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AN ABBREVIATION OF CROISOT'S AXIOM-SYSTEM FOR DISTRIBUTIVE LATTICES WITH I

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In [2] there have been established the axiom-systems which satisfy certain formal requirements defined in that paper for distributive lattices with the constant elements. Unfortunately, only when [2] was already composed and in the final proofs, and, therefore, could not be changed, I unexpectedly obtained a rather interesting result which makes the deductions presented in [2] obsolete, although they are entirely correct. Namely, I have proved that in the sets of postulates given in the assumptions of Theorem 2, *cf.* [2], section **3**, axiom *A17* is redundant.

1 It is obvious, that if an algebraic system

$$\mathfrak{S} = \langle A, \cap, \cup, I \rangle$$

with two binary operations $\cap$ and $\cup$, and with a constant element $I \epsilon A$, is a distributive lattice with I, then the following formulas

S1 $[a] : a \epsilon A . \supset . I = a \cup I$ [i.e. *A1* in [2], section **2**]
S2 $[a] : a \epsilon A . \supset . a = a \cap I$ [i.e. *A2* in [2], section **2**]
S3 $[abc] : a, b, c \epsilon A . \supset . a \cap ((b \cap b) \cup c) = (c \cap a) \cup (b \cap a)$ [i.e. *A4* in [2], section **2**]

are provable in the field of $\mathfrak{S}$. I shall prove here the converse of this statement. Namely:

If the system $\mathfrak{S}$ satisfies the formulas S1, S2 and S3, then it is a distributive lattice with I.

Proof: Let us assume *S1*, *S2* and *S3*. Then:

S4 $[ab] : a, b \epsilon A . \supset . a = (I \cap a) \cup (b \cap a)$ [*S2, S1, S3*; as *A5* in [2], section **2**]
S5 $[abc] : a, b, c \epsilon A . \supset . (b \cap c) \cup (a \cap c) = c \cap ((b \cup a) \cup (b \cup a))$ [*S3, S4, S3, S2*; as *A6* in [2], section **2**]
S6 $[a] : a \epsilon A . \supset . I \cap (a \cup a) = a$ [*S4, S5, S2, S4*; as *A7* in [2], section **2**]
S7 $[a] : a \epsilon A . \supset . I \cup a = I$ [*S2, S3, S1*; as *A8* in [2], section **2**]

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S8 $[ab] : a, b \epsilon A . \supset . a = (b \cap a) \cup (I \cap a)$ [*S2, S7, S2, S3*; as *A9* in [2], section **2**]

S9 $[a] : a \epsilon A . \supset . (a \cap a) = (a \cap a) \cup a$

PR $[a] : a \epsilon A . \supset .$

$(a \cap a) = I \cap ((a \cap a) \cup (a \cap a)) = ((a \cap a) \cap I) \cup (a \cap I) = (a \cap a) \cup a$ [*S6*; *S3*; *S2*]

S10 $[a] : a \epsilon A . \supset . I \cap (a \cap a) = a \cup a$

PR $[a] : a \epsilon A . \supset .$

$I \cap (a \cap a) = I \cap ((a \cap a) \cup a) = (a \cap I) \cup (a \cap I) = a \cup a$ [*S9*; *S3*; *S2*]

S11 $[a] : a \epsilon A . \supset . (a \cup a) \cup (a \cup a) = a \cap a$

PR $[a] : a \epsilon A . \supset .$

$(a \cup a) \cup (a \cup a) = (I \cap (a \cap a)) \cup (I \cap (a \cap a)) = a \cap a$ [*S10*; *S4*]

S12 $[a] : a \epsilon A . \supset . a \cup a = (I \cap a) \cap (I \cap a)$

PR $[a] : a \epsilon A . \supset .$

$a \cup a = ((I \cap a) \cup (I \cap a)) \cup ((I \cap a) \cup (I \cap a)) = (I \cap a) \cap (I \cap a)$ [*S4*; *S11*]

S13 $[a] : a \epsilon A . \supset . a \cap (a \cap a) = (a \cap a) \cup (a \cap a)$

PR $[a] : a \epsilon A . \supset .$

$a \cap (a \cap a) = a \cap ((a \cap a) \cup a) = (a \cap a) \cup (a \cap a)$ [*S9*; *S3*]

S14 $[a] : a \epsilon A . \supset . a \cap a = ((a \cap a) \cup (a \cap a)) \cup (a \cup a)$

PR $[a] : a \epsilon A . \supset .$

$a \cap a = (a \cap (a \cap a)) \cup (I \cap (a \cap a)) = ((a \cap a) \cup (a \cap a)) \cup (a \cup a)$ [*S8*; *S13*; *S10*]

S15 $[a] : a \epsilon A . \supset . a \cup a = a \cap a$

PR $[a] : a \epsilon A . \supset .$

$a \cup a = (I \cap a) \cap (I \cap a)$ [*S12*]

$= (((I \cap a) \cap (I \cap a)) \cup ((I \cap a) \cap (I \cap a))) \cup ((I \cap a) \cup (I \cap a))$ [*S14*]

$= ((a \cup a) \cup (a \cup a)) \cup a = (a \cap a) \cup a = a \cap a$ [*S12*; *S4*; *S11*; *S9*]

S16 $[a] : a \epsilon A . \supset a = a \cap a$

PR $[a] : a \epsilon A . \supset .$

$a = I \cap (a \cup a) = I \cap (a \cap a) = a \cup a = a \cap a$ [*S6*; *S15*; *S10*; *S15*]

S17 $[abc] : a, b, c \epsilon A . \supset . a \cap (b \cup c) = (c \cap a) \cup (b \cap c)$ [*S3*; *S16*]

Since the formulas *S1*, *S2* and *S3* imply *S16* and *S17*, and since, as Croisot has shown, *cf.* [1], p. 27, and [2], section **1**, the set of the formulas *S16*, *S1*, *S2* and *S17* constitutes an axiom-system for distributive lattice with I, we have $\{S1; S2; S3\} \rightleftarrows \{S16, S1; S2; S17\}$. Therefore, the proof is complete. It should be noticed that in the axiomatization presented above the postulate *S3* can be substituted by

*S3** $[abc] : a, b, c \epsilon A : \supset . a \cap (b \cup (c \cap c)) = (c \cap a) \cup (b \cap a)$

Deductions entirely analogous to those given above show without any difficulty that $\{S1; S2; S3^*\} \rightleftarrows \{S16; S1; S3; S17\}$. The matrices $\mathfrak{M}1$, $\mathfrak{M}2$, $\mathfrak{M}3$ and $\mathfrak{M}4$ given in [2], section **4**, *cf.* also [1], pp. 26-27, prove that the axioms *S1*, *S2* and *S3* are mutually independent, and that in this set of postulates *S3* cannot be replaced by *S17*.

2 The fact that Croisot's axiom-system $\{S16; S1; S2; S17\}$ is inferentially equivalent to the shorter axiomatization $\{S1; S2; S3\}$ alters the theorems and the proofs given in [2], as follows:
(1) From the assumptions of Theorem 2 axiom $A17$ should be removed, and the proof of this Theorem should be replaced by the deductions given above in section **1**.
(2) From the assumptions of Theorem 3 axiom $C3$ should be dropped.
(3) The proof of Theorem 1 can be replaced by a simple remark that this Theorem 1 is an immediate consequence of a new version of Theorem 2 and of the self-evident fact that an addition of the formula $A3$, *cf.* [2], section **2**, as a new postulate to the axioms $S1$, $S2$ and $S3$ constitutes an axiom-system for distributive lattices with O and I.

REFERENCES

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