CONDITIONALLY-RESTRICTED OPERATIONS

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In a recent article[1] A. N. Prior discusses a number of examples in which logic of a strictly formal kind imposes unexpected restrictions on what people can say, fear, etc. in certain circumstances. His conclusion is that "we must just let Logic teach us where these blockages will be encountered." Against this I shall argue that Prior's "Logic" is tantamount to working within the paradoxes, not resolving them, that a further explanation is needed and can be found, and that this explanation leads to a broader account than this "Logic" provides of paradox and emptiness, within which these blockages need not be unexpected.

One of the puzzles is this. It can be demonstrated that if Epimenides the Cretan says that all Cretan assertions are false, then some Cretan assertion is true. But it cannot be this one, and so if a Cretan says this there must be some other Cretan assertion. But this is odd, for how can a Cretan's saying of this logically require the occurrence of a different event? Prior's answer is that the logic itself shows that unless some other (and true) assertion is made by some Cretan, Epimenides just cannot say what he is supposed to say: he can utter the words, of course, but he cannot succeed in making an assertion.

However, this is still puzzling. Just how are his words prevented from constituting an assertion? This is even more puzzling when we give certain other interpretations to the operator d in Prior's formula, such as Geach's "It is feared by a schizophrenic that," for it then means that unless something else is feared by a schizophrenic it cannot be feared by a schizophrenic that nothing feared by a schizophrenic is the case. How is he prevented from fearing this? Clearly some further explanation is needed, and and indeed a further explanation can be given as follows:

Let us call what Epimenides is supposed to say S, and let us call all other Cretan assertions C's. Then (putting **T** for "true" and **F** for "false") the truth and falsity conditions for S are:

S is **T** *iff* {(S is **F**) . (All C are **F**)}
S is **F** *iff* {(S is **T**).v.(Some C is **T**)}

Now if there are no other Cretan assertions, these conditions become:

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S is **T** *iff* S is **F**
S is **F** *iff* S is **T**

Also if there are other Cretan assertions, but they are all false, then "All C are **F**" is **T**, and "Some C is **T**" is **F**, and so by the truth-functional rules $(p\,.\,\mathbf{T}) \equiv p$ and $(p \vee \mathbf{F}) \equiv p$ these conditions again become:

S is **T** *iff* S is **F**
S is **F** *iff* S is **T**

That is, in either of these cases S becomes an utterance which is paradoxical in the way in which the simple Liar (Eubulides version) utterance is. Thus we can sum up the position thus:

S is Liar-paradoxical if either there are no C's or all C are **F**, and transposing:

If S is not Liar-paradoxical then there are C's and some C is **T**. This explains the above-mentioned oddity, for what it means is that if Epimenides's utterance is not to be paradoxical in the way in which the simple Liar one is, there must be some other (true) Cretan assertion: it is not his saying of this that logically requires this different event, but his being able to say it non-paradoxically. And exactly the same treatment can be applied to any other interpretation of d. It is not the schizophrenic's fearing that no schizophrenic fear is realised that requires that there should be some other schizophrenic fear: it is merely that this is required if the schizophrenic's fear that no schizophrenic fear is realised is not to be Liar-paradoxical. So if we can assume that we already know how utterances of the simple Liar type come to be paradoxical and that we can resolve this paradox, the analysis given above is all that is needed to explain the further puzzles in the Epimenides version and in all other interpretations of Prior's *T16*.

The policeman-prisoner example which Prior quotes (p. 20) from L. J. Cohen generates a similar puzzle, for we can prove that if both the policeman and the prisoner say what they are supposed to say then necessarily one of them says something else, or, transposing, if neither of them says anything else then *either* the policeman *or* the prisoner does not say what he is supposed to say. This disjunctive consequent is an additional oddity: logic seems to block *either* the policeman's saying *or* the prisoner's, but it doesn't say which is precluded. This puzzle can be resolved in a corresponding way.

