of the remaining rules in Table I; and, finally, let a T-statement be rated provable by means of the structural rules in Table I (to be collectively referred to as **S**) and zero or more of the intelim rules in Table I if it is the

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last entry in a finite column of T-statements each one of which is of the form **R** or follows from one or more previous T-statements in the column by application of **E**, **P**, or one of the intelim rules in question.

TABLE I

Structural rules:

R: $A \vdash A$;

E: *If* $A_1, A_2, \ldots, A_n \vdash B$, *then* $A_1, A_2, \ldots, A_{n+1} \vdash B$;

P: *If* $A_1, A_2, \ldots, A_{n+2} \vdash B$, *then* $A_1, A_2, \ldots, A_{i-1}, A_{i+1}, A_i, A_{i+2}, \ldots, A_{n+2} \vdash B$, *where* $i \leq n + 1$.

Intelim rules for '$\sim$' '$\supset$', '&', 'v', *and* '$\equiv$':

NI: *If* (1) $A_1, A_2, \ldots, A_{n+1} \vdash B$ *and* (2) $A_1, A_2, \ldots, A_{n+1} \vdash \sim B$, *then* $A_1, A_2, \ldots, A_n \vdash \sim A_{n+1}$;

NE: *If* $A_1, A_2, \ldots, A_n \vdash \sim \sim B$, *then* $A_1, A_2, \ldots, A_n \vdash B$;

HI: *If* $A_1, A_2, \ldots, A_{n+1} \vdash B$, *then* $A_1, A_2, \ldots, A_n \vdash A_{n+1} \supset B$;

HE: *If* (1) $A_1, A_2, \ldots, A_n \vdash B \supset C$ *and* (2) $A_1, A_2, \ldots, A_n \vdash (B \supset D) \supset B$, *then* $A_1, A_2, \ldots, A_n \vdash C$;

CI: *If* (1) $A_1, A_2, \ldots, A_n \vdash B$ *and* (2) $A_1, A_2, \ldots, A_n \vdash C$, *then* $A_1, A_2, \ldots, A_n \vdash B \mathbin{\&} C$;

CE: *If* (1) $A_1, A_2, \ldots, A_n \vdash B \mathbin{\&} C$ *and* (2) $A_1, A_2, \ldots, A_n, B, C \vdash D$, *then* $A_1, A_2, \ldots, A_n \vdash D$;

DI: *If* $A_1, A_2, \ldots, A_n \vdash B$, *then* (1) $A_1, A_2, \ldots, A_n \vdash B \vee C$ *and* (2) $A_1, A_2, \ldots, A_n \vdash C \vee B$;

DE: *If* (1) $A_1, A_2, \ldots, A_n \vdash B \vee C$, (2) $A_1, A_2, \ldots, A_n, B \vdash D$, *and* (3) $A_1, A_2, \ldots, A_n, C \vdash D$, *then* $A_1, A_2, \ldots, A_n \vdash D$;

BI: *If* (1) $A_1, A_2, \ldots, A_n, B \vdash C$ *and* (2) $A_1, A_2, \ldots, A_n, C \vdash B$, *then* $A_1, A_2, \ldots, A_n \vdash B \equiv C$;

BE: *If* (1) $A_1, A_2, \ldots, A_n \vdash B$ *and* (2) *either* $A_1, A_2, \ldots, A_n \vdash (D \equiv B) \equiv (D \equiv C)$ *or* $A_1, A_2, \ldots, A_n \vdash (D \equiv C) \equiv (D \equiv B)$, *then* $A_1, A_2, \ldots, A_n \vdash C$.

It is easily shown that:

T1: *If* $A_1, A_2, \ldots, A_n \vdash B$ *is provable, then* $A_1, A_2, \ldots, A_n \vdash B$ *is valid.*

I shall accordingly leave this matter to the reader and restrict myself to proving—as announced before—the following theorem:

T2: *If* $A_1, A_2, \ldots, A_n \vdash B$ *is valid, then* $A_1, A_2, \ldots, A_n \vdash B$ *is provable by means of* **S** *and the intelim rules for such of the connectives* '$\sim$', '$\supset$', '&', 'v', *and* '$\equiv$' *as occur in* $A_1, A_2, \ldots, A_n \vdash B$, from which the converse of **T1**, namely:

T3: *If* $A_1, A_2, \ldots, A_n \vdash B$ *is valid, then* $A_1, A_2, \ldots, A_n \vdash B$ *is provable,*

trivially follows.[2]

Of theorems **T2** and **T3**, the second still holds when the elimination rules for '$\supset$' and '$\equiv$' are phrased in the more traditional fashion:

HE': *If* (1) $A_1, A_2, \ldots, A_n \vdash B$ *and* (2) $A_1, A_2, \ldots, A_n \vdash B \supset C$, *then* $A_1, A_2, \ldots, A_n \vdash C$,

and

BE': *If* (1) $A_1, A_2, \ldots, A_n \vdash B$ *and* (2) *either* $A_1, A_2, \ldots, A_n \vdash B \equiv C$ *or* $A_1, A_2, \ldots, A_n \vdash C \equiv B$, *then* $A_1, A_2, \ldots, A_n \vdash C$.

The first, however, fails, as I shall establish in Section IV.[3]

II

I address myself in this section to the cases where $A_1, A_2, \ldots, A_n \vdash B$ exhibits no connective (Case 1) and to the 15 cases, reduced by various inductions to Case 1, where $A_1, A_2, \ldots, A_n \vdash B$ exhibits any one, any two, any three, or all four of the connectives '$\supset$', '&', 'v', and '$\equiv$'. The conditions under which a wff of *PC* is said in the proof of Case 6 to be in conjunctive normal form and the routine employed to put a wff of *PC* in conjunctive normal form need no rehearsing here. As for the conditions under which an occurrence of a connective in a wff of *PC* is said in the proofs of Cases 2-3 and Cases 7-10 to be either nested or unnested, they read: *Let* A *be a wff of PC of one of the four forms* $B \supset C$, $B \,\&\, C$, $B \vee C$, *and* $B \equiv C$; *then* (1) *every occurrence* (*if any*) *of* '$\supset$', '&', 'v', *or* '$\equiv$' *in* B *or in* C *is a nested occurrence of that connective, and* (2) *every occurrence* (*if any*) *of* '$\supset$', '&', 'v', *or* '$\equiv$' *in* A *that is not nested is unnested.*

Case 1: No connective occurs in $A_1, A_2, \ldots, A_n \vdash B$.

Proof: Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then there is bound to be an i such that A_i is B,[4] in which case $A_1, A_2, \ldots, A_n \vdash B$ follows from $B \vdash B$ (= **R**) by means of **E** and **P**. Hence **T2**.

Case 2: The only connective that occurs in $A_1, A_2, \ldots, A_n \vdash B$ *is* '$\supset$'.

Proof: (a) by induction on p, the number of occurrences of '$\supset$' in B, (b) when $p = 0$, by induction on q, the number of nested occurrences of '$\supset$' in $A_1, A_2, \ldots,$ and A_n, and (c) when $q = 0$, by induction on the number of unnested occurrences of '$\supset$' in $A_1, A_2, \ldots,$ and A_n.

Step 1: $p = 0$.

Step 1.1: $q = 0$. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then (1) there is bound to be an i such that A_i is B, in which case

$$A_1, A_2, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n \vdash B, \tag{2.1}$$

where A_j ($j < i$ or $j > i$) is the left-most one of A_1, A_2, . . . , and A_n to exhibit an occurrence of '$\supset$', is valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **HI**, and **HE**, or (2) there is bound to be an i and there is bound to be a j ($j < i$ or $j > i$) such that A_i is $A_j \supset A_{i_2}$, in which case

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{2.2}$$

is valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **HI**, and **HE**.[5] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (2.1), in one case, by means of **S** and from (2.2), in the other, by means of **S**, **HI**, and **HE**. Hence **T2**.

