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ISOMORPHISMS OF ω-GROUPS

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1. *Introduction** Let ε stand for the set of non-negative integers (*numbers*), V for the class of all subcollections of ε(*sets*), and Λ for the set of isols. A function has as its domain and range subsets of ε. If f is a function we write δf and ρf for its domain and range respectively. The relation of inclusion is denoted by $\subset$ and the sets α and β are recursively equivalent (written: $\alpha \simeq \beta$), if $\delta f = \alpha$ and $\rho f = \beta$ for some function f with a one-to-one partial recursive extension. We denote the recursive equivalence type of α, $\{\sigma \epsilon V | \sigma \simeq \alpha\}$, by $\mathrm{Req}(\alpha)$. The reader is assumed to be familiar with the contents of [3].

The concept of an ω-group was studied by Hassett. He defined two ω-groups to be recursively isomorphic if there is an isomorphism between them which has a one-to-one partial recursive extension. In this paper we will reserve the term recursive isomorphism for a mapping between two r.e. groups and for arbitrary ω-groups we will refer to a recursive isomorphism as an ω-isomorphism (written: $G_1 \cong_\omega G_2$, for ω-groups G_1 and G_2).

It is natural to ask if the partial recursive extension of an ω-isomorphism is itself a recursive isomorphism from a r.e. group onto a r.e. group. For arbitrary ω-groups this question remains open. However, this question can be settled positively in the case of ω-groups of the form $P(\sigma)$ for an isolated set σ. In Proposition P4 we present a proof of this result. It is also of interest to consider when an ω-automorphism of an ω-group can be extended to a recursive automorphism of a r.e. group. This question can also be answered positively, P3, in the case of ω-groups of the form $P(\sigma)$ for σ an isolated set. Finally we show in P8 that every ω-automorphism of $P(\sigma)$, for an immune set σ, is an inner ω-automorphism if and only if $\mathrm{Req}\ (\sigma)$ is multiple-free.

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2. *Basic concepts* We need the following theorem [1, Prop. 1].

(1) *f has a one-to-one partial recursive extension if and only if f and f^{-1} have partial recursive extensions and f is one-to-one.*

Definition. An ω-isomorphism ϕ from an ω-group G_1 onto an ω-group G_2 is *regular* if there exist r.e. supergroups G_1' and G_2' of G_1 and G_2 respectively and a recursive isomorphism ϕ_0 from G_1' onto G_2' such that ϕ_0 is an extension of ϕ.

Definition. A *recursive automorphism* of a r.e. group G is a recursive isomorphism from G onto itself.

Definition. Let ϕ be an automorphism of the π-group $P(\alpha)$. Then ϕ is called

(i) an *ω-automorphism* of $P(\alpha)$, if ϕ is an ω-isomorphism from $P(\alpha)$ onto itself,

(ii) a *strong* ω-automorphism of $P(\alpha)$, if ϕ can be extended to a recursive automorphism ϕ_0 of a π-group of the form $P(\alpha_0)$, where α_0 is some r.e. superset of α.

The question of whether every ω-automorphism of an isolated π-group can be extended to a recursive ω-automorphism of a r.e. supergroup will be settled if we can show that every ω-automorphism of an isolated π-group is strong. This is what we will do. For this purpose we introduce the following concepts.

Definition. A *recursive permutation* is a partial recursive function g such that g is a permutation of the r.e. set δg.

Definition. Let f be a permutation of the set α. Then f is called

(i) an *ω-permutation* of α, if f has a one-to-one partial recursive extension,

(ii) a *regular* ω-permutation of α, if f can be extended to a recursive permutation f_0 of some r.e. superset α_0 of α.

Proposition P1. *Let f be a permutation of the set α. If α is isolated, f is a regular ω-permutation of α if and only if f is an ω-permutation of α.*

Proof. Left to reader.

Notation. In this paper we will denote the group of all finite permutations of a set σ, i.e., those permutations which move only finitely many elements of σ, by $\mathcal{P}(\sigma)$. Also if $f, g \in \mathcal{P}(\sigma)$ and $n \in \sigma$ then we denote f applied to n by $(n)f$ and $(n)(fg) = ((n)f)g$.

Remark. We recall from [3] that a permutation f of a set σ is a member of $\mathcal{P}(\sigma)$ if f moves finitely many elements of σ and $\delta f = \varepsilon$. In the following discussion we will run across permutations, f, whose domain is not all of ε. However, if such permutations move only finitely many elements of σ then it is clear that they can be extended to a member of $\mathcal{P}(\sigma)$. Thus when we say $f \in \mathcal{P}(\sigma)$ we really mean the extension of f to ε, which moves the same elements as f moves, is a member of $\mathcal{P}(\sigma)$; and that the Gödel number f^* of such a permutation f is the Gödel number of this extension to ε.

3. *Strong ω-automorphisms*

Notation. We let $\eta = \{f^* \mid f \in \mathcal{P}(\varepsilon)\}$.

Notation. Let $\mathcal{J} \subset \mathcal{P}(\varepsilon)$, then $\mathcal{J}^* = \{f^* \in \eta \mid f \in \mathcal{J}\}$.

