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A NOTE ON THE GENERALIZED CONTINUUM HYPOTHESIS. II.

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§2*

In [6] I have proved that $\mathfrak{A}$, i.e. the generalized continuum hypothesis, is equivalent to the following formula

A *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{n} < 2^{\mathfrak{m}}$, *then* $\mathfrak{n} \leqslant \mathfrak{m}$

The following convenient abbreviation defined inductively:

For any natural number n, $0 \leqslant n < \infty$, *and any cardinal number* $\mathfrak{m}$

$$\textit{symbol } \overset{\langle n\rangle}{2}\,\mathfrak{m} \textit{ means } \begin{cases} \textit{if } n = 0, \textit{ then } \overset{\langle n\rangle}{2}\,\mathfrak{m} = \mathfrak{m} \\ \textit{if } n > 0, \textit{ then } \overset{\langle n+1\rangle}{2}\,\mathfrak{m} = 2^{\overset{\langle n\rangle}{2}\mathfrak{m}} \end{cases}$$

allows us to express the formulas $\mathfrak{A}$ and **A**, as follows

$\mathfrak{A}$ (= $\mathfrak{A}_0$) *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there exists no* $\mathfrak{n}$ *such that* $\overset{\langle 0\rangle}{2}\,\mathfrak{m} < \mathfrak{n} < \overset{\langle 1\rangle}{2}\,\mathfrak{m}$

A (= $\mathbf{A}_0$) *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{n} < \overset{\langle 1\rangle}{2}\,\mathfrak{m}$, *then* $\mathfrak{n} \leqslant \mathfrak{m}$,

and their particular instances which we obtain by putting $2^{\mathfrak{m}}$, $2^{2^{\mathfrak{m}}}$, $2^{2^{2^{\mathfrak{m}}}}$ *etc.* for $\mathfrak{m}$ in $\mathfrak{A}$ or in **A**, as

*The first part of this paper appeared in *Notre Dame Journal of Formal Logic*, v. III (1962), pp. 274-278. It will be referred to throughout this second part, as [7]. See additional Bibliography given at the end of this part. An acquaintance with [7] is presupposed.

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$\mathfrak{A}_n$ *If* $\mathfrak{m}$ *is a cardinal number which is not finite, then there exists no cardinal* $\mathfrak{n}$ *such that* $\overset{\langle n\rangle}{2}\mathfrak{m} < \mathfrak{n} < \overset{\langle n+1\rangle}{2}\mathfrak{m}$

and

$\mathbf{A}_n$ *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{n} < \overset{\langle n+1\rangle}{2}\mathfrak{m}$, *then* $\mathfrak{n} \leqslant \overset{\langle n\rangle}{2}\mathfrak{m}$

I shall show here that *for any given natural number* $n > 0$, the respective instances of $\mathfrak{A}$ and $\mathbf{A}$, i.e. $\mathfrak{A}_n$ and $\mathbf{A}_n$, are such that

η) $\mathfrak{A}_n$ is equivalent to $\mathbf{A}_n$

δ) $\mathfrak{A}_n$ (or $\mathbf{A}_n$) implies the axiom of choice

On the other hand, although, as we know, $\mathfrak{C}$, i.e. Cantor's hypothesis on alephs, follows from $\mathfrak{A}_0$ (i.e. $\mathfrak{A}$),

ι) I am unable to deduce this principle without the aid either of $\mathbf{E}_1$ or of *D1* or of *C1* from $\mathfrak{A}_n$, for $n > 0$.[4]

(iv) *For a given natural number* $n > 0$, $\mathfrak{A}_n$ *implies the axiom of choice.* Let us assume that

(18) $\mathfrak{m}$ *is an arbitrary cardinal number which is not finite*

Then, put

(19) $\mathfrak{n} = \aleph_0\mathfrak{m}$ and *(20)* $\mathfrak{x} = \overset{\langle n\rangle}{2}\mathfrak{n}$, *for n the same, as assumed in* $\mathfrak{A}_n$

Since, by assumption about $\mathfrak{A}_n$, $n > 0$, and since, by *(19)*, $\mathfrak{n} = \aleph_0\mathfrak{m} = 2\aleph_0\mathfrak{m} = 2\mathfrak{n}$, it follows without any difficulty from *(19)* and *(20)* that

(21) $\mathfrak{x} = \mathfrak{x}^2$ and, *a fortiori*, that *(22)* $\mathfrak{x} = \mathfrak{x} + 1$ and that *(23)* $2^{\mathfrak{x}} = 2^{\mathfrak{x}} + 1$

Whence, by *(22)* and *(23)*,

(24) $\mathfrak{x}$ *and* $2^{\mathfrak{x}}$ *are the transfinite cardinals, i.e. such that* $\mathfrak{x} \geqslant \aleph_0$ *and* $2^{\mathfrak{x}} \geqslant \aleph_0$

Then, we can associate with $\mathfrak{x}$ a Hartogs' aleph $\aleph(\mathfrak{x})$ such that

(25) $\aleph(\mathfrak{x})$ *is the least Hartogs' aleph which is not* $\leqslant \mathfrak{x}$ and $\aleph(\mathfrak{x}) \leqslant 2^{2^{\mathfrak{x}^2}}$

and this can be established without the aid of the axiom of choice.[5] Hence by *(21)* and *(25)*, $\aleph(\mathfrak{x}) \leqslant 2^{2^{\mathfrak{x}}}$ which due to *(23)* gives at once $2^{\mathfrak{x}} \leqslant \aleph(\mathfrak{x}) + 2^{\mathfrak{x}} \leqslant 2^{2^{\mathfrak{x}}}$, i.e., by *(20)*,

