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SYLLOGISTIC WITHOUT EXISTENCE[1]

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Modern logic has been credited with exposing the so-called "existence" assumptions implicit in Aristotelian logic and in many traditional brands of syllogistic. There are, to be sure, dissenting voices. Bocheński, for example, has claimed ([11] p. 425) that the "existence" assumptions are needed only for certain interpretations of syllogistic, interpretations that are by no means the most appropriate. However, the alternative which Bocheński has in mind is Łukasiewicz's axiomatization, a variant of which Bocheński himself has put forth [12]. Now, to refer to Łukasiewicz's system is to beg the question of "existential" import, since that system itself has need of interpretation in terms of basic logical notions. And in fact, one workable interpretation of Łukasiewicz's system would analyze his second axiom, '*Iaa*', as '$\exists x(Ax \,\&\, Ax)$'. In order, then, to make good the claim that Łukasiewicz's axiomatization of syllogistic avoids "existence" assumptions, we must show that on some alternative interpretation the assumptions are indeed avoided. But to give such an interpretation of Łukasiewicz's system is to give an "existence"-free interpretation of syllogistic itself. The latter is what I shall do in this paper.

By 'syllogistic' I shall henceforth mean, when no further qualification is added, Aristotle's theory of valid syllogisms for assertoric categorical statements without negative terms. The interpretation offered is based on a structural analysis of categorical statements which incorporates two features, quantification of the predicate and a binding operator 'such that', somewhat like that of Hailperin [19], which forms common nouns from predicates. On this interpretation, the extension of Aristotle's fourteen valid syllogisms to the traditional 24 is straightforward. On the other hand, no such simple extension is possible to the full traditional theory of negative terms, including obversion and contraposition, unless the system is considerably strengthened.

I. Informal introduction. 1.1 Various closely related kinds of sentences in everyday language are understood by Aristotle and others to express categorical propositions; e.g., 'All men are animalian', 'Everything human is an animal', 'Every man is an animal', 'All men are animals'.

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Which of these are equivalent and how each is best analyzed are by no means the matter-of-course questions they are often thought to be. But Aristotle's decision to treat them all the same may be taken as a decision to regard the form most perspicuously exhibited by one of them as standard for them all. The form which, as we shall see, best fits Aristotle's syllogistic is that exhibited by such statements as

Every man is an animal.
Some dog is a biter.
No Earthling is a Martian.

In these statements, both the subject and the predicate terms are grammatically common nouns. It has been pointed out (e.g. in [13] n. 6, [16] p. 68) that common nouns in natural languages correspond most closely to *variables* in formalized languages.[2] Thus 'man' may be thought of as a variable ranging over men or, somewhat less informally, over the class[3] λx(human x).[4] And indeed, in the above examples the subject terms are fairly obviously variables bound by the initial quantifiers. A first step toward analysis, then, gives us

$\forall$ man (man is an animal)
$\exists$ dog (dog is a biter)
$\bar{\exists}$ Earthling (Earthling is a Martian).

1.2 But what about the common nouns in the predicate? They appear modified by the indefinite article. Now, when the indefinite article accompanies the subject, we naturally translate by means of the particular quantifier:

An animal has been caught.
$\exists$ animal (animal has been caught).

The same procedure is in order when the indefinite article modifies the predicate term. However, considerations of scope require us to place the quantifier farther away from the variable bound:

$\forall$ man $\exists$ animal (man is animal)
$\exists$ dog $\exists$ biter (dog is biter)
$\bar{\exists}$ Earthling $\exists$ Martian (Earthling is Martian)

At this point it becomes clear that the 'is' in these statements is not a copula at all, as so many logic textbooks would have us believe, but rather the sign of identity (cf. [20] p. 510, 2°):

$\forall$ man $\exists$ animal (man = animal)
etc.

For mechanical convenience, I shall take not these but the trivially equivalent forms exemplified by

$\forall$ man $\exists$ animal (animal = man).
$\exists$ dog $\exists$ biter (biter = dog)
$\forall$ Earthling $\forall$ Martian (Martian $\neq$ Earthling)

as standard for **A**, **I**, and **E** categoricals respectively. **O**, of course, looks like this:

$\exists$ woman $\forall$ wife (wife $\neq$ woman).

1.3 Thus an analysis of one traditional ordinary-language form for categoricals leads directly to what has been called "quantification of the predicate". This appellation was meant to distinguish the above kinds of statements from ones like

$\forall$ man (animalian man)

in which only the subject term is quantified. Quantification of the predicate in analyzing categoricals was first advocated in modern times by George Bentham [10] in 1827. The technique was rediscovered, apparently independently, by William Hamilton ([20] pp. 509-559) before 1840. Among the precursors cited by Hamilton, the commentator Ammonius Hermiae of the fifth century A.D. expressly noted the close connection between statements with and those without quantified predicates ([2] pp. 101-8). But taking this connection to be one of equivalence, Ammonius concluded that quantification of the predicate is a superfluous embellishment ([2] p. 106, lines 26-36; tr. [20] p. 550).[5] Recently Timothy Smiley ([26] p. 69) and William T. Parry [24] have called attention anew to the interest which attaches to this treatment of categoricals. As Hamilton and Parry point out, for example, the forms with quantified predicates permit a neat formulation of the traditional doctrine of distribution: 'distributed' $=_{df}$ 'universally quantified'. More important, given elementary properties of quantification and identity, it is easily seen that categoricals thus analyzed satisfy all the laws of the square of opposition without the addition of any "existence" premisses. In particular, subalternation simply becomes a case of the law that a particular quantification follows from a universal quantification.

1.4 These advantages of common-noun analyses of categoricals are exploited by Timothy Smiley [26], who interprets syllogistic by means of many-sorted logic. Smiley takes sortal variables corresponding to our 'man', 'animal', etc., as primitive. He also introduces predicates for each sort, such as 'human', 'animalian'. He then is able to show that whether or not categoricals are interpreted with quantified predicates, Łukasiewicz's system of syllogistic is contained within his system. However, Smiley has not avoided the "existence" assumption, but has kicked it upstairs into the semantics of his system by stipulating a non-empty range for each sortal variable. In a suitably enriched version of his system ([26] p. 68), this assumption finds explicit reflection in such object-language theorems as '$\exists x$(human x)', and so on for each sortal predicate. The peculiar twist which enables us to get around this assumption is achieved by carrying the analysis one step further. Instead of positing primitive sortal variables, I take the corresponding predicates as basic and construct from them variables of restricted range as composite expressions.

1.5 This technique of forming restricted variables was first worked out by Theodore Hailperin [19]. The system introduced below differs in important respects from those of Hailperin (cf. §**2.8**), but much of its versatility rests on the notation which Hailperin invented. By prefixing the operator[6] 'ξx' (Hailperin uses 'νx') to a propositional form 'Fx', we obtain the *restricted variable* '$\xi x Fx$' which, with an important reservation to be explained later on (§**3.3**), may be thought of as ranging over $\lambda x Fx$. Thus, for example, 'ξx(canine x)' has the same role as the common noun 'dog'. It is not required that 'x' occur free in the operand, which thus might be a constant rather than a form. If the operand contains other free variables besides 'x', as e.g. in 'Fxy', it is apparent that '$\xi x Fxy$' is going to be a strange kind of variable: we might call it an "open variable" or a "variable-form". If we need further variables concurrent with '$\xi x Fx$', we may use '$\xi y Fy$', '$\xi z Fz$', etc. It is thus seen that relettering cannot in general be permitted for the ξ-operator. Hailperin prohibits in his formation rules the use of restricted variables as ν-operator variables. No such limitation is imposed here. Thus from the restricted variable 'ξx(canine x)' we can form the still more restricted variable '$\xi \xi x$(canine x) (brown ξx(canine x))', i.e., 'brown dog'. In fact, many syllogistic terms are nested in this way.[7]

1.6 By means of the ξ-formalism, then, common nouns are analyzed into composite terms containing, apart from improper symbols and unrestricted variables, either primitive predicates or complex predicative contexts. I would not want to try to justify this step in the analysis by recourse to ordinary language. Ordinary language contains both a range-restricting mechanism and a great many primitive common nouns. It would be easy enough to set up our formal system so as to reflect this situation. But by assuming only one primitive sort of common nouns, the *summum-genus* individual variables, we effect a formal simplification in the total theory. It may be suspected that this ploy amounts to reparsing ordinary common nouns in much the same way as Quine reparses ordinary proper names. But Quine reparses proper names precisely so that they will not behave as logically primitive proper names would. Our reparsed common nouns, on the other hand, receive the same syntactical treatment as the primitive sortal variables of many-sorted logic.