Notation. Let ϕ be a function from a subfamily $\mathcal{J}$ of $\mathcal{P}(\varepsilon)$ into $\mathcal{P}(\varepsilon)$. Then ϕ^* is the function from a subset of η into η such that

(i) $\delta\phi^* = \mathcal{J}^*$, and
(ii) for $f \in \mathcal{J}$, $\phi(f) = g$ if and only if $\phi^*(f^*) = g^*$.

Definition. A function ϕ from a subfamily of $\mathcal{P}(\varepsilon)$ into $\mathcal{P}(\varepsilon)$ is *effectively computable* if the function ϕ^* from a subset of η into η is partial recursive.

Proposition P2. *Let α be an immune set. (a) If σ is an ω-permutation of α and*

$$\psi^*[f^*] = (\sigma^{-1} f \sigma)^*, \quad \text{for } f^* \in P(\alpha)\,, \tag{2}$$

then ψ^* is a strong ω-automorphism of $P(\alpha)$. (b) For every ω-automorphism ψ^* of $P(\alpha)$, there is exactly one ω-permutation σ of α such that (2) holds.

Proof: Let $\mathcal{G}(\alpha)$ be the group of all permutations of α and $\mathcal{J}(\alpha)$ the group of all restrictions to α of functions in $\mathcal{P}(\alpha)$. Suppose that $\sigma \in \mathcal{G}(\alpha)$. Then the mapping

$$\phi : f \to \sigma^{-1} f \sigma, \quad \text{for } f \in \mathcal{G}(\alpha)\,,$$

is an inner automorphism of $\mathcal{G}(\alpha)$. Since $\mathcal{J}(\alpha)$ is a normal subgroup of $\mathcal{G}(\alpha)$, we see that the restriction of ϕ to $\mathcal{J}(\alpha)$ is an automorphism of $\mathcal{J}(\alpha)$. Thus the mapping

$$\phi_\sigma : f \to \sigma^{-1} f \sigma, \quad \text{for } f \in \mathcal{P}(\alpha)\,,$$

is an automorphism of $\mathcal{P}(\alpha)$, while the mapping ψ^* defined by (2) is an automorphism of $P(\alpha)$. We now make the additional assumption that the set α is immune and σ is an ω-permutation of α. By P1 there exists an extension σ' of σ which is a recursive permutation of a r.e. superset α' of α. Define

$$\phi_{\sigma'}[f] = (\sigma')^{-1} f \sigma', \quad \text{for } f \in \mathcal{P}(\alpha')\,,$$

$$(\psi')^*[f] = ((\sigma')^{-1} f \sigma')^*, \quad \text{for } f^* \in P(\alpha')\,.$$

Then $\phi_{\sigma'}$ is an automorphism of $\mathcal{P}(\alpha')$ which is an extension of ϕ_σ, while $(\psi')^*$ is a recursive automorphism of $P(\alpha')$ which is an extension of ψ^*. Hence ψ^* is a strong ω-automorphism of $P(\alpha)$.

(b) We shall first prove that for every automorphism ψ^* of $P(\alpha)$, there is at most one permutation σ of α related to ψ^* by (2). Let σ_1 and σ_2 be permutations of α. Consider the corresponding automorphisms of $\mathcal{J}(\alpha)$, the mappings ϕ_1 and ϕ_2 such that for $f \in \mathcal{J}(\alpha)$,

$$\phi_1(f) = \sigma_1^{-1} f \sigma_1, \ \phi_2(f) = \sigma_2^{-1} f \sigma_2.$$

We shall use the following lemma, whose proof is left to the reader.

Lemma. *Let σ be a permutation of α and ϕ the automorphism of $\mathcal{F}(\alpha)$ such that*

$$\phi: f \to \sigma^{-1} f \sigma, \ \textit{for } f \in \mathcal{F}(\alpha).$$

If ϕ is the identity mapping on $\mathcal{F}(\alpha)$, then σ is the identity permutation of α.

Now assume $\phi_1 = \phi_2$, i.e., $\phi_1(f) = \phi_2(f)$, for $f \in \mathcal{F}(\alpha)$. Then it follows by the lemma that $\sigma_1\sigma_2^{-1}$ is the identity permutation of α, i.e., that $\sigma_1 = \sigma_2$. Thus for every ω-automorphism ψ^* of $P(\alpha)$, there is at most one ω-permutation σ of α such that (2) holds.

Let, for any ω-permutation σ of α,

$$(\phi_\sigma)^*: f \to (\sigma^{-1} f \sigma)^*, \text{ for } f^* \in P(\alpha).$$

We proceed to prove that for every ω-automorphism ψ^* of $P(\alpha)$, there is at least one ω-permutation σ of α such that $\psi^* = (\phi_\sigma)^*$. Our proof is suggested by Kent's proof of a related but different theorem, [4, p. 360].