(26) $\overset{\langle n+1\rangle}{2}\mathfrak{n} \leqslant \aleph(\mathfrak{x}) + \overset{\langle n+1\rangle}{2}\mathfrak{n} \leqslant \overset{\langle n+2\rangle}{2}\mathfrak{n}$

Now, since we have *(19)*, *(20)* and *(24)*, entirely the same argumentations which Sierpiński used in [3], pp. 434-436, and in [5], show without the help of the axiom of choice that the case $\aleph(\mathfrak{x}) + \overset{\langle n+1\rangle}{2}\,\mathfrak{n} = \overset{\langle n+2\rangle}{2}\,\mathfrak{n}$ of *(26)* implies $\aleph(\mathfrak{x}) = \overset{\langle n+2\rangle}{2}\,\mathfrak{n}$ which, by *(18)*, shows that $\mathfrak{m}$ is an aleph. The second case of *(26)*, i.e. $\overset{\langle n+1\rangle}{2}\,\mathfrak{n} < \aleph(\mathfrak{x}) + \overset{\langle n+1\rangle}{2}\,\mathfrak{n} < \overset{\langle n+2\rangle}{2}\,\mathfrak{n}$, is rejected by $\mathfrak{A}_n$ (for $\mathfrak{m} = 2^{\mathfrak{n}}$). And, the last case of *(26)*, viz. $\overset{\langle n+1\rangle}{2}\,\mathfrak{n} = \aleph(\mathfrak{x}) + \overset{\langle n+1\rangle}{2}\,\mathfrak{n}$, implies, obviously,

$$(27) \quad \aleph(\mathfrak{x}) \leqslant \overset{\langle n+1\rangle}{2}\,\mathfrak{n}$$

which, in virtue of *(22)*, gives at once

$$(28) \quad \overset{\langle n\rangle}{2}\,\mathfrak{n} \leqslant \aleph(\mathfrak{x}) + \overset{\langle n\rangle}{2}\,\mathfrak{n} \leqslant \overset{\langle n+1\rangle}{2}\,\mathfrak{n}$$

Using again the mentioned above argumentation of Sierpiński we can establish that the case $\aleph(\mathfrak{x}) + \overset{\langle n\rangle}{2}\,\mathfrak{n} = \overset{\langle n+1\rangle}{2}\,\mathfrak{n}$ of *(28)* together with *(18)*, *(19)*, *(20)* and *(24)* implies $\aleph(\mathfrak{x}) = \overset{\langle n+1\rangle}{2}\,\mathfrak{n} > \aleph_0\mathfrak{m} \geqslant \mathfrak{m}$, i.e. that $\mathfrak{m}$ is an aleph. The two remaining cases of *(28)*, viz. $\overset{\langle n\rangle}{2}\,\mathfrak{n} < \aleph(\mathfrak{x}) + \overset{\langle n\rangle}{2}\,\mathfrak{n} < \overset{\langle n+1\rangle}{2}\,\mathfrak{n}$ and $\overset{\langle n\rangle}{2}\,\mathfrak{n} = \aleph(\mathfrak{x}) + \overset{\langle n\rangle}{2}\,\mathfrak{n}$, are impossible, since the former is rejected directly by $\mathfrak{A}_n$ and the latter by *(20)* and *(25)*. Since due to $\mathfrak{A}_n$ all possible cases generated by *(25)* imply that $\aleph(\mathfrak{x}) > \mathfrak{m}$, i.e., by *(18)*, that $\mathfrak{m}$ is an aleph, the proof is given that the axiom of choice follows from $\mathfrak{A}_n$, for any given natural $n > 0$.

(v̈) *For any natural number $n > 0$, $\mathfrak{A}_n$ is equivalent to $\mathbf{A}_n$.*

(c) For a given natural number $n > 0$, let us assume $\mathfrak{A}_n$ and the conditions of $\mathbf{A}_n$, viz. that

(29) *$\mathfrak{m}$ and $\mathfrak{n}$ are the arbitrary cardinal numbers which are not finite and such that* $\mathfrak{n} < \overset{\langle n+1\rangle}{2}\,\mathfrak{m}$

Since $\mathfrak{A}_n$ implies the axiom of choice, the conditions given in *(29)* yield that

$$(30) \quad \textit{either } \overset{\langle n\rangle}{2}\,\mathfrak{m} < \mathfrak{n} \textit{ or } \mathfrak{n} \leqslant \overset{\langle n\rangle}{2}\,\mathfrak{m}$$

But, the first case of *(30)*, i.e. $\overset{\langle n\rangle}{2}\,\mathfrak{m} < \mathfrak{n}$, is rejected by *(29)* and $\mathfrak{A}_n$. Hence, the second case of *(30)* holds which shows that $\mathbf{A}_n$ follows from $\mathfrak{A}_n$.

($\mathfrak{d}$) For a given natural number $n > 0$, let us assume $\mathbf{A}_n$ and the condition of $\mathfrak{A}_n$, viz. that

(31) *$\mathfrak{m}$ is an arbitrary cardinal number which is not finite*

Besides, suppose that

(32) *there exists a cardinal number $\mathfrak{n}$ such that* $\overset{\langle n\rangle}{2}\,\mathfrak{m} < \mathfrak{n} < \overset{\langle n+1\rangle}{2}\,\mathfrak{m}$

holds. Since, by *(31)* and *(32)*, $\mathfrak{n}$ is a cardinal number which is not finite, *(32)* together with $\mathbf{A}_n$ implies that $\mathfrak{n} \leqslant \overset{\langle n\rangle}{2}\,\mathfrak{m}$ which shows that the negation of *(32)* must be true. Thus, $\mathbf{A}_n$ implies $\mathfrak{A}_n$, for any given natural number $n > 0$.