Step 1.2: $q > 0$. Then there is bound to be an i such that A_i is of one of the two forms $(A_{i_1} \supset A_{i_2}) \supset A_{i_3}$ and $A_{i_1} \supset (A_{i_2} \supset A_{i_3})$. Now suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid and A_i is of the form $(A_{i_1} \supset A_{i_2}) \supset A_{i_3}$. Then both

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i_2} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{2.3}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{2.4}$$

are bound to be valid and hence–in view of the hypothesis of induction–provable by means of **S**, **HI**, and **HE**.[6] Or suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid and A_i is of the form $A_{i_1} \supset (A_{i_2} \supset A_{i_3})$. Then both

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{2.5}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{2.6}$$

are bound to be valid and hence–in view of the hypothesis of induction–provable by means of **S**, **HI**, and **HE**.[7] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (2.3) - (2.4) in one case and from (2.5) - (2.6) in the other by means of the said rules. Hence **T2**.

Step 2: $p > 0$. Then B is bound to be of the form $B_1 \supset B_2$. Now suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_n, B_1 \vdash B_2 \tag{2.7}$$

is bound to be valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **HI**, and **HE**. But $A_1, A_2, \ldots, A_n \vdash B$ follows from (2.7) by means of the said rules. Hence **T2**.[8]

Case 3: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ *are* '$\supset$' *and* '&'.

Proof: (a) by induction on p, the number of occurrences of '$\supset$' and '&' in B, (b) when $p = 0$, by induction on q, the number of nested occurrences of '$\supset$' and '&' in $A_1, A_2, \ldots, $ and A_n, and (c) when $q = 0$, by induction on the number of unnested occurrences of '&' in $A_1, A_2, \ldots,$ and A_n.

Step 1: $p = 0$.

Step 1.1: $q = 0$. Then there is bound to be an i such that A_i is of the form A_{i_1} & A_{i_2}. Now suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{3.1}$$

is bound to be valid and hence–in view of Case 2 or of the hypothesis of induction–provable by means of **S**, **HI**, **HE**, **CI**, and **CE**. But $A_1, A_2, \ldots, A_n \vdash B$ follows from (3.1) by means of the said rules. Hence **T2**.

Step 1.2: $q > 0$. Then there is bound to be an i such that A_i is of one of the eight forms $(A_{i_1} \supset A_{i_2}) \supset A_{i_3}$, $A_{i_1} \supset (A_{i_2} \supset A_{i_3})$, $(A_{i_1} \,\&\, A_{i_2}) \,\&\, A_{i_3}$, $A_{i_1} \,\&\, (A_{i_2} \,\&\, A_{i_3})$, $(A_{i_1} \supset A_{i_2}) \,\&\, A_{i_3}$, $A_{i_1} \,\&\, (A_{i_2} \supset A_{i_3})$, $(A_{i_1} \,\&\, A_{i_2}) \supset A_{i_3}$, and $A_{i_1} \supset (A_{i_2} \,\&\, A_{i_3})$, where in the last case A_{i_1} is a propositional variable.[9]

Step 1.2.1: A_i is of one of the first two forms listed. Proof similar to the proof of Case 2, Step 1.2, but with **S**, **HI**, **HE**, **CI**, and **CE** doing duty for **S**, **HI**, and **HE**.

Step 1.2.2: A_i is of one of the next four forms listed. Proof similar to the proof of Step 1.1.

Step 1.2.3: A_i is of the form $(A_{i_1} \,\&\, A_{i_2}) \supset A_{i_3}$. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then both

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{3.2}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{3.3}$$

are bound to be valid and hence–in view of Case 2 or of the hypothesis of induction–provable by means of **S**, **HI**, **HE**, **CI**, and **CE**.[10] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (3.2) - (3.3) by means of the said rules. Hence **T2**.

Step 1.2.4: A_i is of the form $A_{i_1} \supset (A_{i_2} \,\&\, A_{i_3})$, where A_{i_1} is a propositional variable. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \supset A_{i_2}, A_{i_1} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{3.4}$$

is bound to be valid and hence–in view of Case 2 or of the hypothesis of induction–provable by means of **S**, **HI**, **HE**, **CI**, and **CE**.[11] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (3.4) by means of the said rules. Hence **T2**.

Step 2: $p > 0$. Then B is bound to be of one of the two forms $B_1 \supset B_2$ and $B_1 \,\&\, B_2$.

Step 2.1: B is of the form $B_1 \supset B_2$. Proof similar to the proof of Case 2, Step 2, but minus the reference to Case 1 and with **S**, **HI**, **HE**, **CI**, and **CE** doing duty for **S**, **HI**, and **HE**.

Step 2.2: B is of the form B_1 & B_2. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then both

$$A_1, A_2, \ldots, A_n \vdash B_1 \tag{3.5}$$

and

$$A_1, A_2, \ldots, A_n \vdash B_2 \tag{3.6}$$

are bound to be valid and hence–in view of Case 2 or of the hypothesis of induction–provable by means of **S**, **HI**, **HE**, **CI**, and **CE**. But $A_1, A_2, \ldots, A_n \vdash B$ follows from (3.5) - (3.6) by means of **CI**. Hence **T2**.

Case 4: The only connective that occurs in $A_1, A_2, \ldots, A_n \vdash B$ is '&'.

Proof by induction on the number of occurrences of '&' in $A_1, A_2, \ldots, A_n$, and B.

Step 1: There is an i such that A_i is of the form A_{i_1} & A_{i_2}. Proof similar to the proof of Case 3, Step 1.1, but with Case 1 doing duty for Case 2 and with **S**, **CI**, and **CE** doing duty for **S**, **HI**, **HE**, **CI**, and **CE**.[12]

Step 2: B is of the form B_1 & B_2. Proof similar to the proof of Case 3, Step 2.2, but with Case 1 doing duty for Case 2 and with **S**, **CI**, and **CE** doing duty for **S**, **HI**, **HE**, **CI**, and **CE**.

Case 5: The only connective that occurs in $A_1, A_2, \ldots, A_n \vdash B$ is 'v'.

Proof by induction on p, the number of occurrences of 'v' in $A_1, A_2, \ldots,$ and A_n, and, when $p = 0$, by induction on the number of occurrences of 'v' in B.

Step 1: $p = 0$. Then B is bound to be of the form $B_1 \vee B_2$. Now suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then (1) there is bound to be an i such that A_i is or occurs in B_1, in which case

$$A_1, A_2, \ldots, A_n \vdash B_1 \tag{5.1}$$

is valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **DI**, and **DE**, or (2) there is bound to be an i such that A_i is or occurs in B_2, in which case

$$A_1, A_2, \ldots, A_n \vdash B_2 \tag{5.2}$$

is valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **DI**, and **DE**.[13] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (5.1) in one case and from (5.2) in the other by means of **DI**. Hence **T2**.

Step 2: $p > 0$. Then there is bound to be an i such that A_i is of the form $A_{i_1} \vee A_{i_2}$. Now suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then both

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i+1}, \ldots, A_n \vdash B \tag{5.3}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{5.4}$$

are bound to be valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **DI**, and **DE**. But $A_1, A_2, \ldots, A_n \vdash B$ follows from (5.3) - (5.4) by means of the said rules. Hence **T2**.[14]

Case 6: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ are 'v' *and* '&'.

Proof (a) by induction on p, the number of wffs among $A_1, A_2, \ldots, A_n$, and B which fail to be in conjunctive normal form, and (b) when $p = 0$, by induction on the number of occurrences of '&' in $A_1, A_2, \ldots, A_n$, and B.

Step 1: $p = 0$.

Step 1.1: There is an i such that A_i is of the form A_{i_1} & A_{i_2}. Proof similar to the proof of Case 3, Step 1.1, but with Case 5 doing duty for Case 2 and with **DI** and **DE** doing duty for **HI** and **HE**.