1.7 It is interesting to note that the natural languages contain equivalents of 'ξ'. Its most literal English translation is 'such', as in '*such* men as are mortal'—'ξ men (men are mortal)'. This use of 'such' is idiomatically restricted to certain contexts, however. Of more general application is 'such that' used between the operator variable and the operand: 'man *such that* he is mortal'.[8] 'Asher' in literary Hebrew (as Geach points out, [18] p. 120) and Yiddish 'vos' in one of its uses are one-word equivalents of 'such that': 'mentsh, *vos* er iz shterblekh'. In simple contexts, natural languages collapse the operator variable of a quantifier and the variable bound into one common noun: '$\forall$ man (man is mortal) '—' Every man is mortal'. The same is true for another translation of 'ξ': 'man *who* is mortal'. 'ξ' thus provides a fitting formalization of essential relative clauses. In consideration of this correspondence to qualifying pronouns and conjunctions in the natural languages, it seems appropriate to call 'ξx' a *qualification* operator or *qualifier*.[9] The most common expression of all for qualification in the natural languages, though it only works in simple contexts, is the qualifying adjective: '*mortal* man'. We begin to see how much more closely the ξ-formalism enables us to parallel the logical structure of ordinary language than do the customary functional calculi.

1.8 Having accustomed ourselves to restricted variables, it is natural next to introduce *restricted constants* as substituends for the various restricted variables. A primitive constant of our formalized language, say 'a', will of course be substitutable for an unrestricted variable 'x', but not in general for a restricted variable, say '$\xi x(\text{canine}\, x)$'. For the latter we are evidently only warranted in substituting proper names of dogs, such as 'Fido'. In order to construct corresponding proper names for our formalized language, we may use a constant-forming operator[10] 'Ξx' which, like 'ξx', is prefixed to propositional forms or sentences. Thus, if 'F' in our formalism translates 'canine', we might use 'ΞxFx' to denote Fido, 'ΞyFy' for Lassie, 'ΞzFz' for Argos, etc. These are all substituends for the variables 'ξxFx', 'ξyFy', ... As with 'ξ', relettering is obviously not permissible. In formalized languages we are accustomed to proper names which are bare labels, unrestricted individual constants. But in ordinary language it seems that restricted constants such as dog constants and girl constants and river constants are much more common than bare labels.

1.9 After this informal explanation of the present approach and its motivations, we are ready to proceed to an axiomatization of qualification. We could start with the two-valued propositional calculus as our base logic. However, the method introduced here is of special interest for interpreting syllogistic in non-classical logic systems which involve a more restrictive notion of implication than the truth-functional one. For besides the usual difficulties of interpreting syllogistic, intensional logics pose the problem of whether to render categorical statements with intensional or with extensional sentence connectives. In the present interpretation, as we have seen, this problem is solved by using *no* sentence connectives to analyze categoricals. The treatment is then easily carried over to the two-valued calculus, for whatever holds in a more restrictive system holds *a fortiori* in two-valued logic.

II. Formal development. The calculus given below is based upon *relevance implication*, as we may call[11] the implication relation formalized almost simultaneously by Moh Shaw-Kwei [23], and then by Church in his weak positive implicational propositional calculus [14, 15]. Although relevance implication has much to recommend it philosophically, I shall not enter into a discussion of its merits here. To the system R of relevance implication with truth functions are added a theory of qualification and a theory of identity.

The system $\mathbf{RXI}_1$

2.1 *Primitives*

Proper: $x\ \ y\ \ z\ \ \ldots$ (unrestricted individual variables)

$a\ \ b\ \ c\ \ \ldots$ (unrestricted individual constants)

$F^n\ G^n\ H^n\ \ldots$ where n is a finite positive integer (n-ary functional constants)

Improper: $\rightarrow$ & - = ξ $\dot{\Xi}$ ( )

2.2 *Formation rules.* 'Γ' and 'Δ' range over expressions of the object language. Improper symbols and juxtaposition are used autonymically.

1. *If* Γ *is an n-ary functional constant and* Δ *a series of n individual variables or constants, then* $\Gamma\Delta$ *is wf.*
2. *If* Γ *is an individual variable and* Δ *a wff, then* $\xi\Gamma\Delta$ *is a restricted individual variable and* $\Xi\Gamma\Delta$ *a restricted individual constant.*
3. *If* Γ *and* Δ *are individual variables or constants, then* $\Gamma = \Delta$ *is wf.*
4. *If* Γ *and* Δ *are wf, so are* $-\Gamma$, $(\Gamma \,\&\, \Delta)$, *and* $(\Gamma \rightarrow \Delta)$.

Individual variables and constants together are called *terms,* which are spoken of as restricted or unrestricted in the same way as variables and constants.

2.3 *Metadefinitions.* As further syntactical notation, let A, B, C, . . . range over wffs and x, y, z, . . . (without subscripts) over terms of the object language. u will occasionally be used as a metavariable ranging over unrestricted (or "universal") terms only. X and X' range over the two symbols ξ and Ξ.

An occurrence of a variable x in a term or wff Γ is a *bound* occurrence of x in Γ if it is a non-stuck occurrence in a term XxA which is a part of Γ. All occurrences of x in Γ which are neither bound nor stuck are *free* occurrences of x in Γ.

A non-bound occurrence in B of a term XxA, where y is free in A, is a *stuck* occurrence of XxA in X'yB; it is also a stuck occurrence in any term or wff of which X'yB is a part. (Unrestricted variables never get stuck.)[12]

An occurrence of a term is *encumbered* iff it is either bound or stuck. (An occurrence of a variable is thus unencumbered iff it is a free occurrence.)

All unencumbered occurrences of a variable x in A are, in XxA, occurrences which are *bound up* with each other and with the occurrence of x as operator variable of X.

Before proceeding, let us extend our syntactical notation to express results of substitution and replacement. Where the same formula variable 'A' occurs more than once in a given context but with different term expressions 'x', 'y' in the argument position, it is to be understood that A(x) is identical with A(y) except for having unencumbered occurrences of x just where A(y) has unencumbered occurrences of y. A(y/x) is the result of replacing *all* unencumbered occurrences of x in A or A(x) by unencumbered occurrences of y. A(y:x) is the result of replacing *any number* of unencumbered occurrences of x in A or A(x) by unencumbered occurrences of y, subject to *two restrictions.* (1) If a replaced occurrence of x is an occurrence in a restricted variable z, then the corresponding occurrence of x in any other occurrence of z which is *bound up* with the first occurrence is also to be replaced by y. (2) If a replaced occurrence of x is in a *free* occurrence of a restricted variable z, then the corresponding occurrence of x in any other free occurrence of z in A is likewise to be replaced by y. Where no ambiguity arises, the parentheses in such notations as the foregoing will be omitted (but not abbreviated by dots).

Alphabetic variant is defined recursively as follows. All unrestricted terms are alphabetic variants of one another. If x and y are variables which are alphabetic variants, then XxAx is an alphabetic variant of X'yAy. The notation x_1, x_2, ... will be used for alphabetic variants of x.

From the formation rules, it is evident that any term x will begin with zero or more occurrences of X's one after the other, followed by an occurrence of an unrestricted term u. Every term an occurrence of which begins with one of these occurrences of an X, as well as u itself, is said to be *buried* in x. Let the metavariables [x], [y], ... range over terms in which respectively x, y, ... are buried.

2.4 *Punctuation conventions*. Parentheses will be omitted under conventions similar to those of Church ([13] pp. 74ff.). It will suffice for the reader to know how to restore them:

1. Moving from left to right, replace successively each dot which occurs at the *left* side of a binary connective by a), putting its com-(at the beginning of the formula or of the parenthetical part in which the dot occurs.
2. Moving from right to left, replace successively each remaining dot by a (, putting its mate at the end of the formula or of the parenthetical part in which the dot occurs.
3. Taking account in the usual way of parentheses already present, restore parentheses for $\leftrightarrow$'s by association to the left, treating clusters of symbols not containing a $\leftrightarrow$ as units.
4. Repeat (3) successively for $\rightarrow$, $\vee$, and &.

The application of these rules may give rise to superfluous brackets, but that should cause no trouble. The connective '-' will often be placed over part or all of the negand.

2.5 *Definitions*

$$(A \leftrightarrow B) =_{df} A \rightarrow B.\&.B \rightarrow A$$
$$(A \vee B) =_{df} -.\bar{A} \,\&\, \bar{B}$$
$$x \neq y =_{df} -x = y$$
$$\exists xA =_{df} A \Xi xA/x$$
$$\forall xA =_{df} A \Xi x\bar{A}/x$$

2.6 *Rules of inference*

mp. $A, A \rightarrow B \vdash B$

adj. $A, B \vdash A \,\&\, B$ where A and B are either both theorems or both (deduced from) hypotheses.

2.7 *Axiom schemata*. Henceforth a syntactical formula is to be understood as standing for only those of the formulas which would otherwise be in its range which are wffs or parts of wffs. Thus, e.g., 'ξaFa' is not a value of 'XxA'.

