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ORDINAL THEORY IN A CONSERVATIVE EXTENSION OF PREDICATE CALCULUS

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Let P' denote the class-set theory which consists of just axioms A1, A2, A3 and theorem M3 (restricted to case $n = 1$) of [2]. Simplifying a little, P' is thus basically a first order theory with equality having two sorts of variables, class variables and set variables, and satisfying an axiom of extensionality and an axiom schema which says the following: for any wff which contains no bound class variables there is a class X of all sets v satisfying φ; in symbols

$$\exists X\ \forall v\ [v \epsilon X \longleftrightarrow \varphi(v)].$$

(As usual for class and set variables we use capital and small letters respectively.) By [3] theory P' is a conservative extension of P, the first-order predicate calculus with equality where the only non-logical symbol is "ϵ" and the individual variables are the set variables.

The purpose of this paper is to show that a surprisingly large portion of the theory of Von-Neumann ordinals and natural numbers can be developed in P'. Such information could be useful in the investigation of any formulation of set theory not using the unrestricted subset axiom

$$\forall Y\ \forall x\ [Y \subseteq x \rightarrow Y \epsilon \mathsf{V}]$$

which involves unrestricted quantification over class variables in an essential way. An example of such a restricted set theory would be a formalization of the set theoretical reasoning used in predicative analysis; cf. [1]. By [3] our results are equally valid for a corresponding conservative class extension K' of any first-order theory K. In such a case one would have in general three types of individual variables: K, set, and class variables.

We say R is a (strict) *linear-ordering* of X (abbrev.: $\mathsf{Lo}_R(X)$) if and only if R is irreflexive, connected and transitive over X; in symbols

$\mathsf{Irr}_R(X)$, i.e., $(\forall u)_X\ \neg(uRu)$
$\mathsf{Con}_R(X)$, i.e., $(\forall u,v)_X\ [uRv \vee u = v \vee vRu]$
$\mathsf{Tr}_R(X)$, i.e., $(\forall u,v,w)_X\ [uRv\ .\ vRw \rightarrow uRw]$

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We now define two notions of well-ordering: (i) R is a (strict) *well-ordering* of class X and (ii) R is a (strict) *strong well-ordering* of X. In symbols we have respectively

(i) $\mathrm{Wo}_R(X) \longleftrightarrow \mathrm{Lo}_R(X)\,.\ \forall y\ [\phi \neq y \subseteq X \rightarrow y$ has an R-first element$]$
(ii) $\mathrm{Wo}_R^*(X) \longleftrightarrow \mathrm{Lo}_R(X)\,.\ \forall Y[\phi \neq Y \subseteq X \rightarrow Y$ has an R-first element$]$

Comment: Usually "uRv" is an abbreviation for "$\langle u,v\rangle \in R$". Now for any two sets x,y the classes

$$\{x,y\} =_{Df} \{u \mid u = x \ \vee\ u = y\}$$

and

$$\langle x,y\rangle =_{Df} \{\{x\},\ \{x,y\}\}$$

are well-defined. However, without axiom A4 (the pairs axiom) we can't show $\{x,y\} \in \mathrm{V}$, hence we can't even prove that $\langle x,y\rangle$ has the ordered pair property. Thus we use "uRv" only suggestively. Actually we will be interested in specific relations R, viz.,

$$\mathrm{E} = \{\langle x,y\rangle \mid x \in y\}$$

and

$$\mathrm{S} = \{\langle x,y\rangle \mid x \subset y\}$$

Thus "$u\mathrm{E}v$" and "$u\mathrm{S}v$" can be considered as an abbreviation of "$u \in v$" and "$u \subset v$" if we don't have axiom A4 or "$\langle u,v\rangle \in \mathrm{E}$" and "$\langle u,v\rangle \in \mathrm{S}$" if we do.