Let us call what the policeman is supposed to say S_1, and what the prisoner is supposed to say S_2, let us call other policeman-assertions P's, and other prisoner-assertions Q's. Then the truth and falsity conditions of S_1 and S_2 are:

S_1 is **T** *iff* $\{(S_2$ is **F**$)$. (All Q are **F**$)\}$
S_1 is **F** *iff* $\{(S_2$ is **T**$)$.v.(Some Q is **T**$)\}$
S_2 is **T** *iff* $\{(S_1$ is **T**$)$.v.(Some P is **T**$)\}$
S_2 is **F** *iff* $\{(S_1$ is **F**$)$. (All P are **F**$)\}$

Now if both {either there are no Q's or all Q are **F**} and {either there are no P's or all P are **F**} then these conditions become:

S_1 is **T** *iff* S_2 is **F**
S_1 is **F** *iff* S_2 is **T**
S_2 is **T** *iff* S_1 is **T**
S_2 is **F** *iff* S_1 is **F**

and hence:

S_1 is **T** *iff* S_1 is **F**
S_2 is **T** *iff* S_2 is **F**

Thus if both {either there are no Q's or all Q are **F**} and {either there are no P's or all P are **F**} then both S_1 and S_2 are Liar-paradoxical (rather like the statements on two sides of a sheet of paper, "The statement on the other side is false" and "The statement on the other side is true"). So, transposing:

If neither S_1 nor S_2 is Liar-paradoxical, then either {there are Q's and some Q is **T**} or {there are P's and some P is **T**}.

In the former case S_1 is **F** and in the latter case S_2 is **T**.

So as before it is not the policeman's and the prisoner's saying of what they are supposed to say that requires that at least one of them should say something else, but their being able to say these things non-paradoxically. Briefly, what we have shown is this:

If neither says anything else then (if both utter what they are supposed to then both utterances are paradoxical). Hence, transposing in the consequent:

If neither says anything else then (if either utterance is non-paradoxical then one or other does not say what he is supposed to).

This formulation shows why "Logic" blocks *either* the policeman's saying *or* the prisoner's, but does not say which is precluded.

The puzzle which Prior borrows from Buridan (pp. 20-21) can be resolved in a similar way, except that the utterances are no longer equally exposed to the risk of paradox. If A says a truth, B says a truth, and C says a falsehood, then if D says that exactly as many truths as falsehoods are uttered on this occasion then D's statement, if it is true, is false, and if it is false, is true, and so in the circumstances it is a variant of the simple Liar utterance: in the circumstances it is a way of saying that what D is thus saying is false. It is not that D cannot say that as many truths as falsehoods are uttered, but that he cannot say it non-paradoxically. In this example we have the conclusion:

If D's utterance is non-paradoxical, then one of the four does not say what he is supposed to say.

Thus Prior is clearly wrong when he comments, "D's saying what is attributed to him is not *more* blocked, as far as this logic goes, by the sayings of A, B, and C than *their* sayings are blocked by what D is supposed to say; and if you hear all these four people together and then ask yourself 'Which of them is it who hasn't really said anything?' there is no more reason for answering '*D*' than there is for answering '*A*', '*B*', or '*C*'." What is symmetrical is that if paradox is to be avoided one–any one–of the four must not say what he is supposed to, but there is an asymmetry in that if they do say this then it is *D*'s utterance, and no-one else's, that is paradoxical. Prior confesses to a feeling that he would like some favouritism here, but he cannot justify it because he confines himself to what "this fragment of logic has to tell us," that is, because in this case as in the others he is working within the paradox, accepting the puzzle set up by the formal proof in each case, and not giving any further explanation of it.

A similar account applies to Mr *X* in Room 7 (pp. 30-32). Mr *X* can, as common sense tells us, reflect at 6 o'clock that nothing that is being thought by anyone in Room 7 at 6 o'clock is the case, but if he is himself (unwittingly) the sole occupant of Room 7 at 6 o'clock then this reflection is paradoxical: its being true would make it false, and *vice versa*.