Implication

R2. $A \rightarrow .A \rightarrow B \rightarrow B$

R3. $A \rightarrow B \rightarrow .B \rightarrow C \rightarrow .A \rightarrow C$
R4. $A \rightarrow .A \rightarrow B. \rightarrow .A \rightarrow B$

Conjunction

R6. $A \mathbin{\&} B \rightarrow B$
R7. $A \rightarrow B.\&.A \rightarrow C.\rightarrow.A \rightarrow B \mathbin{\&} C$

Relating conjunction and negation

R8. $A \mathbin{\&}. B \vee C.\rightarrow A \mathbin{\&} B \vee C$

Negation

R9. $\bar{A} \rightarrow \text{---}\bar{B} \rightarrow .B \rightarrow A$
R10. $\bar{A} \rightarrow A \rightarrow A$

Qualification

X1. $A[x_1]/x \rightarrow A[XxA]/x$
X2. $A[x_1]/\xi xB \mathbin{\&} B[x_1]/x \rightarrow \exists\xi xBA$

Identity

I1. $x = x$
I2. $x = y \rightarrow y = x$
I3. $A \mathbin{\&} x = y \rightarrow Ay{:}x$

2.8 *Historical notes.* Two axiom schemata needed for the system **R** in isolation have been omitted: $A \rightarrow A$ is an instance of X1, and $A \mathbin{\&} B \rightarrow A$ of X2. The implication axioms with $A \rightarrow A$ are those of Moh, [23] p. 63. The conjunction axioms with $A \mathbin{\&} B \rightarrow A$ are as modified by Ackermann ([1] p. 119) from the well-known system of Hilbert and Bernays. Ackermann was also forced to add a variant of R8. The negation axioms are modified from Belnap, [9] p. 1. The qualification axioms are closely related to those used by Hilbert for his ε-operator. X1 is simply a generalization of Hilbert's basic ε-formula, (in our notation) $Ax/u \supset A\varepsilon uA/u$ ([21] p. 13). Since X2 is inessential for characterizing our Ξ-operator, the formal properties of Ξ and ε are very close indeed. The crucial difference is that Hilbert allows relettering for ε. εxFx is thus the same F-thing as εyFy, whereas ΞxFx and ΞyFy are not necessarily identical. It is this close relation between ε and Ξ which enables us to take over Hilbert and Bernays's definitions of the quantifiers ([21] p. 15f.). And for quantifiers thus defined relettering *is* derivable. X1 may also be regarded as a weakened form of Smiley's A5 ([26] p. 59). X2 is essentially the left-to-right half of Hailperin QR4 ([19] p. 23), which corresponds to Smiley A4 (loc. cit). In comparing ξ with Hailperin's ν, we must first take into account a number of differences of detail between the two systems. Hailperin's systems are based on two-valued logic, and he prohibits stuck variables and restricted ν-operator variables. But allowing for these details, it could roughly be said that ξ is to the ν of Hailperin's system $\mathcal{Q}\mathcal{F}\nu$ ([19] pp. 114ff.) as ε is to Hilbert's η-operator ([21] p. 10f.; cf. [19] p. 124).

2.9 *Theorems.* If we were working with material implication, we would need only to deduce Łukasiewicz's axioms for syllogistic in order to show that it is contained in our system. But since **RXI**$_1$ is intensionally based, it will be necessary to show how to prove all the syllogisms. We recall that the four categorical forms are taken to be the following:

A.	$\forall y \exists x \; x = y$	**E**.	$\forall y \forall x \; x \neq y$
I.	$\exists y \exists x \; x = y$	**O**.	$\exists y \forall x \; x \neq y$

Instances of X1 in which x is not free in A (and hence [XxA] not unencumbered in A[XxA]/x) will be cited as "refl". Instances of X2 in which neither ξxB occurs unencumbered in A nor x in B will be cited as "simp".

Th 1. $A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$

By R3, mp twice. Hereafter cited as "chain".

Th 2. $A \& B \rightarrow B \& A$

By R7 from R6 and simp.

Th 3. *If* $\vdash A \rightarrow B$ *then* $\vdash A \& C \rightarrow B \& C$.

0. $A \rightarrow B$	given
1. $A \& C \rightarrow A$	simp
2. $A \& C \rightarrow B$	1, 0, chain
3. $A \& C \rightarrow C$	R6
4. $A \& C \rightarrow B \& C$	2, 3, R7

Forms of Th 3 with either the antecedent or the consequent of the conclusion commuted will also be used, on the authority of Th 2.

Th 4. *If* $\vdash A \rightarrow C$ *and* $\vdash B \rightarrow D$, *then* $\vdash A \& B \rightarrow C \& D$.

0. $A \rightarrow C$	$B \rightarrow D$	given
1. $A \& B \rightarrow C \& B$	$C \& B \rightarrow C \& D$	0, Th 3
2. $A \& B \rightarrow C \& D$		1, chain

Again, commutations will also be assumed.

Th 5. $A \rightarrow --A$

By R9 on $---A \rightarrow ---A$ (refl).

Th 6. $--A \rightarrow A$

1. $\bar{A} \rightarrow ---A$	Th 5
2. $\bar{A} \rightarrow -----A$	1, Th 5, chain
3. $--A \rightarrow A$	2, R9

It will be seen that with the aid of the double-negation theorems, the various forms of contraposition rules follow straightforwardly from R9. These will be used below under the name 'cpos'.

Th 7. *If* $\vdash A \& B \rightarrow C$ *then* $\vdash B \& A \rightarrow C$ (*mutatio praemissarum*). From the premiss by Th 2, chain.

Th 8. *If* $\vdash D \to A$ *and* $\vdash A \;\&\; B \to C$, *then* $\vdash D \;\&\; B \to C$ (replacement of major premiss).

0. $D \to A$	given
1. $A \;\&\; B \to C$	given
2. $D \;\&\; B \to A \;\&\; B$	0, Th 3
3. $D \;\&\; B \to C$	2, 1, chain

Replacement of minor premisses is a corollary by Th 2.

Th 9. $\exists y \exists x\; x = y \to \exists x \exists y\; y = x$, *where* x *is not free in* y *nor vice versa* (conversion of **I**).

1. $\Xi x\; x = \Xi y \Xi x\; x = y = y = \Xi y \Xi x\; x = y = y \to$ $\Xi y \Xi x\; x = y = y = \Xi x\; x = \Xi y \Xi x\; x = y = y$	I2
(*Let* A *abbreviate* $\Xi x\; x = y = y$ *and* B *abbreviate* $x = \Xi y A$.)	
2. $\exists y \exists x\; x = y \to \Xi y A = \Xi x B$	1, df
3. $\to \exists y\; y = \Xi x B$	2, X1, chain
4. $\to \exists x \exists y\; y = x$	3, X1, chain

The last two steps illustrate the use of X1 to give as a special case the principle of existential generalization.

Th 10. $\forall x A \to A[x_1]/x$ (universal instantiation).

1. $\bar{A}[x_1]/x \to \bar{A}\Xi x\bar{A}/x$	X1
2. $A\Xi x A/x \to A[x_1]/x$	1, cpos
3. $\forall x A \to A[x_1]/x$	2, df

Th 11. $\forall y \forall x\; x \neq y \to \forall x \forall y\; y \neq x$, *where* x *is not free in* y *nor vice versa* (conversion of **E**).

1. $\forall y \forall x\; x \neq y \to \forall x\; x \neq \Xi y\text{-}y \neq \Xi x \Xi y\text{-}y \neq x \neq x$	Th 10
(*Let* A *abbreviate* $\text{-}y \neq \Xi x \Xi y\text{-}y \neq x \neq x$.)	
2. $\forall y \forall x\; x \neq y \to \Xi x \Xi y\text{-}y \neq x \neq x \neq \Xi y A$	1, Th 10, chain
(*Let* B *abbreviate* $\Xi y\text{-}y \neq x \neq x$.)	
3. $\Xi y A = \Xi x B \to \Xi x B = \Xi y A$	I2
4. $\Xi x B \neq \Xi y A \to \Xi y A \neq \Xi x B$	3, cpos
5. $\forall y \forall x\; x \neq y \to \Xi y A \neq \Xi x B$	2, 4, chain
6. $\forall y \forall x\; x \neq y \to \forall x \forall y\; y \neq x$	5, df

Theorems 7-11 give us all that is needed to reduce syllogisms to the first figure except for subalternation and *reductio ad impossibile*. Subalternation will be seen to be an instance of X1. The simple conversions proved, together with subalternation, yield the two forms of conversion *per accidens*. *Reductio ad impossibile*, however, is not permissible in R. The principle on which it is based, $A \;\&\; B \to C \to .A \;\&\; \bar{C} \to \bar{B}$, would take us from $A \;\&\; \bar{B} \to A$ to $A \;\&\; \bar{A} \to B$, an implicational paradox. I now proceed to the first-figure syllogisms.

Th 12. $\forall y A y \;\&\; \forall z \exists y\; y = z \to \forall z A z$ (y not free in A nor z; z not free in A nor y).