Step 1.2: B is of the form B_1 & B_2. Proof similar to the proof of Case 3, Step 2.2, but with Case 5 doing duty for Case 2 and with **DI** and **DE** doing duty for **HI** and **HE**.

Step 2: $p > 0$.

Step 2.1: There is an i such that A_i fails to be in conjunctive normal form. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_{i-1}, A_i^*, A_{i+1}, \ldots, A_n \vdash B, \tag{6.1}$$

where A_i^* is any result of putting A_i in conjunctive normal form, is bound to be valid and hence–in view of Step 1 or of the hypothesis of induction–provable by means of **S**, **DI**, **DE**, **CI**, and **CE**. But $A_1, A_2, \ldots, A_n \vdash B$ follows from (6.1) by means of the said rules. Hence **T2**.

Step 2.2: B fails to be in conjunctive normal form. Proof similar to the proof of Step 2.1.

Case 7: The only connective that occurs in $A_1, A_2, \ldots, A_n \vdash B$ is '≡'.

Proof (a) by induction on p, the number of occurrences of '≡' in B, (b) when $p = 0$, by induction on q, the number of nested occurrences of '≡' in $A_1, A_2, \ldots$, and A_n, and (c) when $q = 0$, by induction on the number of unnested occurrences of '≡' in $A_1, A_2, \ldots$, and A_n.

Step 1: $p = 0$.

Step 1.1: $q = 0$. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then (1) there is bound to be an i such that A_i is B, in which case

$$A_1, A_2, \ldots, A_{j-1}, A_{j+1}, \ldots, A_n \vdash B, \tag{7.1}$$

where A_j ($j < i$ or $j > i$) is the left-most one of A_1, A_2, . . . , and A_n to exhibit an occurrence of '$\equiv$', is valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **BI**, and **BE**, or (2) there is bound to be an i and there is bound to be a j ($j > I$ or $j > i$) such that A_i is $A_j \equiv A_{i_2}$, in which case

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{7.2}$$

is valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **BI**, and **BE**, or (3) there is bound to be an i and there is bound to be a j ($j < i$ or $j > i$) such that A_i is $A_{i_1} \equiv A_j$, in which case

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i+1}, \ldots, A_n \vdash B \tag{7.3}$$

is valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **BI**, and **BE**.[15] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (7.1) in the first case by means of **S**, from (7.2) in the second by means of **S**, **BI**, and **BE**, and from (7.3) in the third by means of **S**, **BI**, and **BE**. Hence **T2**.

Step 1.2: $q > 0$. Then there is bound to be an i such that A_i is of one of the two forms $(A_{i_1} \equiv A_{i_2}) \equiv A_{i_3}$ and $A_{i_1} \equiv (A_{i_2} \equiv A_{i_3})$. Now suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then all three of

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i_2} \equiv A_{i_3}, A_{i+1}, \ldots, A_n \vdash B, \tag{7.4}$$

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i_1} \equiv A_{i_3}, A_{i+1}, \ldots, A_n \vdash B, \tag{7.5}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_3}, A_{i_1} \equiv A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{7.6}$$

are bound to be valid and hence–in view of the hypothesis of induction–provable by means of **S**, **BI**, and **BE**.[16] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (7.4) - (7.6) by means of the said rules. Hence **T2**.

Step 2: $p > 0$. Then B is bound to be of the form $B_1 \equiv B_2$. Now suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then both

$$A_1, A_2, \ldots, A_n, B_1 \vdash B_2 \tag{7.7}$$

and

$$A_1, A_2, \ldots, A_n, B_2 \vdash B_1 \tag{7.8}$$

are bound to be valid and hence–in view of Case 1 or of the hypothesis of induction–provable by means of **S**, **BI**, and **BE**. But $A_1, A_2, \ldots, A_n \vdash B$ follows from (7.7) - (7.8) by means of the said rules. Hence **T2**.

Case 8: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ are '$\equiv$' and '$\supset$'.

Proof (a) by induction on p, the number of occurrences of '$\equiv$' and '$\supset$' in B, (b) when $p = 0$, by induction on q, the number of nested occurrences of '$\equiv$' and '$\supset$' in A_1, A_2, . . . , and A_n, and (c) when $q = 0$, by induction on the number of unnested occurrences of '$\supset$' in A_1, A_2, . . . , and A_n.

Step 1: $p = 0$.

Step 1.1: $q = 0$. Then there is bound to be an i such that A_i is of the form $A_{i_1} \supset A_{i_2}$. Now suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then both

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \equiv A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{8.1}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{8.2}$$

are bound to be valid and hence–in view of Case 7 or of the hypothesis of induction–provable by means of **S**, **BI**, **BE**, **CI**, and **CE**.[17] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (8.1) - (8.2) by means of the said rules. Hence **T2**.

Step 1.2: $q > 0$. Then there is bound to be an i such that A_i is of one of the eight forms $(A_{i_1} \equiv A_{i_2}) \equiv A_{i_3}$, $A_{i_1} \equiv (A_{i_2} \equiv A_{i_3})$, $(A_{i_1} \supset A_{i_2}) \supset A_{i_3}$, $A_{i_1} \supset (A_{i_2} \supset A_{i_3})$, $(A_{i_1} \supset A_{i_2}) \equiv A_{i_3}$, $A_{i_1} \equiv (A_{i_2} \supset A_{i_3})$, $(A_{i_1} \equiv A_{i_2}) \supset A_{i_3}$, and $A_{i_1} \supset (A_{i_2} \equiv A_{i_3})$, where in the last case A_{i_1} is a propositional variable.

Step 1.2.1: A_i is of one of the first two forms listed. Proof similar to the proof of Case 7, Step 1.2, but with **S**, **BI**, **BE**, **HI**, and **HE** doing duty for **S**, **BI**, and **BE**.

Step 1.2.2: A_i is of one of the next two forms listed. Proof similar to the proof of Case 2, Step 1.2, but with **S**, **BI**, **BE**, **HI**, and **HE** doing duty for **S**, **HI**, and **HE**.

Step 1.2.3: A_i is of the form $(A_{i_1} \supset A_{i_2}) \equiv A_{i_3}$. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is vălid. Then both

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i_2} \equiv A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{8.3}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \supset A_{i_2}, A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{8.4}$$

are bound to be valid and hence–in view of Case 7 or of the hypothesis of induction–provable by means of **S**, **BI**, **BE**, **HI**, and **HE**.[18] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (8.3) - (8.4) by means of the said rules. Hence **T2**.

Step 1.2.4: A_i is of the form $A_{i_1} \equiv (A_{i_2} \supset A_{i_3})$. Proof similar to the proof of Step 1.2.3, but with A_{i_1} doing duty for A_{i_3}, A_{i_2} doing duty for A_{i_1}, and A_{i_3} doing duty for A_{i_2}.

Step 1.2.5: A_i is of the form $(A_{i_1} \equiv A_{i_2}) \supset A_{i_3}$. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then all three of

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i_2} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B, \tag{8.5}$$

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i_1} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B, \tag{8.6}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{8.7}$$

are bound to be valid and hence–in view of Case 7 or of the hypothesis of induction–provable by means of **S**, **BI**, **BE**, **HI**, and **HE**.[19] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (8.5) - (8.7) by means of the said rules. Hence **T2**.

Step 1.2.6: $A_{i_1} \supset (A_{i_2} \equiv A_{i_3})$, where A_{i_1} is a propositional variable. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then both

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \supset A_{i_2}, A_{i_1} \supset A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{8.8}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2} \equiv A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{8.9}$$

are bound to be valid and hence–in view of Case 7 or of the hypothesis of induction–provable by means of **S**, **BI**, **BE**, **HI**, and **HE**.[20] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (8.8) - (8.9) by means of the said rules. Hence **T2**.

Step 2: $p > 0$. Then B is bound to be one of the two forms $B_1 \equiv B_2$ and $B_1 \supset B_2$.