(vi) *For any given natural number $n > 0$, $\mathfrak{A}_n$ (or, due to* (v), $\mathbf{A}_n$*) together with either $\mathbf{E}_1$ or D1 or C1 implies Cantor's hypothesis on alephs.* First of all we establish that $\mathbf{E}_1$ implies *D1* and that the latter formula infers *C1*. After that, we show that $\mathfrak{A}_n$, for $n > 0$, and *C1* imply $\mathfrak{C}$.

($\mathfrak{e}$) Let us assume $\mathbf{E}_1$ and the conditions of *D1*, i.e. that $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the alephs such that* $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$. Since these conditions imply $\mathfrak{a} < 2^{\mathfrak{b}}$ and $\mathfrak{b} < 2^{\mathfrak{a}}$, by $\mathbf{E}_1$, $\mathfrak{a} \leqslant \mathfrak{b}$ and $\mathfrak{b} \leqslant \mathfrak{a}$ which gives at once $\mathfrak{a} = \mathfrak{b}$. Thus, *D1* follows from $\mathbf{E}_1$. Now, assume *D1* and the conditions of *C1*, i.e. that $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the alephs such that* $\mathfrak{a} < \mathfrak{b}$. Then, since the conditions of *C1* imply $2^{\mathfrak{a}} \leqslant 2^{\mathfrak{b}}$ generally, and since the case $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$ is rejected by *D1*, the conclusion of *C1*, i.e. $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$, holds. Thus, *D1* infers *C1*.

($\mathfrak{f}$) Now, let us assume, for a given natural number $n > 0$, $\mathfrak{A}_n$ and *C1*. Since $\mathfrak{A}_n$ implies $\mathfrak{B}$, *C1* together with $\mathfrak{B}$ gives *C*. Hence, we are able to prove easily that $\mathfrak{A}_n$ and *C* yield formula $\mathfrak{A}_{n-1}$. Namely, assume the conditions of $\mathfrak{A}_{n-1}$, viz. that

(33) *n is the same natural number which is assumed in $\mathfrak{A}_n$*

and that

(34) *$\mathfrak{m}$ is a cardinal number which is not finite*

and, besides, suppose that

(35) *there exists a cardinal number $\mathfrak{n}$ such that* $\overset{\langle n-1\rangle}{2}\,\mathfrak{m} < \mathfrak{n} < \overset{\langle n\rangle}{2}\,\mathfrak{m}$

holds. Hence, in virtue of *C*, *(34)* and *(35)*, we have at once

$$(36) \quad \overset{\langle n\rangle}{2}\,\mathfrak{m} < 2^{\mathfrak{n}} < \overset{\langle n+1\rangle}{2}\,\mathfrak{m}$$

which due to *(33)* and *(34)* is rejected by $\mathfrak{A}_n$. Thus, $\mathfrak{A}_{n-1}$ follows from $\mathfrak{A}_n$ and *C1*. Therefore, since n appearing in $\mathfrak{A}_n$ is a finite number, we have a proof that $\mathfrak{A}_n$ and *C1* imply $\mathfrak{A}_0$, i.e. $\mathfrak{A}$.

Thus, in virtue of ($\mathfrak{e}$) and ($\mathfrak{f}$), we can conclude that

For a given natural number $n \geqslant 0$, $\{\mathfrak{B};\mathfrak{C}\} \rightleftarrows \{\mathfrak{A}_0\} \rightleftarrows \{\mathbf{A}_0\} \rightleftarrows \{\mathfrak{A}_n;\mathbf{E}_1\} \rightleftarrows \{\mathbf{A}_n;\mathbf{E}_1\} \rightleftarrows \{\mathfrak{A}_n;D1\} \rightleftarrows \{\mathbf{A}_n;D1\} \rightleftarrows \{\mathfrak{A}_n;C1\} \rightleftarrows \{\mathbf{A}_n;C1\}$

§3

It is known, *cf.* [6], that the following formulas

$\mathfrak{S}$ *If* $\mathfrak{a}$ *is an arbitrary aleph, then there exists no cardinal number* $\mathfrak{n}$ *such that* $\mathfrak{a} < \mathfrak{n} < 2^{\mathfrak{a}}$

and

C *For any cardinal number* $\mathfrak{n}$ *which is not finite and any cardinal number* $\mathfrak{a}$, *if* $\mathfrak{a}$ *is an aleph and* $\mathfrak{n} < 2^{\mathfrak{a}}$, *then* $\mathfrak{n} \leqslant \mathfrak{a}$

and which, evidently, are obtainable by the simple substitution from $\mathfrak{A}$ and **A** respectively, are such that each of them is equivalent to $\mathfrak{C}$, i.e. Cantor's hypothesis on alephs. But, as it will be shown, an instance of $\mathfrak{S}$, viz.

$\mathfrak{S}_1$ *If* $\mathfrak{a}$ *is an arbitrary aleph, then there exists no aleph* $\mathfrak{b}$ *such that* $\mathfrak{a} < \mathfrak{b} < 2^{\mathfrak{a}}$

and which we can obtain from $\mathfrak{S}$ substituting all cardinal numbers by alephs in that formula, is equivalent to the analogous instance of **C**, namely $\mathbf{E}_1$. And, therefore, as I conjecture, $\mathfrak{S}_1$ is probably weaker than $\mathfrak{C}$.

On the other hand, as I prove here, that there are two simple formulas presented below, **F** and **G**, which are such that

κ) **F** is equivalent to the generalized continuum hypothesis.

λ) **G** is an instance of **F**, obtainable by substitution of all cardinal numbers by alephs in that formula.

and

μ) **G** is equivalent to Cantor's hypothesis on alephs.