1.	$\forall z \exists y\ y = z \rightarrow \exists y\ y = \Xi z \bar{A} z$	Th 10
2.	$\forall z \exists y\ y = z \rightarrow \Xi y\ y = \Xi z \bar{A} z = \Xi z \bar{A} z$	1, df
	(*Let* B *abbreviate* $y = \Xi z \bar{A} z$.)	
3.	$\forall y A y \rightarrow A \Xi y B$	Th 10
4.	$\forall y A y\ \&\ \forall z \exists y\ y = z \rightarrow A \Xi y B\ \&\ \Xi y B = \Xi z \bar{A} z$	3, 2, Th 3
5.	$\rightarrow A \Xi z \bar{A} z$	4, I3, chain
6.	$\rightarrow \forall z A z$	5, df

Putting $\exists x\ x = \ldots$ such that x is not free in y nor z for $A(\ldots)$ in Th 12 gives us Barbara; putting instead $\forall x\ x \neq \ldots$ gives Celarent.

Th 13. $\forall y A y\ \&\ \exists z \exists y\ y = z \rightarrow \exists z A z$ (same restrictions as for Th 12).

1.	$\forall y A y \rightarrow A \Xi y\ y = \Xi z \exists y\ y = z$	Th 10
	(*Let* B *abbreviate* $y = \Xi z \exists y\ y = z$.)	
2.	$\forall y A y\ \&\ \exists z \exists y\ y = z \rightarrow A \Xi y B\ \&\ \exists z \exists y\ y = z$	1, Th 3
3.	$\rightarrow A \Xi y B\ \&\ \Xi y B = \Xi z \exists y\ y = z$	2, df
4.	$\rightarrow A \Xi z \exists y\ y = z$	3, I3, chain
5.	$\rightarrow \exists z A z$	4, X1, chain

Putting $\exists x\ x = \ldots$ and $\forall x\ x \neq \ldots$ for $A(\ldots)$ in Th 13 yields respectively Darii and Ferio.

The usual procedures of reduction given by Aristotle and codified in the traditional mnemonic names may now be applied to all but two of the fourteen Aristotelian syllogisms. The same goes for the ten syllogisms added after Aristotle's time, including the entire fourth figure. The two exceptions are Baroco and Bocardo, in which the 'c' indicates that Aristotle proved them by *reductio ad impossibile*. Since this procedure is not open to us in $\mathbf{RXI}_1$, we must prove the two syllogisms independently.

Th 14. $\forall y \exists x\ x = y\ \&\ \exists z \forall x\ x \neq z \rightarrow \exists z \forall y\ y \neq z$ (Baroco; x, y, z not free in each other).

1.	$\forall x\ x \neq \Xi z \forall x\ x \neq z \rightarrow \Xi x\ x = \Xi y \text{-} y \neq \Xi z \forall x\ x \neq z \neq \Xi z \forall x\ x \neq z$	Th 10
	(*Let* A *abbreviate* $\forall x\ x \neq z$ *and* B *abbreviate* $x = \Xi y \text{-} y \neq \Xi z A$.)	
2.	$\exists z A \rightarrow \Xi x B \neq \Xi z A$	1, df
3.	$\forall y \exists x\ x = y \rightarrow \exists x\ x = \Xi y \text{-} y \neq \Xi z A$	Th 10
4.	$\forall y \exists x\ x = y \rightarrow \Xi x B = \Xi y \text{-} y \neq \Xi z A$	3, df
5.	$\forall y \exists x\ x = y\ \&\ \exists z \forall x\ x \neq z$ $\rightarrow \Xi x B \neq \Xi z A\ \&\ \Xi x B = \Xi y \text{-} y \neq \Xi z A$	4, 2, Th 4
6.	$\rightarrow \Xi y \text{-} y \neq \Xi z A \neq \Xi z A$	5, I3, chain
7.	$\rightarrow \forall y\ y \neq \Xi z A$	6, df
8.	$\rightarrow \exists z \forall y\ y \neq z$	7, X1, chain

Th 15. $\exists z \forall x\ x \neq z\ \&\ \forall z \exists y\ y = z \rightarrow \exists y \forall x\ x \neq y$ (Bocardo; same restrictions as for Th 14).

1.	$\forall z \exists y\ y = z \rightarrow \exists y\ y = \Xi z \forall x\ x \neq z$	Th 10
	(*Let* A *abbreviate* $\forall x\ x \neq z$ *and* B *abbreviate* $y = \Xi z A$.)	

2. $\forall z \exists y\ y = z \rightarrow \Xi yB = \Xi zA$		1, df
3. $\forall z \exists y\ y = z \rightarrow \Xi zA = \Xi yB$		2, I2, chain
4. $\exists z \forall x\ x \neq z\ \&\ \forall z \exists y\ y = z \rightarrow \exists z \forall x\ x \neq z\ \&\ \Xi zA = \Xi yB$		3, Th 3
5.	$\rightarrow \forall x\ x \neq \Xi zA\ \&\ \Xi zA = \Xi yB$	4, df
6.	$\rightarrow \forall x\ x \neq \Xi yB$	5, I3, chain
7.	$\rightarrow \exists y \forall x\ x \neq y$	6, X1, chain

III. Significance of the results. **3.1** *Principles used.* In order to prove the syllogisms, we have by no means used all of the axioms of $\mathbf{RXI}_1$. The reflexivity of implication was used only to get contraposition for use in Th 10 and 11. Of the implication axioms themselves, *only* R3 *was used.* Since R3 is needed only for chain, minimal transivity, $A \rightarrow B.\&.B \rightarrow C.\rightarrow.A \rightarrow C$, would do as well. A great many non-truth-functional implications have been proposed, but I know of none so "strict" as to lack transitivity. This means that our rendering of syllogistic will work for virtually any system of implication, provided one is willing (as many are not) to assume certain of our axioms for the other primitives.[14]

Of the conjunction axioms, all three were used, though not the one relating conjunction and negation (R8). Of the two negation axioms, only R9 was used, but in a form which together with $A \rightarrow A$ gives rise to the double negation equivalence.[15] The reflexivity of identity (I1) was not used.

Of the qualification axioms, X2 was not used, and X1 was used *only to yield properties of quantification.* The essential qualifier in X1 is the X, and in the cases of X1 we used, this was always Ξ. Ξ in turn was used only to build quantifiers. This means that we could have got the same results by using ordinary quantificational axioms and no axioms at all for ξ. This fact may afford some comfort to those who were repelled by the unintuitive definitions of the quantifiers.

3.2 *Existential import.* But what then is the point of introducing the ξ-apparatus? Why not simply put primitive sortal variables in place of our composite restricted variables? The answer is that we want to avoid "existence" assumptions; and as we saw above (§**1.4**), the many-sorted approach fails to do so. In order to gain a clear view of the differences among the various interpretations of syllogistic, we shall do well at this point to recall the ways in which a system may involve "existence" assumptions. For this purpose, it will be useful to speak of *syllogistic terms* and *syllogistic predicates.* Syllogistic predicates are predicates such as '*F*' and '*G*' in the categorical form 'Everything *F* is *G*'. Syllogistic terms are variables (common nouns) such as x and y in the categorical form *Every* x *is a* y. If a syllogistic term x has approximately the same meaning as 'thing that is *F*', then '*F*' will be called the *associated predicate* of x and x the *associated term* of '*F*'. We may then distinguish the following kinds of "existence" assumptions, where Φ is a syllogistic predicate and ϕ is its associated syllogistic term:

1. $\exists u \Phi u$ is a theorem of the system.
2. $\exists u \Phi u$ follows from a categorical in which Φ is a syllogistic predicate or ϕ is a syllogistic term.

3. A syllogism in which Φ is one of the syllogistic predicates (or ϕ one of the syllogistic terms) requires $\exists u \Phi u$ as an additional premiss.
4. It is semantically stipulated that Φ has application.
5. It is semantically stipulated that the range of ϕ is not empty.

In a more exhaustive enumeration, assumptions of the form $\exists u \bar{\Phi} u$ would also be taken into account. Under the currently widespread interpretation of syllogistic more or less after the manner of *Principia Mathematica*, we must assume either (1) or, more parsimoniously, (2) for **I** and **O** and (3) for some syllogisms involving **A** and **E**. Smiley's narrower system makes assumptions (5) and (4); and when he adds universal variables, (1) and certain forms of (2) also come in. The present interpretation of syllogistic assumes only (5), though in a rather peculiar way which I shall now explain.

3.3 *Semantics.* In Smiley's system, assumption (4) goes hand in hand with (5). We can now zero in on the difference between his system and ours by noticing that a restricted variable '$\xi x Fx$' does *not* stand in precisely the same relationship to F as one of Smiley's sortal variables a does to the class denoted by the associated predicate A. To pinpoint the difference: a ranges over the class denoted by A *and* that class is non-empty, whereas '$\xi x Fx$' ranges over F *iff* F is non-empty. What does '$\xi x Fx$' range over if F is empty? Over the complement of F, $\lambda x \bar{F} x$! Thus '$\xi x Fx$' is in either case semantically well-interpreted; but except when 'Fx' is provable or contradictory, the interpretation depends upon a matter of contingent fact. The meaning but not the meaningfulness of a restricted variable may be contingent on what happens to be the case. This peculiarity in the semantics of $\mathbf{RXI}_1$ guarantees a non-empty range for a restricted variable '$\xi x Fx$' whether or not $\exists x Fx$.[16] It is this semantical feature of $\mathbf{RXI}_1$, reflected in the axiom X1, which permits an "existence"-free interpretation of syllogistic in the system. The interpretation is free of "existence" assumptions in the following senses. For no predicative context Φ used in constructing a syllogistic term (or part thereof) $\xi x \Phi x$ need $\exists x \Phi x$ be assumed, whether as an axiom or as a premiss or as a semantical rule. Nor does $\exists x \Phi x$ follow from any categorical containing as (a part of) its subject or predicate term $\xi x \Phi x$.