Prior rejects this kind of explanation[2] because he claims that his formal proof shows that Mr *X* cannot think at all (paradoxically or otherwise) what he is supposed to think. But this is not so. Although his proof appears to use only very innocent and unquestionable logical laws, his logical system implicitly contains the rule that no Liar-paradoxical operation occurs, and it is on this that his proof depends.

I shall first try to show how his proof depends on this. At step *T8*, if we interpret *d* as "It is said by a Cretan that," it has been shown that if it is said by a Cretan that all Cretan assertions are false then there is a true Cretan assertion and there is a false Cretan assertion. From this the conclusion *T16*, that if it is said by a Cretan that all Cretan assertions are false there are at least two Cretan assertions, follows only if we assume that it is not the same assertion that is both true and false. But a Liar-paradoxical assertion is precisely one of which we are compelled to say that it is both true and false. Thus *T16* follows from *T8* only on the assumption that no Cretan assertion is Liar-paradoxical.

This can be brought out more clearly if we translate the earlier steps into a less formal style, letting *C* stand here for "said by a Cretan."

T1. If all *C* are **F**, then if it is *C* that all *C* are **F** then not (all *C* are **F**).

T2. If it is *C* that all *C* are **F**, then if all *C* are **F** then not (all *C* are **F**). - from *T1*

T3. If it is *C* that all *C* are **F**, then not (all *C* are **F**). - from *T2*

T4. If it is *C* that all *C* are **F**, then some *C* are **T**. - from *T3*

T5. If it is *C* that all *C* are **F**, then both (it is *C* that all *C* are **F**) and not (all *C* are **F**). - from *T3*

T6. If both (it is *C* that all *C* are **F**) and not (all *C* are **F**) then some *C* are **F**.

T7. If it is *C* that all *C* are **F**, then some *C* are **F**. - from *T5* and *T6*

Looking over these steps we see that if there were no other *C* this proof would show that the *C* that all *C* are **F** is true (at *T4*, because of *T1*) and that this same *C* is false (at *T7*, because of *T6*). It is only because the logical system does not allow this, but at the same time treats the *C* that all C are **F** as checkable for truth and falsity in the usual way, that "Logic" tells us that if it is C that all C are **F** there must be some other C.

Next I shall try to show how Prior's logical system contains the rule that no Liar-paradoxical operation occurs.

An utterance is Liar-paradoxical if and only if it is such that if we take it as asserting what it appears to, it works out as true if it is false and as false if it is true. Similarly, an apparent command is Liar-paradoxical if and only if it is such that if we take it as a genuine command it works out as obeyed if it is not obeyed and as not obeyed it it is obeyed. And so on. Thus if an operation dp is Liar-paradoxical it is such that if we take it as a genuine operation on p then both (if not p, p) and (if p, not p).

Now the working of Prior's logical system takes dp as a genuine operation on p. (Interpretations of $T1$ and $T6$, as above, constitute ordinary checks of the truth and falsity of what is asserted by reference to its content.) Also the system contains the law "If dp implies that both (if not p, p) and (if p, not p), then not dp." (For it contains the Law of Contradiction $NKpNp$ and hence the law $C(CdpKpNp)Ndp$ and hence the law $C(CdpKCCNppCpNp)Ndp$).

Putting these together, we see that if an operation dp is Liar-paradoxical the working of Prior's system will treat it as a genuine operation; dp will then imply both (if not p, p) and (if p, not p); and the above-stated law will then require that not dp.

Let us look at this in another way. Suppose that, with regard to the simple Liar, a formal proof could be given along the following lines.[3] If someone says that what he is now saying is false, then if what he is saying is false, then it is true—and hence if it is not true it is true, and hence it *is* true—and also if what he is saying is true, then it is false—and hence if it is not false it is false, and hence it *is* false. That is, if anyone says this, it is both true and false. But no statement is both true and false. Therefore no-one does say this. Now a formal proof along these lines would be parallel with the one that Prior gives. Clearly, it would be all right as far as it goes: it would show that we cannot admit that anyone makes a genuine statement that what he is now saying is false. But to show this is merely to present the paradox, not to resolve it. To resolve the paradox we have to go further and explain why a man cannot make this statement, why his uttering of what seem to be the appropriate noises fails to achieve this, and not merely let "Logic" teach us in this way that he cannot.