When working in a definitional extension P^* of P' we say that a defined predicate H is P'-normal if and only if there is a wff φ of P' containing no bound class variables such that in P^*

$$\vdash [H(\mathbf{v}, \mathbf{X}) \longleftrightarrow \varphi(\mathbf{v}, \mathbf{X})]$$

where vectors $\mathbf{v}$ and $\mathbf{X}$ represent all the free set and class variables respectively appearing in H. Likewise one can define the notions of a P'-normal function letter, constant, or restricted variable; cf. [2; p. 12]. Unless stated otherwise, all new defined symbols are P'-normal and all proofs are carried out in (a definitional extension of) P'. We must emphasize here that Wo_R is a P'-normal predicate whereas Wo_R^* isn't. (Of course Wo_R^* is T-normal in any extension T of P' satisfying the unrestricted subset property.)

We say that set x satisfies the *simple subset property* (abbrev.: $\mathrm{Sub}(x)$) if and only if

$$\forall y\ [x \cap y \in \mathrm{V}\ .\ x - y \in \mathrm{V}]$$

We then define On, the class of ordinals as follows:

$$x \in \mathrm{On} \longleftrightarrow \mathrm{Trans}(x)\,.\ \mathrm{Wo}_{\mathrm{E}}(x)\,.\ \mathrm{Sub}(x)\,.\ (\forall y)_x\ \mathrm{Sub}(y)$$

where as usual

$$\mathrm{Trans}(X) \longleftrightarrow \forall u[u \subseteq X \rightarrow u \in X].$$

Variables restricted to the class On will be denoted by small Greek letters. The definition one usually sees, viz.

$x \in \mathrm{On} \leftrightarrow \mathrm{Trans}(x) . \mathrm{Wo}^*_{\mathrm{E}}(x)$,

isn't P'-normal. However, in any extension T of P' satisfying the unrestricted subset property or even a weaker version

$$\forall Y\ \forall \alpha\, [Y \subseteq \alpha \rightarrow Y \epsilon \mathrm{V}],$$

the two definitions are equivalent and T-normal.

Theorem 1. $\mathrm{Irr}_{\mathrm{E}}(\mathrm{On})$, *i.e.*, $\forall \alpha [\alpha \not\epsilon \alpha]$.

Theorem 2. $\mathrm{Trans}(\mathrm{On})$, *i.e.*, $\forall x\, [x \epsilon \alpha \rightarrow x \epsilon \mathrm{On}]$; *in words, every element of an ordinal is an ordinal.*

Proof: Clearly $x \epsilon \alpha \Rightarrow x \subseteq \alpha \Rightarrow \mathrm{Wo}_{\mathrm{E}}(x)$. Now we use the fourth condition in definition of "$x \epsilon \mathrm{On}$" to show $x \epsilon \alpha \Rightarrow \mathrm{Sub}(x)$ and $u \epsilon x \epsilon \alpha \Rightarrow u \epsilon \alpha \Rightarrow \mathrm{Sub}(u)$. To show $\mathrm{Trans}(x)$ consider any $u \epsilon v \epsilon x \epsilon \alpha$. Then $u, v, x \epsilon \alpha$ by $\mathrm{Trans}(\alpha)$. Also $u \mathrm{E} v$ and $v \mathrm{E} x$ implies $u \mathrm{E} x$ by $\mathrm{Tr}_{\mathrm{E}}(\alpha)$, i.e., $u \epsilon x$, as desired.

Corollary 3. $\alpha = \{\beta \mid \beta \epsilon \alpha\}$; *in words, each ordinal equals the set of* ϵ*-smaller ordinals.*

Theorem 4. $\mathrm{Trans}(y) \cdot y \subset \alpha \rightarrow y \epsilon \alpha$.

Proof: Assume $y \subset \alpha$. Now we use the condition $\mathrm{Sub}(\alpha)$ to show that α-y is a set, in fact a non-empty subset of α. Hence by $\mathrm{Wo}_{\mathrm{E}}(\alpha)$, α-y has an E-first element (which must be an ordinal by 2), call it β. We claim that $y = \beta$.

To show $y \subseteq \beta$, consider any $u \epsilon y$. Then $u \epsilon y \subset \alpha$, hence $u \epsilon \alpha$. Likewise $\beta \epsilon \alpha$. Hence:

$$\beta \epsilon u \vee \beta = u \vee u \epsilon \beta$$

by $\mathrm{Con}_{\mathrm{E}}(\alpha)$. If $\beta \epsilon u$, then $\beta \epsilon u \epsilon y$, hence $\beta \epsilon y$ by $\mathrm{Trans}(y)$; if $\beta = u$, then $\beta = u \epsilon y$, hence $\beta \epsilon y$; in either case $\beta \epsilon y$, contradicting choice of $\beta \epsilon \alpha - y$. Thus the only possibility left is $u \epsilon \beta$, as desired.