Moreover, it will be shown that the mentioned above formulas **F** and **G**, i.e. the statements

F *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m}$ *is not finite and* $\mathfrak{m} < \mathfrak{n}$, *then* $2^{\mathfrak{m}} \leqslant \mathfrak{n}$

and

G *For any alephs* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $\mathfrak{a} < \mathfrak{b}$, *then* $2^{\mathfrak{a}} \leqslant \mathfrak{b}$

are such that the following instances of **F**

$\mathbf{F}_1$ *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{m}$, *if* $\mathfrak{a}$ *is an aleph and* $\mathfrak{a} < \mathfrak{n}$, then $2^{\mathfrak{a}} \leqslant \mathfrak{n}$

and

G$_1$ *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{a}$, *if* $\mathfrak{m}$ *is not finite,* $\mathfrak{a}$ *is an aleph and* $\mathfrak{m} < \mathfrak{a}$, *then* $2^{\mathfrak{m}} \leqslant \mathfrak{a}$

are equivalent to **F** (i.e. to $\mathfrak{A}$) and to **G** (i.e. to $\mathfrak{C}$) respectively. And, therefore, the particular forms which **F**$_1$ and **G**$_1$ possess explain in a certain way the relations existing between the generalized continuum hypothesis and Cantor's hypothesis on alephs.

(vii) *Formula* $\mathfrak{D}_1$ *is equivalent to* **E**$_1$. Firstly, we prove that the former formula implies **E**$_1$, and, secondly, that the last formula infers $\mathfrak{D}_1$.

($\mathfrak{e}$) First of all, in order to obtain the required proof we have to show that *C1* follows from $\mathfrak{D}_1$. Hence, let us assume $\mathfrak{D}_1$ and the conditions of *C1*, viz. that $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the arbitrary alephs such that* $\mathfrak{a} < \mathfrak{b}$. Then, obviously, we have $2^{\mathfrak{a}} \leqslant 2^{\mathfrak{b}}$. But, since the first case of this consequence, i.e. $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$, together with the assumptions of *C1* and the general formula $\mathfrak{a} < 2^{\mathfrak{b}}$ gives $\mathfrak{a} < \mathfrak{b} < 2^{\mathfrak{a}}$ which contradicts $\mathfrak{D}_1$, the second case of the discussed consequence, viz. $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$ must be true. Thus, *C1* follows from $\mathfrak{D}_1$.

Now, assume the conditions of **E**$_1$, viz. that $\mathfrak{a}$ *and* $\mathfrak{b}$ *are arbitrary alephs such that* $\mathfrak{b} < 2^{\mathfrak{a}}$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are alephs, we know that *either* $\mathfrak{a} < \mathfrak{b}$ *or* $\mathfrak{b} \leqslant \mathfrak{a}$. Since, in virtue of *C1*, the case $\mathfrak{a} < \mathfrak{b}$ implies $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$ which in its turn together with the conditions of **E**$_1$ infers $\mathfrak{b} < 2^{\mathfrak{a}} < 2^{\mathfrak{b}}$, i.e. the formula rejected by $\mathfrak{D}_1$, the second case of the discussed consequence, viz. $\mathfrak{b} \leqslant \mathfrak{a}$, holds. Therefore, the proof is given that **E**$_1$ follows from $\mathfrak{D}_1$.

($\mathfrak{f}$) Now, let us assume **E**$_1$ and the condition of $\mathfrak{D}_1$, viz. that $\mathfrak{a}$ *is an arbitrary aleph.* Besides, suppose that *there exists an aleph* $\mathfrak{b}$ *such that* $\mathfrak{a} < \mathfrak{b} < 2^{\mathfrak{a}}$. Then, by **E**$_1$, these assumptions give at once that $\mathfrak{b} \leqslant \mathfrak{a}$ which contradicts the assumed fact that $\mathfrak{a} < \mathfrak{b}$. Hence, *there exists no aleph* $\mathfrak{b}$ *such that* $\mathfrak{a} < \mathfrak{b} < 2^{\mathfrak{a}}$, and, therefore $\mathfrak{D}_1$ is a consequence of **E**$_1$.

Thus, we have established that $\{\mathfrak{D}_1\} \rightleftarrows \{\mathbf{E}_1\}$.

(viii) *Each of the formulas* **F** *and* **F**$_1$ *is equivalent to* $\mathfrak{A}$, *and each of the formulas* **G** *and* **G**$_1$ *is equivalent to* $\mathfrak{C}$. It is obvious that **F**$_1$ and **G** are the instances of **F** and **G**$_1$ respectively, and that, on the other hand, **G** is also an instance of **F**$_1$. Hence, it is sufficient to prove 1) that **F** follows from $\mathfrak{A}$, and 2) that $\mathfrak{C}$ implies **G**$_1$, and, in virtue of the fact that $\{\mathfrak{A}\} \rightleftarrows \{\mathfrak{B}; \mathfrak{C}\}$, to show later 3) that **F**$_1$ infers the axiom of choice and 4) that **G** implies Cantor's hypothesis on alephs.

($\mathfrak{g}$) Let us assume $\mathfrak{A}$ and the conditions of **F**, viz. that

(37) $\mathfrak{m}$ *and* $\mathfrak{n}$ *are the arbitrary cardinal numbers such that* $\mathfrak{m}$ *is not finite* and $\mathfrak{m} < \mathfrak{n}$

Hence, by *(37)*,

(38) $\mathfrak{n}$ *and* $2^{\mathfrak{m}}$ *are cardinal numbers which are not finite*

and, therefore, since $\mathfrak{A}$ implies the axiom of choice, *(38)* together with $\mathfrak{B}$ gives

(39) *either* $\mathfrak{n} < 2^{\mathfrak{m}}$ *or* $2^{\mathfrak{m}} \leqslant \mathfrak{n}$

Since the first case of *(39)*, viz. $\mathfrak{n} < 2^{\mathfrak{m}}$, together with *(37)* yields $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$ which is rejected by *(37)* and $\mathfrak{A}$, the second case of *(39)*, viz.