Step 2.1: B is of the form $B_1 \equiv B_2$. Proof similar to the proof of Case 7, Step 2, but minus the reference to Case 1 and with **S**, **BI**, **BE**, **HI**, and **HE** doing duty for **S**, **BI**, and **BE**.

Step 2.2: B is of the form $B_1 \supset B_2$. Proof similar to the proof of Case 2, Step 2, but with Case 7 doing duty for Case 1 and with **S**, **BI**, **BE**, **HI**, and **HE** doing duty for **S**, **BI**, and **BE**.

Case 9: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ are '≡' and '&'.

Proof (a) by induction on p, the number of occurrences of '≡' and '&' in B, (b) when $p = 0$, by induction on q, the number of nested occurrences of '≡' and '&' in $A_1, A_2, \ldots,$ and A_n, and (c) when $q = 0$, by induction on the number of unnested occurrences of '&' in $A_1, A_2, \ldots,$ and A_n.

Step 1: $p = 0$.

Step 1.1: $q = 0$. Then there is bound to be an i such that A_i is of the form A_{i_1} & A_{i_2}, in which case **T2** by the same reasoning as in Case 3, Step 1.1, but with Case 7 doing duty for Case 2 and with **BI** and **BE** doing duty for **HI** and **HE**.

Step 1.2: $q > 0$. Then there is bound to be an i such that A_i is of one of the eight forms $(A_{i_1} \equiv A_{i_2}) \equiv A_{i_3}$, $A_{i_1} \equiv (A_{i_2} \equiv A_{i_3})$, $(A_{i_1} \& A_{i_2}) \& A_{i_3}$, $A_{i_1} \& (A_{i_2} \& A_{i_3})$, $(A_{i_1} \equiv A_{i_2}) \& A_{i_3}$, $A_{i_1} \& (A_{i_2} \equiv A_{i_3})$, $(A_{i_1} \& A_{i_2}) \equiv A_{i_3}$, and $A_{i_1} \equiv (A_{i_2} \& A_{i_3})$, where A_{i_3} in the seventh case and A_{i_1} in the eighth case are propositional variables.

Step 1.2.1: A_i is of one of the first two forms listed. Proof similar to the proof of Case 7, Step 1.2, but with **S**, **BI**, **BE**, **CI**, and **CE** doing duty for **S**, **BI**, and **BE**.

Step 1.2.2: A_i is of one of the next four forms listed. Proof similar to the proof of Step 1.1.

Step 1.2.3: A_i is of the form $(A_{i_1} \& A_{i_2}) \equiv A_{i_3}$, where A_{i_3} is a propositional variable. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then all three of

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \equiv A_{i_3}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B, \tag{9.1}$$

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2} \equiv A_{i_3}, A_{i_1}, A_{i+1}, \ldots, A_n \vdash B, \tag{9.2}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \equiv A_{i_3}, A_{i_2} \equiv A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{9.3}$$

are bound to be valid and hence–in view of Case 7 or of the hypothesis of induction–provable by means of **S**, **BI**, **BE**, **CI**, and **CE**.[21] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (9.1) - (9.3) by means of the said rules. Hence **T2**.

Step 1.2.4: A_i is of the form $A_{i_1} \equiv (A_{i_2} \& A_{i_3})$, where A_{i_1} is a propositional variable. Proof similar to the proof of Step 1.2.3, but with A_{i_1} doing duty for A_{i_3}, A_{i_2} doing duty for A_{i_1}, and A_{i_3} doing duty for A_{i_3}.

Step 2: $p > 0$. Then B is bound to be of one of the two forms $B_1 \equiv B_2$ and $B_1 \& B_2$.

Step 2.1: B is of the form $B_1 \equiv B_2$. Proof similar to the proof of Case 7, Step 2, but minus the reference to Case 1 and with **S**, **BI**, **BE**, **CI**, and **CE** doing duty for **S**, **BI**, and **BE**.

Step 2.2: B is of the form $B_1 \& B_2$. Proof similar to the proof of Case 3, Step 2.2, but with Case 7 doing duty for Case 2 and with **BI** and **BE** doing duty for **HI** and **HE**.

Case 10: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ are '$\equiv$' and '$\vee$'.

Proof (a) by induction on p, the number of occurrences of '$\equiv$' and '$\vee$' in B, (b) when $p = 0$, by induction on q, the number of nested occurrences of '$\equiv$' and '$\vee$' in $A_1, A_2, \ldots,$ and A_n, and (c) when $q = 0$, by induction on the number of unnested occurrences of '$\vee$' in $A_1, A_2, \ldots,$ and A_n.

Step 1: $p = 0$.

Step 1.1: $q = 0$. Then there is bound to be an i such that A_i is of the form $A_{i_1} \vee A_{i_2}$, in which case **T2** by the same reasoning as in Case 5, Step 2, but with Case 7 doing duty for Case 1 and with **S**, **BI**, **BE**, **DI**, and **DE** doing duty for **S**, **DI**, and **DE**.

Step 1.2: $q > 0$. Then there is bound to be an i such that A_i is of one of the eight forms $(A_{i_1} \equiv A_{i_2}) \equiv A_{i_3}$, $A_{i_1} \equiv (A_{i_2} \equiv A_{i_3})$, $(A_{i_1} \vee A_{i_2}) \vee A_{i_3}$, $A_{i_1} \vee (A_{i_2} \vee A_{i_3})$, $(A_{i_1} \equiv A_{i_2}) \vee A_{i_3}$, $A_{i_1} \vee (A_{i_2} \equiv A_{i_3})$, $(A_{i_1} \vee A_{i_2}) \equiv A_{i_3}$, and $A_{i_1} \equiv (A_{i_2} \vee A_{i_3})$, where A_{i_3} in the seventh case and A_{i_1} in the eighth case are propositional variables.

Step 1.2.1: A_i is of one of the first two forms listed. Proof similar to the proof of Case 7, Step 1.2, but with **S**, **BI**, **BE**, **DI**, and **DE** doing duty for **S**, **BI**, and **BE**.

Step 1.2.2: A_i is of one of the next four forms listed. Proof similar to the proof of Step 1.1.

Step 1.2.3: A_i is of the form $(A_{i_1} \vee A_{i_2}) \equiv A_{i_3}$, where A_{i_3} is a propositional variable. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then all three of

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i_3}, A_{i+1}, \ldots, A_n \vdash B, \tag{10.1}$$

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i_3}, A_{i+1}, \ldots, A_n \vdash B, \tag{10.2}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1} \equiv A_{i_3}, A_{i_2} \equiv A_{i_3}, A_{i+1}, \ldots, A_n \vdash B \tag{10.3}$$

are bound to be valid and hence—in view of Case 7 or of the hypothesis of induction—provable by means of **S**, **BI**, **BE**, **DI**, and **DE**.[21bis] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (10.1) - (10.3) by means of the said rules. Hence **T2**.

Step 1.2.4: A_i is of the form $A_{i_1} \equiv (A_{i_2} \vee A_{i_3})$, where A_{i_1} is a propositional variable. Proof similar to the proof of Step 1.2.3, but with A_{i_1} doing duty for A_{i_3}, A_{i_2} doing duty for A_{i_1}, and A_{i_3} doing duty for A_{i_2}.

Step 2: $p > 0$. Then B is bound to be of one of the forms $B_1 \equiv B_2$ and $B_1 \vee B_2$.

Step 2.1: B is of the form $B_1 \equiv B_2$. Proof similar to the proof of Case 7, Step 2, but minus the reference to Case 1 and with **S**, **BI**, **BE**, **DI**, and **DE** doing duty for **S**, **BI**, and **BE**.

Step 2.2: B is of the form $B_1 \vee B_2$. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_n, B_1 \equiv B_2 \vdash B_1 \tag{10.4}$$

is bound to be valid and hence—in view of Case 7 or of the hypothesis of induction—provable by means of **S**, **BI**, **BE**, **DI**, and **DE**.[21ter] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (10.4) by means of the said rules. Hence **T2**.