From this point of view I would consider puzzles of the opposite, Truth-teller, kind. If someone says that what he is now saying is true, then this is not paradoxical in the sense that if we take it as a genuine assertion it will lead to the contradiction that some one thing is both true and false. Yet it is clearly empty in the same way that the simple Liar utterance is. Since it generates no contradiction, there will be no proof with regard to this parallel to the one sketched above for the Liar, and in general there will be no proofs within Prior's "Logic" that statements of this sort are not made. And yet logic, in a wider sense, would reject these:

we can see that the man who utters a Truth-teller remark has failed to make any genuine statement.

Similarly, if a Cretan says that all Cretan assertions are true, then, if we call this S and all other Cretan assertions C's, the truth and falsity conditions for S are:

S is **T** *iff* (S is **T**) . (All C are **T**)
S is **F** *iff* (S is **F**).v.(Some C is **F**)

But then as in the Epimenides above, if either there are no other Cretan assertions or they are all true, these conditions become:

S is **T** *iff* S is **T**
S is **F** *iff* S is **F**

These involve no contradiction but they show that S in these circumstances is undecidable and empty. I maintain, therefore, that if either there are no other Cretan assertions or they are all true, a Cretan cannot say non-emptily that all Cretan assertions are true; and hence that if emptiness is to be avoided then if a Cretan is to say that all Cretan assertions are true there must be some other, false, Cretan assertion. But since what is to be avoided here is only emptiness, not contradiction, "Logic" does now show that a Cretan cannot say this in these circumstances: there is no law within the system of which the rejection of such a saying is an instance.[4] Emptiness can arise similarly in appropriate variants of the other paradoxes. Thus if A says a truth, and B and C say falsehoods, then if D says that exactly as many truths as falsehoods are uttered on this occasion what D says is true if it is true and false if it is false, and there is no further way of deciding which it is. Hence if A, B and C say what they are supposed to say, D cannot say non-emptily what he is supposed to say. And if Mr X, alone in Room 7 at 6 o'clock, thinks that everything that is thought by anyone in Room 7 at 6 o'clock is the case, his thinking is again empty, though not paradoxical, and in these circumstances nothing can decide its truth or falsity. But again "Logic", that is, calculation within the system, will not reveal that there is anything wrong. To detect emptiness we need a wider sort of logic that pays attention to the method of deciding whether a given item is true or false (or obeyed or not obeyed, etc.) and sees whether this method is circular. This has a bearing on the paradoxical cases too; for one way of expressing the solution of a paradox of the Liar type is to show that the paradoxical item is empty and therefore that the contradiction that would be generated if we took it as a genuine operation does not matter, and similarly an item which is contingently paradoxical is also, because of its then crucial self-reference, contingently empty.

We can formulate some general principles that tell us when to expect emptiness and paradoxicality of these kinds. First, if an item is so constructed that its truth-value (or something analogous[5]) depends explicitly wholly upon itself, either positively or negatively, we have a simple Liar or Truth-teller puzzle, that is, emptiness and (in the negative case) paradox. Secondly, if an item is so constructed that its truth-value (etc.) depends explicitly partly upon itself, either positively or negatively (e.g., "Do nothing that I tell you," "Everything that I say is true") then this item is, we may say, partly vacuous and (in the negative case) paradoxical; this is concealed if the circumstances are contingently such that the truth-value is settled by something other than itself, but if they are such that it does finally depend on itself then this item is contingently empty and perhaps paradoxical. Thirdly, if an item is so constructed that in view of some contingent fact its truth-value (etc.) implicitly depends wholly upon itself, then this item is contingently empty and perhaps paradoxical (e.g. if James says that what the President is now saying is false, and James is in fact the President; Buridan's puzzle also belongs here). Fourthly, if an item is so constructed that in view of some contingent fact its truth-value (etc.) implicitly depends partly on itself, then this item is contingently partly empty and (in the negative case) paradoxical, and, as with the second class, further contingent facts may make it actually empty and perhaps paradoxical. (The Epimenides and Mr *X* in Room 7 belong here.) Fifthly, an item may be such that only a contingent fact keeps it out of the third or fourth class (e.g., if James says that what the President is now saying is false, and James is not the President, or if I say that all Cretan assertions are false, and I happen not to be a Cretan). Items in this fifth class are never vacuous or paradoxical, but they are only contingently not so.