To show $\beta \subseteq y$, consider any $u \epsilon \beta$. Then $u \epsilon \beta \epsilon \alpha$, hence $u \epsilon \alpha$. If $u \not\epsilon y$, then $u \epsilon \alpha - y$ and $u \epsilon \beta$, contradicting the choice of β as the E-first element of $\alpha - y$. Thus $u \epsilon y$, as desired.

Corollary 5. $\alpha \subset \beta \leftrightarrow \alpha \epsilon \beta$.

Theorem 6. $\alpha \subset \beta \neq \alpha = \beta \neq \beta \subset \alpha$ *where* "$\neq$" *denotes "exclusive or."*

Proof: Clearly at most one of these holds. Assume none hold, i.e., $\alpha \not\subseteq \beta$ and $\beta \not\subseteq \alpha$, hence $\alpha \cap \beta \subset \alpha$ and $\alpha \cap \beta \subset \beta$. Now $\alpha \cap \beta$ is a set since $\mathrm{Sub}(\alpha)$, hence $\alpha \cap \beta$ is a transitive proper subset of α, hence $\alpha \cap \beta \epsilon \alpha$ by 4. Likewise we have $\alpha \cap \beta \epsilon \beta$. Thus $\alpha \cap \beta \epsilon \alpha \cap \beta$ where $\alpha \cap \beta \epsilon \alpha$, contradicting $\mathrm{Irr}_{\mathrm{E}}(\alpha)$.

Corollary 7. $\alpha \epsilon \beta \neq \alpha = \beta \neq \beta \epsilon \alpha$.

Theorem 8. $\mathrm{Wo}_{\mathrm{E}}(\mathrm{On})$

Proof: We have $\mathrm{Irr}_{\mathrm{E}}(\mathrm{On})$ and $\mathrm{Con}_{\mathrm{E}}(\mathrm{On})$ by 1 and 7 respectively. Also we have $\mathrm{Tr}_{\mathrm{E}}(\mathrm{On})$ since $\alpha \epsilon \beta$ and $\beta \epsilon \gamma$ implies $\alpha \epsilon \gamma$. Consider any $\phi \neq y \subseteq \mathrm{On}$.

Choose any $\alpha \epsilon y$. (A single choice does not require an axiom of choice.) If α is the E-first element of y, we are through. If not, then $\alpha \cap y \neq \phi$. Now we use the condition $\mathsf{Sub}(\alpha)$ to show that $\alpha \cap y$ is a set. Thus $\alpha \cap y$ has an E-first element, say α_0, since $\mathsf{Wo}_{\mathsf{E}}(\alpha)$. Then we easily show that α_0 is the E-first element of y.

Corollary 9. $\mathsf{Wo}_{\mathsf{S}}(\mathsf{On})$

When we speak of an ordering among the ordinals we of course mean the natural ordering, denoted by $<_0$: in symbols,

$$\alpha <_0 \beta \longleftrightarrow \alpha \epsilon \beta \longleftrightarrow \alpha \subset \beta\ .$$

We say that α is a *successor ordinal* if and only if α immediately follows some β. We say that α is a *limit ordinal* if and only if $\alpha \neq \phi$ and for any β less than α we can always find another ordinal between β and α. In symbols we have, respectively,

$$\mathsf{Suc}(\alpha) \longleftrightarrow \exists \beta [\beta <_0 \alpha\ .\ \neg \exists \gamma\ [\beta <_0 \gamma <_0 \alpha\ ]]$$
$$\mathsf{Lim}(\alpha) \longleftrightarrow \phi \neq \alpha\ .\ \forall \beta [\beta <_0 \alpha \ \rightarrow \exists \gamma\ [\beta <_0 \gamma <_0 \alpha\ ]]$$

Theorem 10. $\mathsf{Suc}(\alpha) \longleftrightarrow \bigcup \alpha \subset \alpha$