Case 11: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ *are* '$\supset$' *and* '$\vee$'.

Proof by induction on the number of occurrences of '$\vee$' in $A_1, A_2, \ldots, A_n$, and B.

Step 1: B is of the form $B_1 \vee B_2$, where B_2 does not exhibit any '$\vee$'. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_n \vdash (B_1 \supset B_2) \supset B_2 \tag{11.1}$$

is bound to be valid and hence—in view of Case 2 or of the hypothesis of induction—provable by means of **S**, **HI**, **HE**, **DI**, and **DE**.[22] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (11.1) by means of the said rules. Hence **T2**.

Step 2: B has a component of the form $B_j \vee B_k$, where B_k does not exhibit any '$\vee$'. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_n \vdash B, \tag{11.2}$$

where B is like B except for exhibiting $((B_j \supset B_k) \supset B_k) \supset B_l$ where B exhibits $(B_j \vee B_k) \supset B_l$ or for exhibiting $B_i \supset ((B_j \supset B_k) \supset B_k)$ where B exhibits $B_i \supset (B_j \vee B_k)$, is bound to be valid and hence—in view of Case 2 or of the hypothesis of induction—provable by means of **S**, **HI**, **HE**, **DI**, and **DE**. But $A_1, A_2, \ldots, A_n \vdash B$ follows from (11.2) by means of the said rules. Hence **T2**.

Step 3: There is an i such that A_i is of the form $A_{i_1} \vee A_{i_2}$, where A_{i_2} does not exhibit any '$\vee$'. Proof similar to the proof of Step 1.

Step 4: There is an i such that A_i has a component of the form $A_{i_j} \vee A_{i_k}$, where A_{i_k} does not exhibit any '$\vee$'. Proof similar to the proof of Step 2.

Cases 12-16 are provable along similar lines. Appended is a table (Table IIb) of the various cases to which they reduce and of the transformations—effected for illustration's sake on $B_1 \supset B_2$, B_1 & B_2, $B_1 \vee B_2$, and $B_1 \equiv B_2$—which insure those reductions. A key to the abbreviations used in Table IIb is supplied in Table IIa.

TABLE IIa

T1: $B_1 \supset B_2 \dashrightarrow (B_1 \& B_2) \equiv B_1$

T2: $B_1 \supset B_2 \dashrightarrow (B_1 \vee B_2) \equiv B_2$[23]

T3: $B_1 \& B_2 \dashrightarrow ((B_1 \supset B_2) \supset B_2) \equiv (B_1 \equiv B_2)$

T4: $B_1 \& B_2 \dashrightarrow (B_1 \vee B_2) \equiv (B_1 \equiv B_2)$

T5: $B_1 \vee B_2 \dashrightarrow (B_1 \supset B_2) \supset B_2$

T6: $B_1 \vee B_2 \dashrightarrow (B_1 \& B_2) \equiv (B_1 \equiv B_2)$

T7: $B_1 \equiv B_2 \dashrightarrow (B_1 \supset B_2) \& (B_2 \supset B_1)$

TABLE IIb

Cases	Reducible to Cases	By means of
12: '⊃', '&', and 'v'	3	*T5*
13: '⊃', '&', and '≡'	3	*T7*
	8	*T3*
14: '⊃', 'v', and '≡'	8	*T5*
	10	*T2*
15: '&', 'v', and '≡'	9	*T6*
	10	*T4*
16: '⊃', '&', 'v', and '≡'	3	*T5* and *T7*
	8	*T3* and *T5*
	9	*T1* and *T6*
	10	*T2* and *T4*

III

I complete in this section the proof of **T2** by solving the case where $A_1, A_2, \ldots, A_n \vdash$ exhibits only '$\sim$' (Case 17) and reducing to Case 17 the 15 cases where $A_1, A_2, \ldots, A_n \vdash B$ exhibits besides '$\sim$' any one, any two, any three, or all four of '⊃', '&', 'v', and '≡'.

Case 17: The only connective that occurs in $A_1, A_2, \ldots, A_n \vdash B$ *is* '$\sim$'.

Proof by induction on p, the number of wffs among $A_1, A_2, \ldots, A_n$, and B which consist of two or more occurrences of '$\sim$' followed by a propositional variable.

Step 1: $p = 0$. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then (1) there is bound to be an i such that A_i is B, in which case $A_1, A_2, \ldots, A_n \vdash B$ follows from $B \vdash B$ (= **R**) by means of **E** and **P**, or (2) there is bound to be an i and there is bound to be a j ($j < i$ or $j > i$) such that A_i is $\sim A_j$, in which case $A_1, A_2, \ldots, A_n \vdash B$ follows from $A_j \vdash A_j$ and $\sim A_j \vdash \sim A_j$ (= **R**) by means of **E**, **P**, **NI**, and **NE**.[24] Hence **T2**.

Step 2: $p > 0$. Then there is bound to be an i such that A_i is of the form $\underbrace{\sim\sim\ldots\sim}_{k\ times} A_i^*$, where $k \geq 2$ and A_i^* is a propositional variable, or B is bound to be of the form $\underbrace{\sim\sim\ldots\sim}_{k\ times} B^*$, where $k \geq 2$ and B^* is a propositional variable.

Step 2.1: A_i is of the form $\underbrace{\sim\sim\ldots\sim}_{k\ times} A_i^*$. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid and k is even. Then

$$A_1, A_2, \ldots, A_{i-1}, A_i^*, A_{i+1}, \ldots, A_n \vdash B \tag{17.1}$$

is bound to be valid and hence–in view of Step 1 or of the hypothesis of induction–provable by means of **S**, **NI**, and **NE**.[25] Or suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid and k is odd. Then

$$A_1, A_2, \ldots, A_{i-1}, \sim A_i^*, A_{i+1}, \ldots, A_n \vdash B \tag{17.2}$$

is bound to be valid and hence–in view of Step 1 or of the hypothesis of induction–provable by means of **S**, **NI**, and **NE**.[26] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (17.1) in one case and from (17.2) in the other by means of the said rules. Hence **T2**.

Step 2.2: B is of the form $\underbrace{\sim\sim\ldots\sim}_{k\ times} B^*$. Proof similar to the proof of Step 2.1.

Case 18: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ are '$\sim$' and '$\supset$'.

Proof by induction on the number of occurrences of '$\supset$' in $A_1, A_2, \ldots, A_n$, and B.

Step 1: There is an i such that A_i is of the form $\underbrace{\sim\sim\ldots\sim}_{k\ times}(A_{i_1} \supset A_{i_2})$ where $k \geq 0$.

Step 1.1: k equals 0 or k is even. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then both

$$A_1, A_2, \ldots, A_{i-1}, \sim A_{i_1}, A_{i+1}, \ldots, A_n \vdash B \tag{18.1}$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{18.2}$$

are bound to be valid and hence–in view of Case 17 or of the hypothesis of induction–provable by means of **S**, **NI**, **NE**, **HI**, and **HE**.[27] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (18.1) - (18.2) by means of the said rules. Hence **T2**.

Step 1.2: k is odd. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_{i_1}, \sim A_{i_2}, A_{i+1}, \ldots, A_n \vdash B \tag{18.3}$$

is bound to be valid and hence—in view of Case 17 or of the hypothesis of induction—provable by means of **S**, **NI**, **NE**, **HI**, and **HE**.[28] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (18.3) by means of the said rules. Hence **T2**.

Step 2: B is of the form $\underbrace{\sim\sim\ldots\sim}_{k\ times}(B_1 \supset B_2)$, where $k \geq 0$.

Step 2.1: k equals 0 or k is even. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then

$$A_1, A_2, \ldots, A_n, B_1 \vdash B_2 \tag{18.4}$$

is bound to be valid and hence—in view of Case 17 or of the hypothesis of induction—provable by means of **S**, **NI**, **NE**, **HI**, and **HE**.[29] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (18.4) by means of the said rules. Hence **T2**.