(40) $2^{\mathfrak{m}} \leqslant \mathfrak{n}$

holds. Thus, $\mathfrak{A}$ implies **F**.

($\mathfrak{h}$) *Formula* $\mathfrak{C}$ *infers* $\mathbf{G}_1$. Let us assume $\mathfrak{C}$ and the conditions of $\mathbf{G}_1$, viz. that

(41) $\mathfrak{m}$ *and* $\mathfrak{a}$ *are arbitrary cardinal numbers such that* $\mathfrak{m}$ *is not finite,* $\mathfrak{a}$ *is an aleph, and* $\mathfrak{m} < \mathfrak{a}$

Hence, in virtue of *(41)*, we know that $\mathfrak{m}$ *is an aleph* which means that

(42) *there exists an ordinal number* α *such that* $\mathfrak{m} = \aleph_\alpha$ *and* $\aleph_\alpha < \mathfrak{a}$

Furthermore, due to [6] and the point ($\mathfrak{e}$) of this paper (§2, (vi)) we know that $\mathfrak{C}$ implies **C** and *C1*. Hence, by *(41)*, *(42)* and *C1*, we obtain $2^{\mathfrak{m}} < 2^{\mathfrak{a}}$ which together with **C** and the fact that $2^{\mathfrak{m}}$ is a cardinal number allows us to establish that

(43) $2^{\mathfrak{m}} \leqslant \mathfrak{a}$

holds. Thus, $\mathbf{G}_1$ is a consequence of $\mathfrak{C}$.

($\mathfrak{i}$) *The axiom of choice follows from* $\mathbf{F}_1$. Let us assume $\mathbf{F}_1$ and that

(44) $\mathfrak{m}$ *is an arbitrary cardinal number which is not finite*

Moreover, put

(45) $\mathfrak{n} = \aleph_0 \mathfrak{m}$

Then, we can associate with $\mathfrak{n}$ a Hartogs' aleph $\aleph(\mathfrak{n})$ such that

(46) $\aleph(\mathfrak{n})$ *is the least Hartogs' aleph which is not* $\leqslant \mathfrak{n}$

and this, as we know, can be established without the aid of the axiom of choice. Besides, and also without the help of the said axiom, we can prove that it follows from *(46)* that

(47) $\mathfrak{n} < \mathfrak{n} + \aleph(\mathfrak{n})$

and that

(48) *there exists no cardinal* $\mathfrak{x}$ *such that* $\mathfrak{n} < \mathfrak{x} < \mathfrak{n} + \aleph(\mathfrak{n})$

Moreover, generally, we have

(49) $\aleph(\mathfrak{n}) \leqslant \mathfrak{n} + \aleph(\mathfrak{n})$

Now, let us suppose that the first case of *(49)*, i.e.

(50) $\aleph(\mathfrak{n}) < \mathfrak{n} + \aleph(\mathfrak{n})$

holds. Then, since, obviously, $\aleph(\mathfrak{n})$ and $\mathfrak{n} + \aleph(\mathfrak{n})$ are cardinal numbers and, moreover, $\aleph(\mathfrak{n})$ is not finite, *(50)* and $\mathbf{F}_1$ imply

(51) $2^{\aleph(\mathfrak{n})} \leqslant \mathfrak{n} + \aleph(\mathfrak{n})$

Since, by *(45)*, $\mathfrak{n} = 2\mathfrak{n}$, it follows from this fact, *(51)* and *(47)* that

(52) $\mathfrak{n} < \mathfrak{n} + 2^{\aleph(\mathfrak{n})} \leqslant \mathfrak{n} + \aleph(\mathfrak{n})$

But, since the first case of *(52)*, i.e. $\mathfrak{n} < \mathfrak{n} + 2^{\aleph(\mathfrak{n})} < \mathfrak{n} + \aleph(\mathfrak{n})$, contradicts *(48)*, the second part of *(52)*, viz.

(53) $\mathfrak{n} + 2^{\aleph(\mathfrak{n})} = \mathfrak{n} + \aleph(\mathfrak{n})$

must hold. In virtue of the known theorem, which says that

T1 *If* $\mathfrak{m}$, $\mathfrak{p}$ *and* $\mathfrak{q}$ *are cardinal numbers such that* $\mathfrak{m} + \mathfrak{p} = \mathfrak{m} + \mathfrak{q}$, *then there exist cardinal numbers* $\mathfrak{n}$, $\mathfrak{p}_1$ *and* $\mathfrak{q}_1$ *such that* $\mathfrak{p} = \mathfrak{n} + \mathfrak{p}_1$; $\mathfrak{q} = \mathfrak{n} + \mathfrak{q}_1$; $\mathfrak{m} + \mathfrak{p}_1 = \mathfrak{m} = \mathfrak{m} + \mathfrak{q}_1$

and which is provable without the aid of the axiom of choice,[6] and the points *(53)*, *(45)* and *(46)*, we can establish that

there exist cardinal numbers $\mathfrak{x}$, $\mathfrak{y}$ *and* $\mathfrak{t}$ *such that*

(54) $\aleph(\mathfrak{n}) = \mathfrak{x} + \mathfrak{y}$; *(55)* $2^{\aleph(\mathfrak{n})} = \mathfrak{x} + \mathfrak{t}$; *(56)* $\mathfrak{n} + \mathfrak{y} = \mathfrak{n} = \mathfrak{n} + \mathfrak{t}$

Since, by *(54)*, $\mathfrak{x}$ and $\mathfrak{y}$ are alephs and since, by *(56)*, $\mathfrak{n} \geqslant \mathfrak{y}$, due to *(46)* we know that