Proof: We have $\mathsf{Suc}(\alpha)$

$$\Longleftrightarrow \exists \beta [\beta \epsilon \alpha\ .\ \neg \exists \gamma\ [\beta \epsilon \gamma \epsilon \alpha]]$$

$$\Longleftrightarrow \exists \beta [\beta \epsilon \alpha\ .\ \beta \notin \bigcup \alpha]$$

$$\Longleftrightarrow \bigcup \alpha \subset \alpha \text{ (since } \bigcup \alpha \subseteq \alpha \text{ by } \mathsf{Trans}(\alpha)).$$

Theorem 11. $\mathsf{Lim}(\alpha) \longleftrightarrow \bigcup \alpha = \alpha \neq \phi$

Proof: We have $\mathsf{Lim}(\alpha)$

$$\Longleftrightarrow \alpha \neq \phi\ .\ \forall \beta [\beta \epsilon \alpha \ \rightarrow \exists \gamma [\beta \epsilon \gamma \epsilon \alpha]]$$

$$\Longleftrightarrow \alpha \neq \phi\ .\ \forall \beta [\beta \epsilon \alpha \ \rightarrow \beta \epsilon \bigcup \alpha]$$

$$\Longleftrightarrow \alpha \neq \phi\ .\ \alpha \subseteq \bigcup \alpha$$

$$\Longleftrightarrow \alpha \neq \phi\ .\bigcup \alpha = \alpha \text{ (since } \bigcup \alpha \subseteq \alpha \text{ by } \mathsf{Trans}(\alpha)).$$

Theorem 12. $\alpha = \phi \not\equiv \mathsf{Suc}(\alpha) \not\equiv \mathsf{Lim}(\alpha)$.

Proof: $\mathsf{Trans}(\alpha) \Longrightarrow \bigcup \alpha \subseteq \alpha$

$$\Longrightarrow \bigcup \alpha \subset \alpha \not\equiv \bigcup \alpha = \alpha$$
$$\Longrightarrow \bigcup \alpha \subset \alpha \not\equiv [(\alpha = \phi . \bigcup \alpha = \alpha) \not\equiv (\alpha \neq \phi . \bigcup \alpha = \alpha)]$$
$$\Longrightarrow \mathsf{Suc}(\alpha) \not\equiv \alpha = \phi \not\equiv \mathsf{Lim}(\alpha)\ .$$

Let us take a little closer look at successor ordinals. Let us say that β *is a successor of* α if and only if

$$\alpha < \beta\ .\ \neg \exists \gamma [\alpha < \gamma < \beta].$$

Clearly, if $\mathrm{Suc}(\beta)$, then β is the successor of a unique ordinal α. However given any ordinal α we can't prove in P' that there is some β which is the successor of α. But we can say a few things. Let $X^+ = X \cup \{X\}$.

Theorem 13. *β is the successor of $\alpha \longleftrightarrow \beta = \alpha^+$.*

Proof: If $\alpha < \beta$ we have $\alpha \in \beta$, hence $\alpha \subseteq \beta$, hence $\alpha \cup \{\alpha\} \subseteq \beta$. Conversely if $\gamma \in \beta$, then $\gamma \leq \alpha$ since $\neg(\alpha < \gamma < \beta)$, hence $\gamma \in \alpha$ or $\gamma = \alpha$, hence $\gamma \in \alpha \cup \{\alpha\}$; thus $\beta \subseteq \alpha \cup \{\alpha\}$.

If $\beta = \alpha \cup \{\alpha\}$, then $\alpha \in \beta$, hence $\alpha < \beta$. Also if $\gamma < \beta$, then $\gamma \in \beta$, hence $\gamma \in \alpha$ or $\gamma = \alpha$, hence $\gamma \leq \alpha$, hence $\neg(\alpha < \gamma < \beta)$.

Corollary 14. $\mathrm{Suc}(\beta) \rightarrow \beta = \alpha^+$ *where* $\alpha = \bigcup \beta$

Proof: Clearly $\beta = \alpha^+$ for some unique α, by 13. It is straightforward to show that $\alpha = \bigcup \beta$.

Corollary 15. $\bigcup \beta \in \mathrm{On}$ *for any ordinal* β.