Step 2.2: k is odd. Suppose $A_1, A_2, \ldots, A_n \vdash B$ is valid. Then both

$$A_1, A_2, \ldots, A_n \vdash B_1 \tag{18.5}$$

and

$$A_1, A_2, \ldots, A_n \vdash \sim B_2 \tag{18.6}$$

are bound to be valid and hence—in view of Case 17 or of the hypothesis of induction—provable by means of **S**, **NI**, **NE**, **HI**, and **HE**.[30] But $A_1, A_2, \ldots, A_n \vdash B$ follows from (18.5) - (18.6) by means of the said rules. Hence **T2**.

Case 19: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ are '$\sim$' and '&'.

Proof similar to the proof of Case 18, but with '&' doing duty for '$\supset$', **CI** and **CE** doing duty for **HI** and **HE**,

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B$$

doing duty for (18.1) - (18.2),

$$A_1, A_2, \ldots, A_{i-1}, \sim A_{i_1}, A_{i+1}, \ldots, A_n \vdash B$$

and

$$A_1, A_2, \ldots, A_{i-1}, \sim A_{i_2}, A_{i+1}, \ldots, A_n \vdash B$$

doing duty for (18.3),

$$A_1, A_2, \ldots, A_n \vdash B_1$$

and

$$A_1, A_2, \ldots, A_n \vdash B_2$$

doing duty for (18.4), and

$$A_1, A_2, \ldots, A_n, B_1 \vdash \sim B_2$$

doing duty for (18.5) - (18.6).

Case 20: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ are '$\sim$' and '$\vee$'.

Proof similar to the proof of Case 18, but with '$\vee$' doing duty for '$\supset$', **DI** and **DE** doing duty for **HI** and **HE**,

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i+1}, \ldots, A_n \vdash B$$

and

$$A_1, A_2, \ldots, A_{i-1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B$$

doing duty for (18.1) - (18.2),

$$A_1, A_2, \ldots, A_{i-1}, \sim A_{i_1}, \sim A_{i_2}, A_{i+1}, \ldots, A_n \vdash B$$

doing duty for (18.3),

$$A_1, A_2, \ldots, A_n, \sim B_1 \vdash B_2$$

doing duty for (18.4), and

$$A_1, A_2, \ldots, A_n \vdash \sim B_1$$

and

$$A_1, A_2, \ldots, A_n \vdash \sim B_2$$

doing duty for (18.5) - (18.6).

Case 21: The only two connectives that occur in $A_1, A_2, \ldots, A_n \vdash B$ are '$\sim$' and '$\equiv$'.

Proof similar to the proof of Case 18, but with '$\equiv$' doing duty for '$\supset$', **BI** and **BE** doing duty for **HI** and **HE**,

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B$$

and

$$A_1, A_2, \ldots, A_{i-1}, \sim A_{i_1}, \sim A_{i_2}, A_{i+1}, \ldots, A_n \vdash B$$

doing duty for (18.1) - (18.2),

$$A_1, A_2, \ldots, A_{i-1}, A_{i_1}, \sim A_{i_2}, A_{i+1}, \ldots, A_n \vdash B$$

and

$$A_1, A_2, \ldots, A_{i-1}, \sim A_{i_1}, A_{i_2}, A_{i+1}, \ldots, A_n \vdash B$$

doing duty for (18.3),

$$A_1, A_2, \ldots, A_n, B_1 \vdash B_2$$

and

$$A_1, A_2, \ldots, A_n, B_2 \vdash B_1$$

doing duty for (18.4), and

$$A_1, A_2, \ldots, A_n, \sim B_1 \vdash B_2$$

and

$$A_1, A_2, \ldots, A_n, \sim B_2 \vdash B_1$$

doing duty for (18.5) - (18.6).

Cases 22-32 are provable in the same manner as Case 11. Appended is a table (Table IIIb) of the various cases to which they reduce and of the transformations—effected for illustration's sake on $B_1 \supset B_2$, B_1 & B_2, $B_1 \vee B_2$, and $B_1 \equiv B_2$—which insure those reductions. A key to the abbreviations used in Table IIIb is supplied in Table IIIa.

TABLE IIIa

T1: $B_1 \supset B_2 \dashrightarrow \sim(B_1 \,\&\, \sim B_2)$

T2: $B_1 \supset B_2 \dashrightarrow \sim B_1 \vee B_2$

T3: $B_1 \,\&\, B_2 \dashrightarrow \sim(B_1 \supset \sim B_2)$

T4: $B_1 \,\&\, B_2 \dashrightarrow \sim(\sim B_1 \vee \sim B_2)$

T5: $B_1 \vee B_2 \dashrightarrow \sim B_1 \supset B_2$

T6: $B_1 \vee B_2 \dashrightarrow \sim(\sim B_1 \,\&\, \sim B_2)$

T7: $B_1 \equiv B_2 \dashrightarrow \sim((B_1 \supset B_2) \supset \sim(B_2 \supset B_1))$

T8: $B_1 \equiv B_2 \dashrightarrow \sim(B_1 \,\&\, \sim B_2) \,\&\, \sim(B_2 \,\&\, \sim B_1)$

T9: $B_1 \equiv B_2 \dashrightarrow \sim(\sim(\sim B_1 \vee B_2) \vee \sim(\sim B_2 \vee B_1))$

TABLE IIIb

Cases	Reducible to Cases	By means of
22: '$\sim$', '$\supset$', and '&'	18	*T3*
	19	*T1*
23: '$\sim$', '$\supset$', and 'v'	18	*T5*
	20	*T2*
24: '$\sim$', '$\supset$', and '$\equiv$'	18	*T7*
25: '$\sim$', '&', and 'v'	19	*T6*
	20	*T4*

TABLE IIIb (Continued)

Cases	Reducible to Cases	By means of
26: '$\sim$', '&', and '$\equiv$'	19	*T8*
27: '$\sim$', 'v', and '$\equiv$'	20	*T9*
28: '$\sim$', '$\supset$', '&' and 'v'	18	*T3* and *T5*
	19	*T1* and *T6*
	20	*T2* and *T4*
29: '$\sim$', '$\supset$', '&', and '$\equiv$'	18	*T3* and *T7*
	19	*T1* and *T8*
30: '$\sim$', '$\supset$', 'v', and '$\equiv$'	18	*T5* and *T7*
	20	*T2* and *T9*
31: '$\sim$', '&', 'v', and '$\equiv$'	19	*T6* and *T8*
	20	*T4* and *T9*
32: '$\sim$', '$\supset$', '&', 'v', and '$\equiv$'	18	*T3*, *T5*, and *T7*
	19	*T1*, *T6*, and *T8*
	20	*T2*, *T4*, and *T9*

IV

My main theorem, **T2**, fails, as I remarked in Section II, when the elimination rules for '$\supset$' and '$\equiv$' are respectively made to read like **HE'** and **BE'**. $p \supset q, (p \supset r) \supset q \vdash q$, for example, though classically valid, is not intuitionistically valid; **R**, **E**, **P**, **HI**, and **HE'**, on the other hand, are all intuitionistically sound; $p \supset q, (p \supset r) \supset q \vdash q$ is therefore not provable by means of **R**, **E**, **P**, **HI**, and **HE'**.[31] Similarly, $p, (r \equiv p) \equiv (r \equiv q) \vdash q$ and $p, (r \equiv q) \equiv (r \equiv p) \vdash q$, though classically valid, are not intuitionistically valid; **R**, **E**, **P**, **BI**, and **BE'**, on the other hand, are all intuitionistically sound; neither $p, (r \equiv p) \equiv (r \equiv q) \vdash q$ nor $p, (r \equiv q) \equiv (r \equiv p) \vdash q$ is therefore provable by means of **R**, **E**, **P**, **BI**, and **BE'**.