(57) *it cannot be* $\aleph(\mathfrak{n}) = \mathfrak{y}$

Hence, by *(54)* and *(57)*,

(58) $\aleph(\mathfrak{n}) = \mathfrak{x}$

and, therefore, by *(55)*,

(59) $2^{\aleph(\mathfrak{n})} = \aleph(\mathfrak{n}) + \mathfrak{t}$

which, in virtue of the known theorem, which says that

T2 *For any cardinal number* $\mathfrak{m}$ *such that* $\mathfrak{m} \geqslant \aleph_0$: $2^{\mathfrak{m}} - \mathfrak{m} = 2^{\mathfrak{m}}$

and which is provable without the aid of the axiom of choice, and the obvious fact that $\aleph(\mathfrak{n}) \geqslant \aleph_0$, allows us to establish that

(60) $2^{\aleph(\mathfrak{n})} = \mathfrak{t}$

i.e. due to *(56)* that

(61) $\mathfrak{n} \geqslant 2^{\aleph(\mathfrak{n})}$

Since *(61)* gives immediately $\mathfrak{n} > \aleph(\mathfrak{n})$ which due to *(46)* is impossible, we can conclude that *(61)* is not true, and, therefore, the point which generated *(61)*, i.e. *(53)* must be also rejected. Consequently, we have to reject the points *(52)* and *(51)* which means that our supposition *(50)* is also not true. Therefore, the second case of *(49)*, viz.

(62) $\aleph(\mathfrak{n}) = \mathfrak{n} + \aleph(\mathfrak{n})$

must hold, which together with *(45)* and *(44)* gives at once $\aleph(\mathfrak{n}) \geqslant \mathfrak{n} = \aleph_0\mathfrak{m} \geqslant \mathfrak{m}$, i.e. that our arbitrary cardinal number $\mathfrak{m}$ which is not finite is an aleph. Thus, the proof is given that $\mathbf{F}_1$ implies the axiom of choice.

(j) *Formula* **G** *implies Cantor's hypothesis on alephs.* Let us assume **G** and the condition of $\mathfrak{C}$, viz. that α *is an arbitrary ordinal number.* Then, banally, we have $\aleph_\alpha < \aleph_{\alpha+1}$ which together with **G** allows us to establish that $2^{\aleph_\alpha} \leqslant \aleph_{\alpha+1}$. Since the first case of this consequence, viz. $2^{\aleph_\alpha} < \aleph_{\alpha+1}$, infers an impossible conclusion, namely $\aleph_\alpha < 2^{\aleph_\alpha} < \aleph_{\alpha+1}$, the second case, viz.

(63) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$

must hold. Thus, $\mathfrak{C}$ follows from **G** which, therefore, in its turn is equivalent to $\mathbf{G}_1$. Hence, the proof is given that $\{\mathfrak{A}\} \rightleftarrows \{\mathbf{F}\} \rightarrow \{\mathbf{F}_1\}$ and that $\{\mathfrak{C}\} \rightleftarrows \{\mathbf{G}\} \rightleftarrows \{\mathbf{G}_1\}$.

§4

It is shown in [6] that the axiom of choice is equivalent to the following formula

B *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$ *which are not finite, if* $\mathfrak{n} < 2^{\mathfrak{m}}$, *then either* $\mathfrak{n} \leqslant \mathfrak{m}$ *or* $\mathfrak{m} < \mathfrak{n}$

and which possesses a structure analogous to **A** and **C**. A natural question arises whether a similar analog of **F** and **G**, namely

H *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{n}$, *if* $\mathfrak{m}$ *is not finite and* $\mathfrak{m} < \mathfrak{n}$, *then either* $2^{\mathfrak{m}} \leqslant \mathfrak{n}$ *or* $\mathfrak{n} < 2^{\mathfrak{m}}$

and the following instance of **H**

$\mathbf{H}_1$ *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{n}$, *if* $\mathfrak{a}$ *is an aleph and* $\mathfrak{a} < \mathfrak{n}$, *then either* $2^{\mathfrak{a}} \leqslant \mathfrak{n}$ *or* $\mathfrak{n} < 2^{\mathfrak{a}}$

are such that each of them is equivalent to the axiom of choice.

This problem remains open, but I was able to prove that

ν) $\mathbf{H}_1$ together with the formula given below *B2* which is an instance of

B and which at the same time is also a consequence of Cantor's hypothesis on alephs alone implies $\mathfrak{B}$ and, moreover, that

$\mathfrak{z})$ the conjunction of $\mathbf{H}_1$ and $\mathbf{E}_1$ is equivalent to $\mathfrak{A}$.

If it were true that $\mathbf{H}_1$ and $\mathbf{E}_1$ are really weaker than $\mathfrak{B}$ and $\mathfrak{C}$ respectively, this result may be of some interest, because in such a case it would be shown that the generalized continuum hypothesis is equivalent to the conjunction of the fragment of the axiom of choice and of the fragment of Cantor's hypothesis on alephs.

Since formula $\mathbf{H}$ is nothing else than a weak formulation of the law of trichotomy for cardinals, it is obvious that $\mathbf{H}$ follows from $\mathfrak{B}$. Hence, it is sufficient to prove: 1) that the formula presented below $B2$ follows from $\mathfrak{C}$, 2) that $\mathbf{H}_1$ together with $B2$ infers $\mathfrak{B}$, and 3) that $\mathfrak{C}$ is a consequence of $\mathbf{H}_1$ and $\mathbf{E}_1$.