3.4 *Comparison with Aristotle.* That Aristotle did admit terms the associated predicates of which have no application is clear from the following passage:

> . . . when in the adjunct [composing a subject term] there is some opposite which involves a contradiction, the predication of the simple term is impossible. Thus it is not right to call a dead man a man. When, however, this is not the case, it is not impossible. ([5] 21a21-23)

Our interpretation of syllogistic enables us to make some sense of this passage. Let $\xi x.A \;\&\; \bar{A}$, for short c, be a self-contradictory term. For this term we cannot prove both $\exists \xi x\bar{A}\ \xi x\bar{A} = c$ and $\exists \xi x\bar{A}\ \xi x\bar{A} = c$, since the only way to get these would be from $\vdash Ac/x \;\&\; \bar{A}c/x$ as follows:

0. $Ac/x \;\&\; \bar{A}c/x$		given
1. Ac/x	$\bar{A}c/x$	0, simp/R6

2. $c = c$		I1
3. $c = c \,\&\, Ac/x$	$c = c \,\&\, \bar{A}c/x$	2, 1, adj
4. $\exists\xi xA\ \xi xA = c$	$\exists\xi x\bar{A}\ \xi x\bar{A} = c$	3, X2

But of course the premiss is a contradiction. Nevertheless, we *can* prove $\exists\xi xA\ \xi xA = c \vee \exists\xi x\bar{A}\ \xi x\bar{A} = c$ from excluded middle, $Ac/x \vee \bar{A}c/x$. (I omit the proof here; it follows similar lines to the deduction just given, but depends on various standard properties of disjunction as well.) Thus we must disagree with Aristotle to this extent: *one* of the simple terms *is* predicable of the self-contradictory term. But this is just a consequence of excluded middle, which holds for all terms. Notice by the way that in our system the simple term ξxA may fail to be predicable of $\xi x.A \,\&\, B$ even though $A \,\&\, B$ is not contradictory, namely when, on whatever grounds, $\exists x.A \,\&\, B$ fails. Aristotle allows for this possibility in the sequel to the passage already quoted:

> Yet the facts of the case might rather be stated thus: when some such opposite elements are present, resolution is never possible, but when they are not present, resolution is nevertheless not always possible. (21a23-25)

Thus some of the peculiarities of $\mathbf{RXI}_1$ approximate Aristotle's own doctrines.[17]

IV. Negative terms. After Aristotle's time syllogistic was extended to include the possibility of negative terms in categoricals. In the extended theory, the equivalences of obversion and contraposition play an essential role. Aristotle himself considers negative terms in some detail and expressly recognizes certain modes of obversion ([5] 20a20-40; 51b41-52a8). In spite of this, he never develops a syllogistic theory of negative terms. Not even in the positive syllogistic, where obversion could cut the number of syllogisms to be dealt with in half (by subsuming Celarent under Barbara, etc.), does Aristotle avail himself of this tool. This reason seems to be that he regarded obversions as valid in only one direction, so that no equivalences arose.[18] But whatever the historical explanation may be for Aristotle's abstinence from obversion and contraposition, these "equivalences" take on a problematic character in our interpretation of syllogistic. In fact, they seem to fail entirely in $\mathbf{RXI}_1$. I can offer no proof of this conjecture, but I shall support it indirectly in the following way. I present two systems involving stronger (looser) theories of identity in which certain forms of obversion become provable. The proofs of obversion are then seen to depend essentially on principles not available in $\mathbf{RXI}_1$. At the conclusion of this part I present a still stronger system based on material implication.

4.1 *Term-negation.* The syllogistic presented so far has been highly general as to the sorts of terms admitted. For a treatment of negative terms, we must introduce some limitations. To begin with, there is no direct way to form the negatives of unrestricted variables, so these will have to be excluded as syllogistic terms. For restricted variables, we must distinguish two kinds of negatives, a *relative* and an *absolute* negative. For example, the negative of 'brown dog' *relative to the class of dogs* is 'non-brown dog', whereas the *absolute negative* would be 'non-(brown dog)'. For a term of the form $\xi u\bar{A}$ (u unrestricted), the two negatives coincide in $\xi u\bar{A}$. But for terms of the form $\xi\xi xAB$, only the relative negative $\xi\xi xA\bar{B}$ can be formulated in our notation. Since the theory of negative terms has traditionally taken term-negation in the *absolute* sense, the only variables we can construct corresponding to the terms of that theory are of the form ξuA. However, the theory can be extended to cover relative term-negation in certain circumstances. Consider, e.g., the statement

Every calico cat is a female cat.

If the absolute contraposition of this statement to

Every non-(female cat) is a non-(calico cat)

is a valid inference, then so too, it would seem, is the relative contraposition to

Every non-female cat is a non-calico cat.

On the other hand, to contrapose

Every calico cat is a female animal

to

Every non-female animal is a non-calico cat

is patently wrong. We notice that in the first case, both negations are relative to the class of cats, a condition not satisfied by the second example. We can generalize this condition by requiring that if a categorical with subject term ξyB and predicate term ξxA is to be obverted or contraposed, then x and y must be alphabetic variants. Then the negatives $\xi y\bar{B}$ and $\xi x\bar{A}$ will always be relative to the same class, the domain of x and y.

Let us use a, $\bar{a}$, b, $\bar{b}$ as abbreviations for ξxA, $\xi x\bar{A}$, ξx_1B, $\xi x_1\bar{B}$ respectively, where a is not free in B nor b in A. The obverses of the four categorical forms may then be stated as follows:

obverse of **A**: $\forall b \forall \bar{a}\ \bar{a} \neq b$ obverse of **E**: $\forall b \exists \bar{a}\ \bar{a} \neq b$
obverse of **I**: $\exists b \forall \bar{a}\ \bar{a} \neq b$ obverse of **O**: $\exists b \exists \bar{a}\ \bar{a} \neq b$

We need not treat contraposition specially, for it would be derivable from obversion and conversion. To contrapose **A**{**O**}, first obvert it to **E**{**I**}, apply simple conversion, and then obvert the result back to **A**{**O**} again.

4.2 *The system* $\mathbf{RXI_2}$. The first strengthened system we shall consider is just like $\mathbf{RXI_1}$ except that in place of I1 and I2 it has the following axiom schema for identity:

$$\text{I1'}.\quad A \rightarrow x = x$$

The next two theorems suffice to show that $\mathbf{RXI_2}$ contains $\mathbf{RXI_1}$:

$\mathrm{Th_2}16.$ $x = x$ (I1)

Let A in I1' be any theorem and detach.

$Th_2 17$. $x = y \rightarrow y = x$ (I2)

1. $x = y \rightarrow x = x$	I1'
2. $x = y \rightarrow x = y$	refl
3. $x = y \rightarrow x = x \;\&\; x = y$	1, 2, R7
4. $x = y \rightarrow y = x$	3, I3, chain

Since $\mathbf{RXI}_2$ contains $\mathbf{RXI}_1$, we have at our disposal all the theorems thus far proved for use in $\mathbf{RXI}_2$.

$Th_2 18$. $A[x_1]/x \rightarrow \exists a\; a = [x_1]$

1. $A[x_1]/x \rightarrow [x_1] = [x_1]$	I1'
2. $A[x_1]/x \rightarrow A[x_1]/x$	refl
3. $A[x_1]/x \rightarrow [x_1] = [x_1] \;\&\; A[x_1]/x$	1, 2, R7
4. $A[x_1]/x \rightarrow \exists a\; a = [x_1]$	3, X2, chain

The intuitive import of $Th_2 18$ is best seen in such instances as 'runs $x \rightarrow \exists \xi y(\text{runs } y)\; \xi y(\text{runs } y) = x$', i.e. 'If x runs, then x is a runner.' It is interesting to note that R plus X1, $Th_2 18$, and I3 constitute a sufficient axiom system for $\mathbf{RXI}_2$. I chose to start with the less economical axiomatization because it separates the properties of qualification from those of identity.