It should, nonetheless, be noted that **HE** follows from **HE'** by means of **R**, **E**, **P**, **HI**, **NI**, and **NE** or by means of **R**, **E**, **P**, **HI**, **BI**, and **BE**, and hence may occasionally give way to **HE'**. Similarly, **BE** follows from **BE'** by means of **R**, **E**, **P**, **BI**, **HI**, and **HE** or by means of **R**, **E**, **P**, **BI**, **NI**, and **NE**, and hence may occasionally give way to **BE'**. Finally, **NE** follows from the intuitionist elimination rule for '$\sim$', namely:

NE': *If* (1) $A_1, A_2, \ldots, A_n \vdash B$ *and* (2) $A_1, A_2, \ldots, A_n \vdash \sim B$, *then* $A_1, A_2, \ldots, A_n \vdash C$, by means of **R**, **E**, **P**, **NI**, **HI**, and **HE** or by means of **R**, **E**, **P**, **NI**, **BI**, and **BE**, and hence may occasionally give way to **NE'**.[32]

Now for my second conjecture. Suppose $A_1, A_2, \ldots, A_n \vdash B$ exhibits no connective or exhibits no connective other than '&' and 'v'. It follows from **T1** and **T2** that if $A_1, A_2, \ldots, A_n \vdash B$ is provable or, as I shall now put it, classically provable, then $A_1, A_2, \ldots, A_n \vdash B$ is provable by means of **R**, **E**, **P**, **CI**, **CE**, **DI**, and **DE**. But all seven of those rules—I just noted—are intuitionistically sound. Hence if $A_1, A_2, \ldots, A_n \vdash B$ is classically provable, then $A_1, A_2, \ldots, A_n \vdash B$ is intuitionistically provable as well.[33] I conjectured that in view of this result one cannot convert a set of structural and intelim rules fit for PC_I, the intuitionist variant of PC, into one fit for PC by altering the intelim rules for '&' or 'v'.[34] The surmise is of some interest since we have long known how to bridge the gap between PC_I and PC by altering the intelim rules for '$\sim$' and have recently learned how to bridge that gap by altering the intelim rules for '$\supset$' or those for '$\equiv$'. Proof of it is now available, but must be saved for another occasion.[35]

NOTES

1. See "Etudes sur les Règles d'Inférence dites Règles de Gentzen, Première Partie," *Dialogue*, vol. I, no. 1, pp. 56-66, where I offered the conjecture for a slightly different, but equivalent, set of structural and intelim rules. Four cases of my conjecture (Cases 1, 4, 5, and 6) have been studied independently by Nuel D. Belnap, Jr. and R. H. Thomason; see footnote 2.

2. In view of **T1** and **T2** a T-statement T, when provable at all, is bound to be provable by means of **S** and the intelim rules for such of the connectives '$\sim$', '$\supset$', '&', 'v', and '$\equiv$' as occur in T. Now let the following structural rule:

C: *If* (1) $A_1, A_2, \ldots, A_n, B \vdash C$ *and* (2) $A_1, A_2, \ldots, A_n \vdash B$, *then* $A_1, A_2, \ldots, A_n \vdash C$,

be appended in Table I to **R**, **E**, and **P**; let a T-statement T be rated derivable from n ($n \geq 0$) T-statements $T_1, T_2, \ldots,$ and T_n if T is the last entry in a finite column of T-statements each one of which is a T_i, or is of the form **R** in Table I, or follows from one or more previous T-statements in the column by application of one of the remaining rules in Table I; and let the same T-statement T be rated derivable from the same T-statements $T_1, T_2, \ldots,$ and T_n by means of **S** and zero or more of the intelim rules in Table I if T is the last entry in a finite column of T-statements each one of which is a T_i, or is of the form **R** in Table I, or follows from one or more previous T-statements in the column by application of **E**, **P**, **C**, or one of the intelim rules in question. Belnap and Thomason have recently proved of any $n+1$ T-statements $T, T_1, T_2, \ldots,$ and T_n which exhibit no connective or exhibit no connective other than '&' and 'v' that T, when derivable at all from $T_1, T_2, \ldots,$ and T_n, is derivable from them by means of **S** and the intelim rules for such of the connectives '&' and 'v' as occur in $T, T_1, T_2, \ldots,$ and T_n; see "A

Rule-completeness Theorem," *Notre Dame Journal of Formal Logic*, vol. IV, no. 1 (1963), pp. 39-43. I would conjecture, to generalize upon this result, that a T-statement T, when derivable at all from n T-statements T_1, T_2, . . . , and T_n, is derivable from them by means of **S** and the intelim rules for such of the connectives '$\sim$', '$\supset$', '&', 'v', and '$\equiv$' as occur in T, T_1, T_2, . . . , and T_n. Rule **C**, by the way, is redundant in the presence of the intelim rules for any one of the connectives '$\sim$', '$\supset$', '&', 'v', and '$\equiv$'.

3. Version **HE** of the elimination rule for '$\supset$' was suggested to me by Professor Stig Kanger.

4. Note that if there were no i such that A_i is B, then $A_1, A_2, \ldots, A_n \vdash B$ would come out false when the truth-value **T** is assigned to every one of A_1, A_2, . . . , and A_n, and the truth-value **F** is assigned to B.

5. Note that if there were no i such that A_i is B and there were no two i and j such that A_i is $A_j \supset A_{i_2}$, then $A_1, A_2, \ldots, A_n \vdash B$ would come out false when **F** is assigned to B, **F** is assigned to the left-hand component of every conditional among A_1, A_2, . . . , and A_n whose right-hand component is assigned **F**, and **T** is assigned to every other propositional variable that may occur in A_1, A_2, . . . , and A_n. Note also that $(A_j \,\&\, (A_j \supset A_{i_2})) \equiv (A_j \,\&\, A_{i_2})$ is valid

6. Note that $((A_{i_1} \supset A_{i_2}) \supset A_{i_3}) \equiv ((A_{i_1} \,\&\, (A_{i_2} \supset A_{i_3})) \vee A_{i_3})$ is valid, a point which was brought to my attention by Professor Henry Hiż and Professor Belnap and proved crucial to the solution of Case 2.

7. Note that $(A_{i_1} \supset (A_{i_2} \supset A_{i_3})) \equiv ((A_{i_1} \supset A_{i_3}) \vee (A_{i_2} \supset A_{i_3}))$ is valid.

8. Case 2 could be proved somewhat more simply if I modified it to read: "The only connective (if any) that occurs in $A_1, A_2, \ldots, A_n \vdash B$ is '$\supset$'," and did not insist on reducing it to Case 1. The same holds true of a few other cases in this section.

9. Note that when A_{i_1} is a conditional, then the A_i in question is of the first form listed, and when A_{i_1} is a conjunction, then the A_i in question is of the seventh form listed. That the eight forms listed (and like ones in the proofs of Cases 8-10) are exhaustive was pointed out to me by Professor Belnap and proved crucial to the solution of Case 3.

10. Note that $((A_{i_1} \,\&\, A_{i_2}) \supset A_{i_3}) \equiv ((A_{i_1} \supset A_{i_3}) \vee (A_{i_2} \supset A_{i_3}))$ is valid.

11. Note that $(A_{i_1} \supset (A_{i_2} \,\&\, A_{i_3})) \equiv ((A_{i_1} \supset A_{i_2}) \,\&\, (A_{i_1} \supset A_{i_3}))$ is valid; note also that, so long as A_{i_1} is a propositional variable, $A_{i_1} \supset A_{i_2}$ and $A_{i_1} \supset A_{i_3}$ jointly exhibit one nested occurrence of '$\supset$' and '&' less than $A_{i_1} \supset (A_{i_2} \,\&\, A_{i_3})$ does.