Proof: If $\beta = \phi$ or $\mathrm{Lim}(\beta)$, then $\bigcup \beta = \beta$, hence $\bigcup \beta \in \mathrm{On}$. If $\mathrm{Suc}(\beta)$, then $\bigcup \beta = \alpha$ where $\alpha^+ = \beta$, hence $\bigcup \beta \in \mathrm{On}$.

Let us now define the class ω of natural numbers in the usual fashion:

$$x \in \mathrm{K}_{\mathrm{I}} \longleftrightarrow (x = \phi \vee \mathrm{Suc}(x)) \,.\, x \in \mathrm{On}$$
$$x \in \omega \longleftrightarrow x \in \mathrm{K}_{\mathrm{I}} \,.\, (\forall u)_x\, u \in \mathrm{K}_{\mathrm{I}}\,.$$

Let i, j, k, l, m, n denote integers. For most of the propositions 1 - 15 there are corresponding propositions which one obtains by replacing ordinals and the class On by integers and the class ω. Denote these corresponding propositions by 1_ω - 15_ω. Now 1_ω and 4_ω - 10_ω follow immediately from 1 and 4 - 10 respectively since $\omega \subseteq \mathrm{On}$. To prove 2_ω, i.e., $\mathrm{Trans}(\omega)$, consider any $x \in n$. We need to show $x \in \omega$. But clearly $x \in \mathrm{K}_{\mathrm{I}}$ and

$$u \in x \in n \Longrightarrow u \in n \text{ (by 2)} \Longrightarrow u \in \mathrm{K}_{\mathrm{I}}\,.$$

Of course 3_ω follows from 2_ω. By definition of K_{I} and ω we have that none of natural numbers are limit ordinals hence 11_ω is vacuously true and pointless. Corresponding to 12 we have the following:

Theorem 12_ω. $\forall n[n = \phi \not\equiv \mathrm{Suc}(n)]$.

Finally 13_ω - 15_ω follow easily from 13 - 15.

There are many forms of induction theorems we can prove.

Theorem 16. *Assume X is a transitive class of ordinals. Then* $\forall x[(\forall \beta)_X[\beta \subseteq x \rightarrow \beta \in x] \rightarrow X \subseteq x]$

Proof: Assume we have $(\forall \beta)_X[\beta \subseteq x \rightarrow \beta \in x]$, yet $X \not\subseteq x$. Choose any $\beta \in X - x$. If β is the least ordinal in $X - x$, let $\beta_0 = \beta$. Otherwise there are ordinals less than β in $X - x$, i.e., $(\beta \cap X) - x \neq \phi$. But $\beta \cap X = \beta$ by $\mathrm{Trans}(X)$. Hence $\beta - x \neq \phi$. But $\beta - x$ is a set by $\mathrm{Sub}(\beta)$, hence $\beta - x$ has a first element, say β_0 which clearly is also the least ordinal in $X - x$. In any case we can say that $X - x$ has an E-first element β_0, hence $\beta_0 \subseteq x$. But then $\beta_0 \in x$ by hypothesis; contradiction.

Corollary 17_ω. $\forall x[\forall n[n \subseteq x \rightarrow n \epsilon x] \rightarrow \omega \subseteq x]$

Theorem 18_ω. $\forall x[0 \epsilon x . \forall n[n \epsilon x \rightarrow n^+ \epsilon x] \rightarrow \omega \subseteq x]$.

Proof: Assume $\omega \nsubseteq x$. As in 16, we can show that $\omega - x$ has a unique first element, say n_0. Now $n_0 \neq 0$ since $0 \epsilon x$. Hence $\mathrm{Suc}(n_0)$ by 12_ω, hence $n_0 = m_0^+$ for some unique $m_0 \epsilon \omega$ by 14_ω. By definition of n_0, we must have $m_0 \epsilon x$, hence $m_0 = n_0 \epsilon x$ by hypothesis, a contradiction.

Of course to show the actual existence of an ordinal, natural number of any set requires more axioms of set theory. In P' we can show the existence of at least one proper class, viz. Russell's class

$$\mathrm{R} = \{x \mid x \notin x\}.$$

To show class On is proper seems to require the additional weak axiom

$$\forall x, y\ [x \cap y \epsilon \mathrm{V}].$$

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