# A THEOREM ON MAXIMAL SETS

JOSEPH S. ULLIAN

We presuppose familiarity with [2]. Let $\mathfrak{A}$ be the class of recursively enumerable (r.e.) sets with infinite complements. A r.e. set $\mathbf{M}$ is *maximal* if $\mathbf{M} \in \mathfrak{A}$ and every r.e. superset of $\mathbf{M}$ which is in $\mathfrak{A}$ differs only finitely from $\mathbf{M}$. Existence of maximal sets is established in [1].

*Theorem*: For every maximal set $\mathbf{W}_z$, there is a set in $\mathfrak{A}$ every recursive permutation of which lacks at most finitely much of $\overline{\mathbf{W}}_z$

*Proof*: Otherwise, there is a maximal set $\mathbf{W}_z$ such that for every r.e. set $\mathbf{W}_x$,

$$\mathbf{W}_x \in \mathfrak{A} \rightleftarrows (\exists y)(\phi_y \text{ is a recursive permutation } \& \\ \overline{\mathbf{W}}_z - \phi_y(\mathbf{W}_x) \text{ is infinite}).$$

Since $\mathbf{W}_z$ is maximal, $\overline{\mathbf{W}}_z - \phi_y(\mathbf{W}_x)$ is infinite just in case $\phi_y(\mathbf{W}_x) - \mathbf{W}_z$ is finite. Thus

$$\mathbf{W}_x \in \mathfrak{A} \rightleftarrows (\exists y)(\phi_y \text{ is a recursive permutation } \& \\ \phi_y(\mathbf{W}_x) - \mathbf{W}_z \text{ is finite}). \tag{1}$$

Now it is established in [3] that $\{x \mid \phi_x \text{ is a recursive permutation}\} \equiv \overline{\mathbf{S}^{(2)}} \in \Pi_2$, and it is easily verified that $\{x \mid \phi_y(\mathbf{W}_x) - \mathbf{W}_z \text{ is finite}\} \in \Sigma_3$. But then $\{x \mid \text{r.h.s. of (1) holds}\} \in \Sigma_3$, while we know from [2] that $\{x \mid \mathbf{W}_x \in \mathfrak{A}\} \equiv \overline{\mathbf{S}^{(3)}} \notin \Sigma_3$.

This result may be of help in determining whether or not every set in $\mathfrak{A}$ has a maximal superset.

## REFERENCES

[1] R. M. Friedberg, Three theorems on recursive enumeration, *Journal of Symbolic Logic*, vol. 23 (1958), pp. 309-316.

*Received May 16, 1961*

[2] H. Rogers, Jr., Computing Degrees of Unsolvability, *Mathematische Annalen*, vol. 138 (1959), pp. 125-140.

[3] H. Rogers, Jr., *Theory of Recursive Functions and Effective Computability*, vol. 1, mimeographed by Massachusetts Institute of Technology.

*University of California*
*Berkeley, California*