(i$\mathfrak{x}$) As we know, we are unable to prove formula B without the aid of $\mathfrak{B}$.[7] Now, consider the following three instances of B

$B1$ *For any cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{n}$, *if* $\mathfrak{a}$ *is an aleph and* $2^{\mathfrak{a}} < 2^{\mathfrak{n}}$, *then* $\mathfrak{a} < \mathfrak{n}$

$B2$ *For any cardinal numbers* $\mathfrak{m}$ *and* $\mathfrak{a}$, *if* $\mathfrak{a}$ *is an aleph and* $2^{\mathfrak{m}} < 2^{\mathfrak{a}}$, *then* $\mathfrak{m} < \mathfrak{a}$

and

$B3$ *For any alephs* $\mathfrak{a}$ *and* $\mathfrak{b}$, *if* $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$, *then* $\mathfrak{a} < \mathfrak{b}$

As in the case of B, we need the axiom of choice in order to obtain $B1$. On the other hand, $B2$ and $B3$ follow not only from B, but also from $\mathfrak{C}$ and general set theory respectively. We prove it as follows:

($\mathfrak{k}$) *Cantor's hypothesis on alephs implies B2.* Let us assume $\mathfrak{C}$ and the conditions of $B2$, viz. that $\mathfrak{m}$ *and* $\mathfrak{a}$ *are the arbitrary cardinal numbers such that* $\mathfrak{a}$ *is an aleph and* $2^{\mathfrak{m}} < 2^{\mathfrak{a}}$. If $\mathfrak{m}$ is a finite cardinal, then, since $\mathfrak{a}$ is an aleph, we have that $\mathfrak{m} < \mathfrak{a}$. Hence, assume that $\mathfrak{m}$ is not finite. Since $\mathfrak{a}$ is an aleph, in virtue of $\mathfrak{C}$, $2^{\mathfrak{a}}$ *is also an aleph*, and, therefore due to condition $2^{\mathfrak{m}} < 2^{\mathfrak{a}}$ and the assumption that $\mathfrak{m}$ is not finite we know that $\mathfrak{m}$ *is an aleph*. Since, as we know, $C1$ is a consequence of $\mathfrak{C}$, an application of $C1$ to the deductions presented above yields that $\mathfrak{m} < \mathfrak{a}$. Thus, $B2$ follows from $\mathfrak{C}$ alone.

It should be noted that although $C1$ is a consequence of $\mathbf{E}_1$ (or $\mathfrak{D}_1$), we are unable to prove $B2$ using only this latter formula.

($\mathfrak{l}$) *Proof of B3.* Let us assume the conditions of $B3$, viz. that $\mathfrak{a}$ *and* $\mathfrak{b}$ *are the arbitrary alephs such that* $2^{\mathfrak{a}} < 2^{\mathfrak{b}}$. Since $\mathfrak{a}$ and $\mathfrak{b}$ are alephs, we have *either* $\mathfrak{a} = \mathfrak{b}$ *or* $\mathfrak{b} < \mathfrak{a}$ *or* $\mathfrak{a} < \mathfrak{b}$. Since the first and the second case of this consequence, viz. $\mathfrak{a} = \mathfrak{b}$ and $\mathfrak{b} < \mathfrak{a}$, imply $2^{\mathfrak{a}} = 2^{\mathfrak{b}}$ and $2^{\mathfrak{b}} \leqslant 2^{\mathfrak{a}}$ respectively, which contradicts the condition of $B3$, the third case of the discussed consequence, viz. $\mathfrak{a} < \mathfrak{b}$, holds. Hence, $B3$ is provable in the field of general set theory.

(ẍ) *Formula* $\mathbf{H}_1$ *together with B2 implies the axiom of choice.* Let us assume $\mathbf{H}_1$, *B2* and that

(64) $\mathfrak{m}$ *is an arbitrary cardinal number which is not finite*

Furthermore, put

(65) $\mathfrak{n} = \mathfrak{m} + \aleph_0$

and

(66) $\mathfrak{x} = 2^{\mathfrak{n}}$

Since, by *(65)*, $\mathfrak{n} = \mathfrak{n} + 1$, *(66)* gives at once

(67) $\mathfrak{x} = 2\mathfrak{x}$

Besides, as we know, we can, without the aid of the axiom of choice:

a) associate with $\mathfrak{x}$ a Hartogs' aleph $\aleph(\mathfrak{x})$ such that

(68) $\aleph(\mathfrak{x})$ *is the least Hartogs' aleph which is not* $\leqslant \mathfrak{x}$

and

b) in virtue of *(68)*, prove that

(69) $\mathfrak{x} < \mathfrak{x} + \aleph(\mathfrak{x})$

and that

(70) *there exists no cardinal* $\mathfrak{f}$ *such that* $\mathfrak{x} < \mathfrak{f} < \mathfrak{x} + \aleph(\mathfrak{x})$

Moreover, we have generally

(71) $\aleph(\mathfrak{x}) \leqslant \mathfrak{x} + \aleph(\mathfrak{x})$

Because the first case of *(71)*, viz.

(72) $\aleph(\mathfrak{x}) = \mathfrak{x} + \aleph(\mathfrak{x})$

due to *(64)*, *(65)* and *(66)* gives at once $\aleph(\mathfrak{x}) \geqslant \mathfrak{x} = 2^{\mathfrak{n}} > \mathfrak{n} = \mathfrak{m} + \aleph_0 \geqslant \mathfrak{m}$, i.e. that our arbitrary cardinal number $\mathfrak{m}$ which is not finite is an aleph, and, therefore that this case implies the axiom of choice, it remains only to analyze the second case of *(71)*, viz.