$Th_2 19$. $\forall a\; a \neq [x_1] \rightarrow \bar{A}[x_1]/x$

1. $A[x_1]/x \rightarrow \exists a\; a = [x_1]$	$Th_2 18$
2. $A[x_1]/x \rightarrow \Xi a\; a = [x_1] = [x_1]$	1, df
3. $\Xi a\; a = [x_1] \neq [x_1] \rightarrow \bar{A}[x_1]/x$	2, cpos
4. $\forall a\; a \neq [x_1] \rightarrow \Xi a\; a = [x_1] \neq [x_1]$	Th 10
5. $\forall a\; a \neq [x_1] \rightarrow \bar{A}[x_1]/x$	4, 3, chain

We are now in a position to prove certain modes of obversion in $\mathbf{RXI}_2$:

$Th_2 20$. $\forall b \forall \bar{a}\; \bar{a} \neq b \rightarrow \forall b \exists a\; a = b$ (converse obversion of **E**).

1. $\forall b \forall \bar{a}\; \bar{a} \neq b \rightarrow \forall \bar{a}\; \bar{a} \neq \Xi b \bar{\exists} a\; a = b$	Th 10
2. $\rightarrow -\bar{A} \Xi b \bar{\exists} a\; a = b/x$	1, $Th_2 19$, chain
3. $\rightarrow A \Xi b \bar{\exists} a\; a = b/x$	2, Th 6, chain
4. $\rightarrow \exists a\; a = \Xi b \bar{\exists} a\; a = b$	3, $Th_2 18$, chain
5. $\rightarrow \forall b \exists a\; a = b$	4, df

By a similar proof, which I do not give here, we have

$Th_2 21$. $\forall b \forall a\; a \neq b \rightarrow \forall b \exists \bar{a}\; \bar{a} = b$ (obversion of **E**).

I shall assume that **A** and **O**, **I** and **E** are contradictories; proof is left to the reader. Given this assumption, the last two theorems contrapose respectively to

$Th_2 22$. $\exists b \forall a\; a \neq b \rightarrow \exists b \exists \bar{a}\; \bar{a} = b$ (obversion of **O**).
$Th_2 23$. $\exists b \forall \bar{a}\; \bar{a} \neq b \rightarrow \exists b \exists a\; a = b$ (converse obversion of **O**).

In $\mathbf{RXI}_2$, then, we can prove one half of each obversion equivalence, viz. the half which takes us from a negative to an affirmative statement. But these proofs cannot be made to work in $\mathbf{RXI}_1$, depending as they do on $Th_2 18$, which makes essential use of I1'. The closest we can come in $\mathbf{RXI}_1$ to $Th_2 20$, e.g., is $\Xi b \bar{\exists} a\ a = b = \Xi b \bar{\exists} a\ a = b \ \& \ \forall b \forall \bar{a}\ \bar{a} \neq b \rightarrow \forall b \exists a\ a = b \vee \Xi b \bar{\exists} a\ a = b \neq \Xi b \bar{\exists} a\ a = b$. Unfortunately, the forms of obversion which work in $\mathbf{RXI}_2$ are in the main just the opposite of those recognized by Aristotle. However, curiously enough, there is one passage where he accepts obversion from the negative to the affirmative:

> Thus, if the question were asked 'Is Socrates wise?' and the negative answer were the true one, the positive inference 'Then Socrates is unwise' is correct. (20a25-7)

The inference countenanced here is of the form

$$\bar{\exists} a\ a = x \vdash \exists \bar{a}\ \bar{a} = x.$$

If this general form is valid, it is enough to give us (in the contect of $\mathbf{RXI}_1$) $Th_2 20$-23. The trouble is that Aristotle elsewhere explicitly denies the general validity of such an inference (52a4, 20f.). The other halves of the obversion equivalences, viz. obversion and converse obversion of affirmative statements, do not seem to be provable even in $\mathbf{RXI}_2$.

4.3 *The system* $\mathbf{RXI}_3$. This system contains a still different strengthened theory of identity. $\mathbf{RXI}_3$ is just like $\mathbf{RXI}_1$ except that in place of I2 and I3 it has

$$\text{I3}'.\quad x = y \rightarrow .A \rightarrow Ay:x$$

It is easily seen that $\mathbf{RXI}_3$ contains $\mathbf{RXI}_1$, with the help of an auxiliary theorem from R:

$Th_3 16$. $A \rightarrow .B \rightarrow C \vdash B \rightarrow .A \rightarrow C$ (permutation).

1. $A \rightarrow .B \rightarrow C$	hypothesis
2. $B \rightarrow C \rightarrow C \rightarrow .A \rightarrow C$	1, R3
3. $B \rightarrow .B \rightarrow C \rightarrow C$	R2
4. $B \rightarrow .A \rightarrow C$	3, 2, chain

$Th_3 17$. $x = y \rightarrow y = x$ (I2)

1. $x = y \rightarrow .x = x \rightarrow y = x$	I3'
2. $x = x \rightarrow .x = y \rightarrow y = x$	1, $Th_3 16$
3. $x = x$	I1
4. $x = y \rightarrow y = x$	3, 2, mp

$Th_3 18$. $A \ \& \ x = y \rightarrow Ay:x$ (I3)

1. $x = y \rightarrow .A \rightarrow Ay:x$	I3'
2. $A \ \& \ x = y \rightarrow x = y$	R6
3. $A \ \& \ x = y \rightarrow .A \rightarrow Ay:x$	2, 1, chain
4. $A \rightarrow .A \ \& \ x = y \rightarrow Ay:x$	3, $Th_3 16$
5. $A \ \& \ x = y \rightarrow A$	simp
6. $A \ \& \ x = y \rightarrow .A \ \& \ x = y \rightarrow Ay:x$	5, 4, chain
7. $A \ \& \ x = y \rightarrow Ay:x$	6, R4

We therefore have at our disposal for **RXI**$_3$ Th 1-15.

$\mathrm{Th}_3 19.\ A \to B \to .C \to A \to .C \to B$

By $\mathrm{Th}_3 16$ on R3.

$\mathrm{Th}_3 20.\ A \to B \to .\bar{B} \to \bar{A}$

1.	$B \to --B$	Th 5
2.	$B \to ----B$	1, Th 5, chain
3.	$--A \to B \to .--A \to ----B$	2, $\mathrm{Th}_3 19$
4.	$--A \to B \to .\bar{B} \to \bar{A}$	3, R9, chain
5.	$--A \to A$	Th 6
6.	$A \to B \to .--A \to B$	5, R3
7.	$A \to B \to .\bar{B} \to \bar{A}$	6, 4, chain

$\mathrm{Th}_3 21.\ A[x_2]/x \to .\bar{A}[x_1]/x \to \forall a\ a \neq [x_1]$

1.	$\Xi a\text{-}a \neq [x_1] = [x_1] \to .A\Xi a\text{-}a \neq [x_1]/x \to A[x_1]/x$	I3'
2.	$A\Xi a\text{-}a \neq [x_1]/x \to .\Xi a\text{-}a \neq [x_1] = [x_1] \to A[x_1]/x$	1, $\mathrm{Th}_3 16$
3.	$A[x_2]/x \to A\Xi a\text{-}a \neq [x_1]/x$	X1
4.	$A[x_2]/x \to .\Xi a\text{-}a \neq [x_1] = [x_1] \to A[x_1]/x$	3, 2, chain
5.	$A[x_2]/x \to .\bar{A}[x_1]/x \to \Xi a\text{-}a \neq [x_1] \neq [x_1]$	4, $\mathrm{Th}_3 20$, chain
6.	$A[x_2]/x \to .\bar{A}[x_1]/x \to \forall a\ a \neq [x_1]$	5, df

$\mathrm{Th}_3 22.\ A \to .B \to C,\ D \to .C \to E \vdash D \to .A \to .B \to E$

1.	$A \to .B \to C$	hypothesis
2.	$D \to .C \to E$	hypothesis
3.	$A \to .C \to E \to .B \to E$	1, R3, chain
4.	$C \to E \to .A \to .B \to E$	3, $\mathrm{Th}_3 16$
5.	$D \to .A \to .B \to E$	2, 4, chain

In the following theorem, let a stand for ξxAx (rather than for ξxA):

$\mathrm{Th}_3 23.\ \bar{A}[x_2]/x \to .A[x_3]/x \to .\forall b \exists a\ a = b \to \forall b \forall \bar{a}\ \bar{a} \neq b$
(obversion of **A**).

