ON THE FORMALIZATION OF TWO MODAL THESES

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I. In medieval times, when a flowering of modal logic was in progress among scholastic logicians, two modal theses were formulated, and accorded widespread acceptance:

($T1$) The "mere possibility" of a proposition cannot entail its factuality (*a posse ad esse non valet consequentia*).

($T2$) No "mere fact" or "merely contingent proposition" entails a necessary proposition (*a esse ad necesse non valet consequentia*).

It is the objective of the present paper to examine the issues raised in the questions: How are these theses to be articulated within the framework of modern formalizations of modal logic? What symbolic rendition is appropriate for them? What special assumptions, if any, are requisite if the appropriate formalized versions of these theses are to enjoy the status of acceptable modal principles?

II. For the purposes of the present discussion, we assume a symbolic system of modal logic based on the operators "$\Diamond$", "$\Box$", and "$\rightarrow$" (representing possibility, necessity, and strict implication, respectively). This modal system is assumed to be "normal" in the sense that at least the following definitions and laws obtain, in addition to *modus ponens* and a substitution:

($D1$) $\Box p =_{Df} \sim\Diamond\sim p$

($R1$) $\Box p \rightarrow p$

($R2$) $(p \,\&\, q) \rightarrow p$

($R3$) $\Diamond(p \,\&\, q) \rightarrow \Diamond p$

($R4$) $(p \rightarrow q) \rightarrow (\sim q \rightarrow \sim p)$

($R5$) $(p \rightarrow q) \rightarrow \sim\Diamond(p \,\&\, \sim q)$

Derivatively (in view of $D1$ and $R1$) we have:

($R6$) $p \rightarrow \Diamond p$

Received February 2, 1960

We will assume also that propositional quantification is possible in this system of modal logic, and is governed by rules of quantification regarding which no special assumptions need be made for our purposes, other than that the familiar duality between universal and existential quantification obtains.

III. We now can turn to an investigation of the first of our modal theses, *T1*, to the effect that the "mere possibility" of a proposition cannot entail its factuality. It would appear that the most straightforward symbolic rendition of *T1* is:

$(T1.1)\ \sim (\exists p)(\Diamond p \rightarrow p).$

Equivalently, we can reformulate this as,

(1) $(p) \sim (\Diamond p \rightarrow p).$

A literal linguistic interpretation of (1) might be: "There is no proposition whose possibility entails its factuality." Thus the foregoing symbolization appears to be adequate to its task.

It is important to observe, however, that the symbolic transcription of *T1* as *T1.1* can *not* be sustained in a modal system in which the converse of *R5* obtains, as happens when "$\rightarrow$" is not taken as primitive, but is defined by,

$(D3)\quad p \rightarrow q = Df \sim \Diamond (p \;\&\; \sim q).$

For if *D3* is accepted, then *T1.1* leads to the following consequences:

(2) $(p) \Diamond (\Diamond p \;\&\; \sim p)$ from (1), *D3*

(3) $(p) \Diamond \sim p$ from (2), *R3*

But it is clear that (3), or equivalently $\sim (\exists p) \Box p$ must be rejected, since it denies the existence of necessary propositions.

In a modal system in which *D3* is accepted as definition of the strict implication relationship "$\rightarrow$", a *different* mode of implication, in addition to "$\rightarrow$", is required to represent the mode of entailment at issue in the thesis *T1*. To assure that the paradoxical consequence (3) does not follow, we can use in symbolizing *T1* the mode of entailment obtained by modifying *D3* so as to eliminate from it the "degenerate" cases in which $\sim \Diamond (p \;\&\; \sim q)$ is true merely because $\sim \Diamond p$ or $\sim \Diamond \sim q$, i.e. when $p \rightarrow q$ obtains simply on the grounds that p is impossible (self-contradictory) or that q is necessary. Thus we define,

$(D4)\quad p \Rightarrow q = Df\ (p \rightarrow q) \;\&\; \Diamond p \;\&\; \Diamond \sim q.$

And we now reformulate *T1.1* as,

$(T1.2)\ \sim (\exists p)(\Diamond p \Rightarrow p).$

It is readily seen that—even if *D3* is accepted as definition of "$\rightarrow$"—this formulation of *T1* does not entail untenable consequences. For *T1.2* simply leads to the following chain of inferences:

(4)	$(p) \sim (\Diamond p \Rightarrow p)$	from $T1.2$
(5)	$(p) \sim ([\Diamond p \rightarrow p] \,\&\, \Diamond\Diamond p \,\&\, \Diamond \sim p)$	from (4), $D4$
(6)	$(p) ([\Diamond p \rightarrow p] \supset \sim [\Diamond\Diamond p \,\&\, \Diamond \sim p])$	from (5)
(7)	$(p) ([\Diamond p \rightarrow p] \supset [\sim \Diamond\Diamond p \vee \sim \Diamond \sim p])$	from (6)
(8)	$(p) ([\Diamond p \rightarrow p] \supset [\Box\Box \sim p \vee \Box p])$	from (7), $D1$
(9)	$(p) ([\Diamond p \rightarrow p] \supset [\Box \sim p \vee \Box p])$	from (8), $R1$

Thus (9) guarantees that $\Diamond p \rightarrow p$ – or $\sim \Diamond(\Diamond p \,\&\, \sim p)$, by $D3$ – is at any rate not possible for contingent propositions. This result, far from being unacceptable, seems perfectly natural.

The upshot of our discussion of the formalization of the thesis $T1$ is now readily summarized. Generally speaking, $T1.1$ provides an adequate formalization of this thesis. However, if "$\rightarrow$" is taken as defined by definition $D3$, then the implication relationship "$\rightarrow$" of $T1.1$ must be replaced by the non-degenerate implication relationship "$\Rightarrow$" as given in definition $D4$, with the result that $T1$ is formalized by $T1.2$.

IV. Let us now turn to the investigation of the second modal thesis, $T2$, to the effect that no "mere fact" or "merely contingent proposition" entails a necessary proposition. It would seem on first view that this thesis can be rendered in a direct and straightforward way as:

$(T2.1)$ $(p)\,(q)\,([(p \rightarrow q) \,\&\, \sim \Box p] \rightarrow \sim \Box q)$.

This could equivalently be re-cast as:

(1)	$(p)\,(q)\,([(q \rightarrow p) \,\&\, \Diamond p] \rightarrow \Diamond q)$	from $T2.1$, $D1$, $R4$

However, despite its apparent suitability, this formalization will not serve.

First of all, in the special case that "$\rightarrow$" is governed by the definition $D3$, $T2.1$ is actually self-contradictory. For it leads to the following line of reasoning: Assume q is self-contradictory, i.e. that $q \rightarrow \sim q$, and consequently (by $R3$ and $R5$) that $\sim \Diamond q$. But $q \rightarrow r$ obtains for *any* proposition r, in accordance with $D3$ and $R3$. Thus if we let r by any proposition p for which $\Diamond p$ is true, (1) yields $\Diamond q$. But this contradicts the foregoing.

This result shows that, if we wish to accept $D3$, we must reject $T2.1$. Further, it strongly suggests that we again reformulate $T2.1$ by use of the implication relationship "$\Rightarrow$" from which the "degenerate" entailments admitted by "$\rightarrow$" have been extruded. Consequently we would reformulate $T2.1$ as:

$(T2.2)$ $(p)\,(q)\,([(p \Rightarrow q) \,\&\, \sim \Box p] \rightarrow \sim \Box q)$.

But before we pursue this possibility any farther, we should stop to recognize that $T2.1$ is unacceptable as it stands, even without adoption of $D3$ as definition of "$\rightarrow$".

To establish the general unacceptability of $T2.1$, it suffices to observe that it leads to the following chain of consequences:

(2) $(p)\,(q)\,([([p \,\&\, q] \rightarrow p) \,\&\, \Diamond p] \rightarrow \Diamond [p \,\&\, q])$
from (1), letting "q" be "p & q"

(3) $(p)\,(q)\,(\Diamond p \rightarrow \Diamond [p \,\&\, q]$ from (2), $R2$

But (3) is patently unacceptable.

Having thus demonstrated that *T2.1* is *not* an acceptable formalization of *T2*, it remains to be investigated whether *T2.2* will serve. Now in view of *D4*, *T2.2* amounts to:

(4) $(p)\,(q)\,([p \rightarrow q] \,\&\, \Diamond p \,\&\, \Diamond \sim q \,\&\, \sim \Box p] \rightarrow \sim \Box q)$.

But since the antecedent of the principal implication of (4) is a conjunction containing $\Diamond \sim q$, or equivalently $\sim \Box q$ (by *R1*), (4) is an immediate consequence of *R2*. Thus (4), and equivalently *T2.2* is an inescapable assertion of the "normal" modal systems. It follows that *T2.2* is eminently qualified to serve as an acceptable formalization of the thesis *T2*.

V. In concluding, a brief word of retrospect will suffice. We have seen that the two venerable modal theses investigated in the present discussion can readily be assimilated within the framework of modern systems of modal logic in symbolic articulation. In each case, however, it appeared that a more natural accommodation of these theses requires a concept of strict implication from which the possibility of degenerate entailments has been eliminated in some such manner as that represented by the modified strict implication relationship given in definition *D4*.

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