(73) $\aleph(\mathfrak{x}) < \mathfrak{x} + \aleph(\mathfrak{x})$

which, by *(64)*, *(65)* and *(66)*, and in virtue of $\mathbf{H}_1$, yields that

(74) *either* $2^{\aleph(\mathfrak{x})} \leqslant \mathfrak{x} + \aleph(\mathfrak{x})$ *or* $\mathfrak{x} + \aleph(\mathfrak{x}) < 2^{\aleph(\mathfrak{x})}$

Since we have *(64)*, *(65)*, *(66)*, *(68)*, *(69)* and *(70)*, the entirely same reasonings which allowed us to reject point *(51)* in (ἰ) previously, enable us now to show that the first case of *(74)*, viz. $2^{\aleph(\mathfrak{x})} \leqslant \mathfrak{x} + \aleph(\mathfrak{x})$, is impossible, because it leads to a contradiction. Hence, the second case of *(74)*, namely

(75) $\mathfrak{x} + \aleph(\mathfrak{x}) < 2^{\aleph(\mathfrak{x})}$

holds which, by *(66)*, gives at once

(76) $2^{\mathfrak{n}} < 2^{\aleph(\mathfrak{x})}$

Hence, due to *(76)* and *B2* we obtain

(77) $\aleph(\mathfrak{x}) > \mathfrak{n}$

which, by *(65)* and *(64)*, implies $\aleph(\mathfrak{x}) > \mathfrak{n} = \mathfrak{m} + \aleph_0 \geqslant \mathfrak{m}$, i.e. that our arbitrary cardinal number $\mathfrak{m}$ which is not finite is an aleph, and, therefore, that the second case of *(71)* also implies the axiom of choice.

Thus, the proof is given that $\mathbf{H}_1$ and *B2* imply $\mathfrak{B}$. Moreover, since *B2* is an instance of *B*, we can conclude that $\{\mathbf{H}_1;B\} \rightleftarrows \{\mathfrak{B}\}$. But, it should be noted that the mutual independence of $\mathbf{H}_1$ and *B* is not proved.

(xi) *Formulas* $\mathbf{H}_1$ *and* $\mathbf{E}_1$ *infers Cantor's hypothesis on alephs.* Let us assume $\mathbf{H}_1$ and $\mathbf{E}_1$ and the condition of $\mathfrak{C}$, viz. that

(78) α *is an arbitrary ordinal number*

Hence, we have $\aleph_\alpha < \aleph_{\alpha+1}$ which, in virtue of $\mathbf{H}_1$ gives

(79) *either* $2^{\aleph_\alpha} < \aleph_{\alpha+1}$ *or* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ *or* $\aleph_{\alpha+1} < 2^{\aleph_\alpha}$

But, the first and third cases of *(79)* are impossible, since the first case, i.e. $2^{\aleph_\alpha} < \aleph_{\alpha+1}$, implies $\aleph_\alpha < 2^{\aleph_\alpha} < \aleph_{\alpha+1}$, and the third case together with $\mathbf{E}_1$ gives $\aleph_{\alpha+1} \leqslant \aleph_\alpha$. Hence, the second case of *(79)*, viz.

(80) $2^{\aleph_\alpha} = \aleph_{\alpha+1}$

holds. Thus, $\mathfrak{C}$ follows from $\mathbf{H}_1$ and $\mathbf{E}_1$.

As we know, $\mathfrak{C}$ implies *B2* and the latter formula together with $\mathbf{H}_1$ yields $\mathfrak{B}$. This allows us to establish that $\{\mathfrak{A}\} \rightleftarrows \{\mathfrak{B};\mathfrak{C}\} \rightleftarrows \{\mathbf{H}_1;\mathbf{E}_1\}$. But, since it is proved in [6], that $\{\mathfrak{A}\} \rightleftarrows \{\mathfrak{B};\mathbf{E}_1\}$, it must be stressed here that the result presented above is interesting only in the case, if $\mathbf{H}_1$ is really a weaker formula than $\mathfrak{B}$.

NOTES

4. It is clear that we must discern formulas $\mathfrak{A}_n$ from the following theorem $\mathfrak{G}$: *For a given cardinal number* $\mathfrak{m}$ *let us denote by* $G(\mathfrak{m})$ *the proposition that there exists no cardinal number* $\mathfrak{n}$ *such that* $\mathfrak{m} < \mathfrak{n} < 2^{\mathfrak{m}}$. *Then*

from the formulas $G(\mathfrak{m})$, $G(2^{\mathfrak{m}})$ *and* $G(2^{2^{\mathfrak{m}}})$ *it follows that cardinal numbers* $\mathfrak{m}$, $2^{\mathfrak{m}}$ *and* $2^{2^{\mathfrak{m}}}$ *are alephs.* This theorem is announced without proof in [2], p. 314, theorem **89**, and the proof that $\mathfrak{G}$ implies that $\mathfrak{m}$ is an aleph is given in [3], p. 437. From the given there remark of Sierpiński that for any cardinal number $\mathfrak{m} > 1$, formula $G(\mathfrak{m})$ implies without the aid of $\mathfrak{B}$ that $\mathfrak{m} > \aleph_0$ it follows immediately that $2^{\mathfrak{m}}$ and $2^{2^{\mathfrak{m}}}$ are also alephs. It is easy to prove that in the theorem $\mathfrak{G}$ the definition of $G(\mathfrak{m})$ can be substituted by: *For a given cardinal number* $\mathfrak{m}$, *if a cardinal number* $\mathfrak{n}$ *is such that* $\mathfrak{n} < 2^{\mathfrak{m}}$, *then* $\mathfrak{n} \leqslant \mathfrak{m}$.

5. Concerning the Hartogs' alephs and their properties used in this paper, especially in the given below points *(26)*, *(46)*, *(47)*, *(48)*, *(68)*, *(69)* and *(70)*, *cf.* [8], [2], pp. 310-311, [3], pp. 407-409 and 413, [1], pp. 220-221, and [10], pp. 28-29.

6. Concerning this theorem and the given below theorem **T2**, *cf.* [2], theorems **6** and **56**, [5], [9], [3], pp. 161 and 168-170, and [1], p. 118.

7. *Cf.* [7], note 1.

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To be continued

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