1.	$\forall b \exists a\ a = b \to \exists a\ a = \Xi b\overline{\forall \bar{a}}\ \bar{a} \neq b$	Th 10
	(*Let* C *abbreviate* $\overline{\forall \bar{a}}\ \bar{a} \neq b$.)	
2.	$\forall b \exists a\ a = b \to \Xi a\ a = \Xi bC = \Xi bC$	1, df
3.	$\forall b \exists a\ a = b \to .A\Xi a\ a = \Xi bC \to A\Xi bC$	2, I3', chain
4.	$A\Xi a\ a = \Xi bC \to .\forall b \exists a\ a = b \to A\Xi bC$	3, $\mathrm{Th}_3 16$
5.	$A[x_3]/x \to A\Xi a\ a = \Xi bC$	X1
6.	$A[x_3]/x \to .\forall b \exists a\ a = b \to A\Xi bC$	5, 4, chain
7.	$A\Xi bC \to -\bar{A}\Xi bC$	Th 5
8.	$\forall b \exists a\ a = b \to A\Xi bC \to .\forall b \exists a\ a = b \to -\bar{A}\Xi bC$	7, $\mathrm{Th}_3 19$
9.	$A[x_3]/x \to .\forall b \exists a\ a = b \to -\bar{A}\Xi bC$	6, 8, chain
10.	$\bar{A}[x_2]/x \to .-\bar{A}\Xi bC \to \forall \bar{a}\ \bar{a} \neq \Xi bC$	$\mathrm{Th}_3 21$
11.	$\bar{A}[x_2]/x \to .A[x_3]/x \to .\forall b \exists a\ a = b \to \forall \bar{a}\ \bar{a} \neq \Xi bC$	9, 10, $\mathrm{Th}_3 22$
12.	$\to \forall b \forall \bar{a}\ \bar{a} \neq b$	11, df

Here, then, is an example of the sort of obversion which is provable in **RXI**$_3$. Notice that the conditions $\bar{A}[x_2]/x$ and $A[x_3]/x$ are in effect "existential'' premisses; $Th_3 23$ is deductively equivalent to $\exists x\bar{A}x \rightarrow .\exists xAx \rightarrow \ldots$, which is in fact an instance of it. By a similar proof, differing mainly in details of double negation, we can get

$Th_3 24$. $A[x_2]/x \rightarrow .\bar{A}[x_3]/x \rightarrow .\forall b\exists\bar{a}\ \bar{a} = b \rightarrow \forall b\forall a\ a \neq b$
(converse obversion of **A**).

Assuming again the relations of contradiction, we can apply $Th_3 20$ to transform the last two theorems respectively to

$Th_3 25$. $\bar{A}[x_2]/x \rightarrow .A[x_3]/x \rightarrow .\exists b\exists\bar{a}\ \bar{a} = b \rightarrow \exists b\forall a\ a \neq b$
(converse obversion of **I**).

$Th_3 26$. $A[x_2]/x \rightarrow .\bar{A}[x_3]/x \rightarrow .\exists b\exists a\ a = b \rightarrow \exists b\forall\bar{a}\ \bar{a} \neq b$ (obversion of **I**).

Thus in $\mathbf{RXI}_3$ we can prove the other half of each obversion equivalence, the half which takes us from an affirmative to a negative statement, though only for categoricals whose predicate term a is neither universal nor null in application relative to the range of x. The proofs depend essentially on the exported form of I3', which is available in neither $\mathbf{RXI}_1$ nor $\mathbf{RXI}_2$. The closest we can come to, say, $Th_3 27$ in those two systems is $A[x_2]/x\ \&\ \forall b\exists a\ a = b \rightarrow \forall b\forall\bar{a}\ \bar{a} \neq b \vee A[x_3]/x$. The modes of obversion provable in $\mathbf{RXI}_3$ are the ones that correspond most closely to those recognized by Aristotle. In 20a20f. he gives an example of converse obversion of **A**($Th_3 24$), and in 20a22f. he accepts an instance of abversion of **I**($Th_3 26$). In 51b41-52a1 Aristotle gives as valid the inference scheme

$$\exists\bar{a}\ \bar{a} = x \vdash \bar{\exists} a\ a = x$$

and farther down (52a6-8) he approves its contrapositive. (Notice that this is the converse of the inference cited in the previous section.) In the context of $\mathbf{RXI}_1$, all forms of obversion and converse obversion from affirmative to negative statements could be derived from this principle. That is just the situation we have in $\mathbf{RXI}_3$, with the important exclusion of universal and null terms. In other words, we can interpret in $\mathbf{RXI}_3$ what seems to be Aristotle's predominant view on negative terms. From this standpoint, his theory of negative terms differs from his positive syllogistic in that it requires ''existence'' assumptions.

4.4 *Two-valued logic.* Since contraposition requires obversion in both directions, it may be presumed to fail in both $\mathbf{RXI}_2$ and $\mathbf{RXI}_3$. Neither of the strengthened systems, then, permits an interpretation of the full traditional theory of negative terms. The move which naturally suggests itself is to combine $\mathbf{RXI}_2$ and $\mathbf{RXI}_3$ into one system in which the theorems of both will be available. The resulting system—call it TXI—turns out to be none other than the two-valued propositional calculus (T) extended to qualification and identity. To see that this is so, we have only to note that I1', $B \rightarrow x = x$, and I3', $x = x \rightarrow .A \rightarrow A$, together give $B \rightarrow .A \rightarrow A$, which permutes by $Th_3 16$ to $A \rightarrow .B \rightarrow A$. The latter suffices to turn relevance implication into material implication. Seeing this, we can proceed to give a much more economical formulation of TXI as follows. Formation rules are the same as for $\mathbf{RXI}_1$, except that $\supset$ replaces $\rightarrow$ and the clause for & drops out. To the definitions is added a definition of &. Adj is deleted from the rules of inference. The R axioms are replaced by some suitable formulation of T, say Łukasiewicz's. The qualification axioms are the same, though X2 may now be given in exported form. The identity axioms are I1 and I3'. I leave it to the reader to verify the following theorem of TXI:

$$\bar{A}[x_2]/x \ \& \ A[x_3]/x \ \& \ B[x_4]/x_1 \ \& \ \bar{B}[x_5]/x_1 \supset . \forall b \exists a \ a = b \equiv \forall \bar{a} \exists \bar{b} \ \bar{b} = \bar{a}$$

(contraposition of **A**)

A similar conditioned equivalence is provable for **O.** Thus syllogistic with negative terms is interpretable in TXI if for each term a we make two "existence" assumptions, $\exists xA$ and $\exists x\bar{A}$. If we are content to use extensional logic, then, we may say that Aristotelian syllogistic differs from syllogistic with negative terms in requiring no "existence" assumptions. But if we interpret the main connective of syllogisms to be relevance implication, then it seems we must reject the traditional theory of negative terms as invalid. This is one of the ways in which such an interpretation fits Aristotle's practice.

V. Concluding remarks. **5.1** *Drawbacks.* The system R and its extension to quantification were developed in an attempt to eliminate various unintuitive theorems of the classical calculi. These include the much discussed paradoxes (or "paradoxes") of material and strict implication, Peirce's law, and others. One such theorem which is avoided in the most natural quantificational extension of R[19] is

$$\exists y . Ay \rightarrow \forall x Ax$$

for the case when Ax contains free x. Aside from its intuitive oddness, it is natural to reject this principle from an extension of R because its analogue in terms of disjunctive and conjunctive expansions, viz.

$$(A \rightarrow A \ \& \ B \ \& \ldots) \vee (B \rightarrow A \ \& \ B \ \& \ldots) \vee \ldots,$$

is not provable in R. But the offending principle is a theorem of $\mathbf{RXI_1}$: it follows by "existential" generalization (i.e. X1) from $A \Xi x \bar{A} x \rightarrow A \Xi x \bar{A} x$ (itself an instance of X1). It is of course evident that this "paradox" is directly bound up with our definition of the universal quantifier. But even if we took universality as primitive, we would still get an equivalence corresponding to that definition.

A still graver difficulty with X1 rears its head when we try to extend $\mathbf{RXI_1}$ (or any of the stronger systems) to include a treatment of modalities. Standard systems of quantification and modality contain theorems analogous to

$$\exists x \Box A \rightarrow \Box \exists x A$$

and

$$\Diamond \forall x A \rightarrow \forall x \Diamond A$$

but, for obvious reasons, reject the converses. Now, the above theorems would follow straightforwardly in a modal extension of $\mathbf{RXI_1}$, but *so would their converses.* For

$$\Box\exists xA \rightarrow \exists x\Box A$$

is by definition the same as

$$\Box A \Xi xA/x \rightarrow \exists x\Box A,$$

an instance of "existential" generalization (X1). Similarly,

$$\forall x\Diamond A \rightarrow \Diamond\forall xA$$

is just

$$\forall x\Diamond A \rightarrow \Diamond A\Xi x\bar{A}/x,$$

which is a case of Th 10, universal instantiation. These theorems, which would in no way depend on the particular principles of modality used in the system, clearly conflict with our intuitive notions of necessity and possibility. It is interesting to note that their quantificational analogue,

$$\forall x\exists yA \rightarrow \exists y\forall xA$$

i.e., by definition,

$$\forall xA\Xi yA/y \rightarrow \exists y\forall xA$$

or

$$\forall x\exists yA \rightarrow \exists yA\Xi x\bar{A}/x,$$

is rejected in $\mathbf{RXI}_1$ for the case where A contains free $\underline{x}$ and y. For in that case, ΞyA $\{\Xi x\bar{A}\}$ would be *stuck* in $\forall xA\Xi yA/y$ $\{\exists yA\Xi x\bar{A}/x\}$, and the formula would fail to fall under X1 {Th 10}. The modal analogue of stuckness and bondage is *obliqueness*. We might consider restricting X1 (and hence Th 10) to prohibit instantiation into oblique contexts, but such a restriction would block many desirable modal principles as well. The conclusion appears inescapable that qualification of the brand formalized in $\mathbf{RXI}_1$ is incompatible with modality.