These principles cover all the members of Prior's "family of paradoxes," most of which fall in the third and fourth classes. A particular group of those in the second and fourth classes is covered by four special rules which we can combine as follows:

$$\text{If it is } C \text{ that } \left\{\begin{array}{l} \text{all } C \text{ are } \mathbf{T} \\ \text{some } C \text{ is } \mathbf{T} \\ \text{all } C \text{ are } \mathbf{F} \\ \text{some } C \text{ is } \mathbf{F} \end{array}\right\} \text{ then unless } \left\{\begin{array}{l} \text{some other } C \text{ is } \mathbf{F} \\ \text{some other } C \text{ is } \mathbf{T} \\ \text{some other } C \text{ is } \mathbf{T} \\ \text{some other } C \text{ is } \mathbf{F} \end{array}\right\}$$

$$\text{this } C \text{ is } \left\{\begin{array}{l} \text{empty} \\ \text{empty} \\ \text{empty and paradoxical} \\ \text{empty and paradoxical} \end{array}\right\}$$

These rules hold for any suitable *C*.[6]

Items in these four classes are obviously empty or paradoxical in rather different ways. There is something blatantly wrong with an item in the first class, and, as Prior remarks (p. 30) no-one in his senses falls into this sort of trap. But "Do everything that I tell you" (which belongs to the second class) can be disobeyed and it can be partly obeyed. It is only at the last step, when all my other orders have been obeyed, that there ceases to be any way of either obeying or not obeying this one. With an item in the third class, the emptiness or paradoxicality is only extensional, not intensional: if James does not know that he is the President, he has no difficulty in saying that what the President is now saying is false, and meaning it; but since he is the President there is nothing independent that can make his statement true or false, and its own falsity would make it true, and *vice versa*. And items in the fourth class have features in common with those of both the second and third classes, so that Mr X in Room 7, whose thinking belongs in this fourth class, will, as common sense tells us, have no difficulty in thinking what he is supposed to think; the paradoxicality here is only extensional and doubly contingent, contingent both upon his being in Room 7 and upon there being no other, correct, thoughts occurring there at 6 o'clock.

In stating these five principles that warn us of places in which emptiness and paradoxicality may be found I am not proposing rules for excluding such items from a language. If we wanted an absolute, non-contingent, guarantee against emptiness and paradox of these kinds then we would need some kind of type-restriction that would prevent items of all five classes from even being considered, but for most purposes this would be far too sweeping: It would prevent us from considering operations that actually occur. Rather we should take these principles as indicating regions in which emptiness and paradox are to be expected, and are to be guarded against or understood as the occasion demands.

NOTES

1. "On a family of paradoxes," *Notre Dame Journal of Formal Logic*, Vol. II, No. 1, (1961), pp. 16-32.

2. We have discussed it in a correspondence to which he refers (p. 31).

3. It is difficult to give such a formal proof because of the difficulty of formulating the self-reference and justifying the logical moves which it permits without introducing the complication of quotation.

4. Prior's justification for calling such pseudo-laws "Mackie's formulae" (p. 23) is that I once rashly, in correspondence, hazarded the guess that there might be such laws.

5. That is, for a command, its being obeyed or not, for a fear, its being fulfilled or not, and so on.

6. Of these four rules, the last two correspond to rules that are provable within a system that excludes paradoxical operations, as Prior's does, but the first two do not. The third corresponds to Prior's *T16*, *CdUpCdpNpE(2+)pdp* and the fourth to one which he mentions as having been pointed out by Geach (p. 18) and which would be symbolised as *CdEpKdpNpE(2+)pdp*.

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