12. The above proof of Step 1, presupposing as it does rule **C** of footnote 2, no longer goes through when the elimination rule for '&' is phrased in the more traditional fashion:

CE': *If* $A_1, A_2, \ldots, A_n \vdash B \& C$, *then* (1) $A_1, A_2, \ldots, A_n \vdash B$ *and* (2) $A_1, A_2, \ldots, A_n \vdash C$,

since **CE'** does not yield **C**. Professor Belnap has obtained a proof of Step 1 which eschews **CE** in favor of **CE'**. The proof, however, does not suit my declared strategy of reducing all of Cases 2-16 to Case 1. In view of the conjecture of footnote 2, I also prefer of two elimination rules the one which yields **C**. **CE** was suggested to me as a substitute for **CE'** by Professor Belnap.

13. Note that if there were no i such that A_i is or occurs in B_1 or B_2, then $A_1, A_2, \ldots, A_n \vdash B$ would come out false when **T** is assigned to every one of $A_1, A_2, \ldots$, and A_n, and **F** is assigned to every propositional variable that occurs in B.

14. The above proof of Case 5 still goes through when the elimination rule for 'v' is phrased in the more traditional fashion:

DE': *If* (1) $A_1, A_2, \ldots, A_n, B \vdash D$ *and* (2) $A_1, A_2, \ldots, A_n, C \vdash D$, *then* $A_1, A_2, \ldots, A_n, B \vee C \vdash D$.

In view, however, of the conjecture of footnote 2, I prefer **DE**, which yields rule **C** of that footnote, to **DE'**, which does not. **DE** was suggested to me as a substitute for **DE'** by Professor Belnap.

15. Note that if there were no i such that A_i is B and there were no two i and j such that A_i is $A_j \equiv A_{i_2}$ or $A_{i_1} \equiv A_j$, then $A_1, A_2, \ldots, A_n \vdash B$ would come out false when **F** is assigned to B, **F** is assigned to the left-hand (right-hand) component of every biconditional among $A_1, A_2, \ldots$, and A_n whose right-hand (left-hand) component is assigned **F**, and **T** is assigned to every other propositional variable that may occur in $A_1, A_2, \ldots$, and A_n. Note also that $(A_j \& (A_j \equiv A_{i_2})) \equiv (A_j \& A_{i_2})$ is valid.

16. Note that $((A_{i_1} \equiv A_{i_2}) \equiv A_{i_3}) \equiv (((A_{i_1} \& (A_{i_2} \equiv A_{i_3})) \vee (A_{i_2} \& (A_{i_1} \equiv A_{i_3}))) \vee (A_{i_3} \& (A_{i_1} \equiv A_{i_2})))$ is valid, a point which was brought to my attention by Professor Belnap and proved crucial to the solution of Case 7.

17. Note that $(A_{i_1} \supset A_{i_2}) \equiv ((A_{i_1} \equiv A_{i_2}) \vee A_{i_2})$ is valid.

18. Note that $((A_{i_1} \supset A_{i_2}) \equiv A_{i_3}) \equiv ((A_{i_1} \& (A_{i_2} \equiv A_{i_3})) \vee ((A_{i_1} \supset A_{i_2}) \& A_{i_3}))$ is valid.

19. Note that $((A_{i_1} \equiv A_{i_2}) \supset A_{i_3}) \equiv (((A_{i_1} \& (A_{i_2} \supset A_{i_3})) \vee (A_{i_2} \& (A_{i_1} \supset A_{i_3}))) \vee A_{i_3})$ is valid.

20. Note that $(A_{i_1} \supset (A_{i_2} \equiv A_{i_3})) \equiv (((A_{i_1} \supset A_{i_2}) \,\&\, (A_{i_1} \supset A_{i_3})) \vee (A_{i_2} \equiv A_{i_3}))$ is valid.

21. Note that $((A_{i_1} \,\&\, A_{i_2}) \equiv A_{i_3}) \equiv ((((A_{i_1} \equiv A_{i_3}) \,\&\, A_{i_2}) \vee ((A_{i_2} \equiv A_{i_3}) \,\&\, A_{i_1})) \vee ((A_{i_1} \equiv A_{i_3}) \,\&\, (A_{i_2} \equiv A_{i_3})))$ is valid.

21.[bis] Note that $((A_{i_1} \vee A_{i_2}) \equiv A_{i_3}) \equiv (((A_{i_1} \,\&\, A_{i_3}) \vee (A_{i_2} \,\&\, A_{i_3})) \vee ((A_{i_1} \equiv A_{i_3}) \,\&\, (A_{i_2} \equiv A_{i_3})))$ is valid.

21.[ter] Note that $(B_1 \vee B_2) \equiv ((B_1 \equiv B_2) \supset B_1)$ is valid.

22. Note that $(B_1 \vee B_2) \equiv ((B_1 \supset B_2) \supset B_2)$ is valid. The need for the restriction 'where B_2 does not exhibit any 'v'' (and like ones in Steps 2-4) was pointed out to me by Professor Belnap.

23. Or, less familiarly, $(B_1 \equiv B_2) \vee B_2$.

24. Note that if there were no i such that A_i is B and there were no two i and j such that A_i is $\sim A_j$, then $A_1, A_2, \ldots, A_n \vdash B$ would come out false when **F** is assigned to every propositional variable that is prefaced in A_1, A_2, . . . , and A_n by '$\sim$', **T** is assigned to every other propositional variable that may occur in A_1, A_2, and A_n, and **T** or **F** is assigned to the propositional variable that occurs in B according as that variable is prefaced or not by '$\sim$'.

25. Note that, where k is even, $\underbrace{\sim\sim\ldots\sim}_{k\ times} A_i^* \equiv A_i^*$ is valid.

26. Note that, where k is odd, $\underbrace{\sim\sim\ldots\sim}_{k\ times} A_i^* \equiv \sim A_i^*$ is valid.

27. Note that, where k is 0 or even, $\underbrace{\sim\sim\ldots\sim}_{k\ times}(A_{i_1} \supset A_{i_2}) \equiv (\sim A_{i_1} \vee A_{i_2})$ is valid.

28. Note that, where k is odd, $\underbrace{\sim\sim\ldots\sim}_{k\ times}(A_{i_1} \supset A_{i_2}) \equiv (A_{i_1} \,\&\, \sim A_{i_2})$ is valid.

29. Note that, where k is 0 or even, $\underbrace{\sim\sim\ldots\sim}_{k\ times}(B_1 \supset B_2) \equiv (B_1 \supset B_2)$ is valid.

30. Note that, where k is odd, $\underbrace{\sim\sim\ldots\sim}_{k\ times}(B_1 \supset B_2) \equiv (B_1 \,\&\, B_2)$ is valid.

31. **HE**, by the way, is nothing but a combined version of **HE'** and Peirce's Law.

32. For proofs of some of those results, see E. W. Beth and H. Leblanc, "A Note on the Intuitionist and the Classical Propositional Calculus,"

Logique et Analyse, no. 11-12 (1960), pp. 174-176, H. Leblanc and Nuel D. Belnap, Jr., "Intuitionism Reconsidered," *Notre Dame Journal of Formal Logic*, vol. III, no. 2 (1962), pp. 79-82, and H. Leblanc, "Etudes sur les Règles d'Inférence dites Règles de Gentzen, Première Partie."

33. Similarly, $A_1, A_2, \ldots, A_n \vdash B$, when classically valid, is intuitionistically valid as well, so long as $A_1, A_2, \ldots, A_n \vdash B$ exhibits no connective or exhibits no connective other than '&' and 'v'.

34. See the last paper of mine mentioned in footnote 32.

35. See N. D. Belnap, Jr., H. Leblanc, and R. H. Thomason, "On not strengthening intuitionistic logic," forthcoming in this journal. R. E. Vesley's disproof of the conjecture in "On strengthening intuitionistic logic," this journal, vol. IV, no. 1 (1963), p. 80, uses an intelim rule for 'v' which violates the requirements implicitly placed here upon an intelim rule. Weak forms of the conjecture have already been proved by D. H. J. de Jongh and by Belnap and Thomason; see in connection with the latter two the paper mentioned in footnote 2.

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