5.2 *Further tasks*. The exact significance of the above interpretation of syllogistic will not be clear until a satisfactory semantics is developed for qualification. In the case of $\mathbf{RXI}_{1-3}$, this task is aggravated by our present lack of a semantics for **R**. But it is reasonable to call for an attack on the problem in regard to TX and TXI. In §**3.3** I gave a rough sketch of how we might approach the semantics of $\mathbf{RXI}_1$. The approach involved the possibility of two different ranges for a restricted variable 'ξxFx', depending upon whether or not $\exists xFx$. For TX, however, it seems likely that a simpler interpretation might be possible, according to which 'ξxFx' would always range over $\lambda x.\,\exists yFy \supset Fx$, and similarly for more complicated restricted variables. This would open up the prospect of interpreting TXI in an enriched version of Hailperin's $\mathcal{L}\mathcal{F}\nu$ by means of the definition

$$\xi xA =_{df} \nu x.\,\exists xA \supset A.$$

It is, however, questionable whether X2 would follow on the basis of this definition. Such an interpretation might in turn permit the elimination of ξ-operators as a corollary of a generalization of Hailperin's ν-elimination theorem. Needless to say, this result would trivialize our "existence"-free interpretation of syllogistic as far as TXI were concerned; for a categorical such as $\forall b \exists a\ a = b$ would come out equivalent to $\forall x . \exists x_1 B \supset Bx/x_1 \supset . \exists x A \supset A$. The latter would seem a rather far-fetched rendering of *All* b's *are* a's.

5.3 *Summary.* An interpretation which closely fits Aristotle's development of syllogistic can be achieved if we take subject and predicate terms to be real *terms*, i.e. common nouns or variables of various ranges (**1.1-2**). If the predicate term is to be a variable, its quantity must be made explicit (quantification of the predicate, **1.3**). Although common nouns are primitive in ordinary language, we gain better control over the "existential" commitment involved in their use if we analyze them as qualification expressions—'man' as 'thing such that it is human' or 'ξx(human x)' and so on (**1.4-7**). Using the qualification axioms put forth, with identity and the system of relevance implication as a base logic (**2.1-7**), we can prove all the syllogisms of Aristotelian and traditional positive syllogistic (**2.9**) without any "existence" assumptions so far as the syllogistic predicates are concerned (**3.2**). From the conventional point of view, the syllogistic terms range over non-empty domains; but what those domains are depends upon whether or not the associated syllogistic predicates have application (**3.3**). Obversion and contraposition do not work, but they become derivable in successive strengthenings of the system (**4.2-4**). These strengthenings culminate in classical two-valued logic with qualification and identity, which is capable of subsuming the full traditional theory of negative terms, though only with extensive "existence" assumptions (**4.4**). Although the semantics of qualification remain to be worked out (**5.2**), the results already attained open up new perspectives on the long-standing problem of interpreting syllogistic. These results are offered as one example of the many fruitful applications of qualification theory to the philosophy of language.

NOTES

1. I am indebted to Professors Alan Ross Anderson, Milton Fisk, and Rulon S. Wells for helpful discussion of earlier drafts of parts of this paper. A summary [7] of parts II and III was read before the Association for Symbolic Logic, New York, 28 December 1965.

2. Cf. note 8 below.

3. I use 'class' in Church's sense, [13] p. 29, to mean a one-valued singularly propositional function in extension. It may well be that a sufficiently sophisticated semantics would demand a specification of the *sense range* for complete characterization of a variable. In the case of the above example, the sense range of 'man' would be the property humanity or manhood, in a certain sense of these words. My resorting to the 'class' locution, then, is not meant to suggest any identification of properties with classes.

4. Throughout the informal presentation I allow myself for heuristic purposes to mix English with the formalism. For similar reasons I follow the familiar practice of using adjectives as if they were verbs, an identification which might turn out on a finer-grained analysis to be insupportable. The operator 'λ', which is not part of any system presented here, is to be understood in the usual way.

5. In the cases where quantification of the predicate gives rise to statements entirely different from the conventional **A**, **I**, **O**, and **E**, Ammonius finds other objections. Since Hamilton (and more recently Parry) was specially interested in these unorthodox forms, he was justified in claiming not to have been anticipated by Ammonius. What Ammonius did clearly see was the possibility of expressing categorical propositions by means of statements with quantified predicate terms.

6. As Church points out, [13] n. 112, this is not an operator in his sense, since it yields neither a constant nor a form as usually conceived. But it seems appropriate to follow Hailperin in calling it an operator, since it shares with accustomed operators the feature of containing the occurrence of a variable whose function is to *bind* other occurrences of that variable in the expression to which it is prefixed.—See also the historical reference in Church's note.

7. Although they are not treated in this paper, we should not overlook the fact that ordinary language also contains common nouns ranging over polyadic propositional functions, e.g. 'married couple', which we might formalize as '$\xi^2 xy(x$ is married to $y)$'. Cf. on this subject [6], p. 331.

8. In parsing '*x* such that . . . ' as a qualification operator forming restricted individual variables, I consciously oppose the practice which has arisen of construing it as an *abstraction* operator forming predicates or class expressions. Thus it is misleading to read '$\lambda x(Fx)y$' as '*y* is *an x such that Fx*'. This practice is based on the same confusion as the treatment of common nouns such as 'man' in 'Socrates is a man' as *predicates*. The little quantifier 'a' must not be passed over lightly.

9. This term is not perfect, since it is not the operand but rather the operator variable (or what that variable stands for, its range) which gets qualified. The suggested terminology also conflicts with the traditional use of 'quality' to refer to the affirmative or negative status of statements.

10. Interestingly enough, 'Ξx', unlike 'ξx', is an operator in Church's sense. Cf. n. 6 above.

11. The name makes reference to Anderson and Belnap's syntactical completeness proof for systems involving this brand of implication. (The gist of the proof is sketched in [3] p. 38.) This appellation is to be preferred to 'weak implication', as it is the calculus which is weakened by strengthening the implication relation thereof. Cf. a review by W. T. Parry, [25] p. 257.

12. These *prima facie* circular characterizations of 'bound' and 'stuck' are condensed recursive definitions. A basis clause arises when the occurrence of 'stuck' in the characterization of 'bound' drops out for the case in which x is unrestricted. Our "stuck" variables arise when Hailperin's "subordinate" variables are bound. In Hailperin's system, this is ruled out by the formation rules. As Church points out, [13] n. 94, the admission of stuck variables presents semantic difficulties, the gravity of which I do not mean to minimize. However, the stuck variables (as opposed to stuck constants) play no essential role in the present interpretation of syllogistic.

13. For the system as given, I have not been able to derive a rule of replacement of equivalents. Though such a rule is not needed for syllogistic, it may be that the rule or a third qualification axiom is necessary if ξ and Ξ are to have their intended properties. Candidates might be $\forall$u.Au $\leftrightarrow$ Bu.& C $\rightarrow$ C[XxBx] : [XxAx] or the stronger $\forall$u.Au $\leftrightarrow$ Bu. $\rightarrow$ [XxAx] = [XxBx], where [XxBx] is the result of replacing Ax in [XxAx] by Bx.

14. In particular, our interpretation of syllogistic works for the system E of Anderson and Belnap (e.g. [4] p. 14), which differs from R only in having $A \rightarrow A.\&.B \rightarrow B. \rightarrow .C \rightarrow C$ in place of R2.

15. The interpretation thus works for the system of Fitch [17], whose negation is of just this sort. (Viz., R10 and the ensuing *reductio* variants fail, but contraposition and double-negation elimination hold.)

16. In order not to introduce too many strange considerations at once, the semantic sketch just given is based on the conventional interpretation of the unrestricted variables 'x', 'y', . . . as ranging over a non-empty domain. However, following Lejewski [22] I hold that ordinary quantification theory, as well as the system presented here, is equally valid for empty domains. Thus, I would not ultimately admit even to assumption (5) of §3.2. Now, if the range of 'x' is empty, it follows that the ranges of 'ξxFx' and of all other common nouns will likewise be empty. But in this case, '$\exists xFx$' no longer holds iff λxFx is non-empty: the former may be true, while the latter is false. Thus the account given above may be rendered valid for empty domains too by replacing clauses of the form λxA *is non-empty* by clauses of the form $\exists$xA. This standpoint accounts for my use of double quotes around 'existence'.

17. It was these passages in Aristotle which first suggested to me as an axiom an instance of X1, $\exists uA \rightarrow A\xi uA/u$.

18. Textual evidence on this point seems somewhat conflicting. For a detailed discussion of the relevant passages, see Manley Thompson [27].

19. The system referred to is the one suggested at the end of [8], viz. R'' minus the axiom there numbered (15). (15) is in fact deductively equivalent to the shorter $(\exists y)(fy \rightarrow (x)fx)$.

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