Assume that ψ^* is an ω-automorphism of $P(\alpha)$. Then ψ^* has a one-to-one partial recursive extension, say $(\psi_0)^*$. Since the set of all Gödel numbers of finite permutations is recursive, we may assume without loss of generality that $\delta(\psi_0)^*$ and $\rho(\psi_0)^*$ consist of Gödel numbers of finite permutations. Let

$$\text{for } f^* \in P(\alpha), \ \psi(f) = g \text{ mean: } \psi^*(f^*) = g^*,$$

$$\text{for } f^* \in \delta(\psi_0)^*, \ \psi_0(f) = g \text{ mean: } \psi_0^*(f^*) = g^*.$$

Thus ψ is an automorphism of $\mathcal{P}(\alpha)$ and ψ_0 is a one-to-one extension of ψ which is effectively computable. We point out three properties of the mapping ψ.

(i) Let the *order*, $o(f)$, of a finite permutation f be the unique number n such that $f^n = i$, $f^m \neq i$, for $0 < m < n$, where i is the identity permutation. Then $o(\psi(f)) = o(f)$, for $f \in \mathcal{P}(\alpha)$.

(ii) Let $M_1, M_2 \subset \mathcal{P}(\alpha)$. If $\psi(M_1) \subset M_2$ and $\psi^{-1}(M_2) \subset M_1$, then $\psi(M_1) = M_2$.

(iii) If C is the conjugacy class of $\mathcal{P}(\alpha)$, so is $\psi(C)$.

Let C_t be the conjugacy class of $\mathcal{P}(\alpha)$ which consists of all transpositions of elements in α (completed to functions defined on ε). We say that a subset $\mathcal{S}$ of $\mathcal{P}(\alpha)$ has property Γ, if

$$(f_1, f_2, g_1, g_2 \in \mathcal{S}) \text{ and } (o(f_1 g_1) = o(f_2 g_2)) \text{ then } f_1 g_1 \text{ is conjugate to } f_2 g_2.$$

We claim that C_t is the one and only conjugacy class of $\mathcal{P}(\alpha)$ which has property Γ and consists of elements of order 2. First of all, it is readily seen that C_t has property Γ, and C_t trivially consists of elements of order 2. Now suppose $\mathcal{D}$ is a conjugacy class of $\mathcal{P}(\alpha)$ which contains only elements of order 2 and such that $\mathcal{D} \neq C_t$. Let $f \in \mathcal{D}$ and $f = \gamma_1 \ldots \gamma_k$ be the decomposition of f into disjoint cycles. Then $o(f) = 2$ implies $o(\gamma_i) = 2$, for $1 \leq i \leq k$. Also $\mathcal{D} \neq C_t$ entails $k \geq 2$.

Let h_n be the principal function of α, i.e., the function which enumerates α in increasing order. Put

$$\begin{aligned} f_1 &= (h_1, h_2)(h_3, h_4) \ldots (h_{2k-1}, h_{2k}), \\ g_1 &= (h_{2k+1}, h_{2k+2}) \ldots (h_{4k-1}, h_{4k}), \\ f_2 &= f_1, \\ g_2 &= (h_1, h_2)(h_{2k+1}, h_{2k+2}) \ldots (h_{4k-3}, h_{4k-2}). \end{aligned}$$

Hence f_1, f_2, g_1, $g_2 \in \mathfrak{D}$, since they belong to the same cycle class as f. However,

$$\begin{aligned} f_1 \cdot g_1 &= (h_1, h_2) \ldots (h_{4k-1}, h_{4k}), \\ f_2 \cdot g_2 &= (h_3, h_4) \ldots (h_{4k-3}, h_{4k-2}) \end{aligned}$$

both have order 2, but are not conjugate. Thus $\mathfrak{D}$ does not have property Γ.

We now consider $\psi(\mathcal{C}_t)$. This is a conjugacy class of $\mathcal{P}(\alpha)$ by (iii) and consists of elements of order 2 by (i). It is easily seen that property Γ is preserved under ψ, hence $\psi(\mathcal{C}_t)$ has property Γ. Therefore $\psi(\mathcal{C}_t) = \mathcal{C}_t$. Define

$$T_n = \{(n, x) \mid x \in \alpha \text{ and } x \neq n\}, \text{ for } n \in \alpha.$$

We observe that

(a) the product of any two distinct elements of T_n has order 3,

(b) if the product of two transpositions is of order 3, they have exactly one element in common.

Using (a), (b) and (ii) we obtain

$$n \in \alpha \Rightarrow (\exists m)[m \in \alpha \text{ and } \psi(T_n) = T_m].$$

Let, for $n \in \alpha$, $(n)\sigma = m$ mean: $\psi(T_n) = T_m$. It follows that σ maps α into itself. For $p, q \in \alpha$,

$$p \neq q \Rightarrow T_p \neq T_q \Rightarrow \psi(T_p) \neq \psi(T_q) \Rightarrow T_{(p)\sigma} \neq T_{(q)\sigma} \Rightarrow (p)\sigma \neq (q)\sigma.$$

Moreover, since ψ^{-1} is also an automorphism of $\mathcal{P}(\alpha)$,

$$\begin{aligned} m \in \alpha &\Rightarrow (\exists n)\,[n \in \alpha \text{ and } T_n = \psi^{-1}(T_m)] \\ &\Rightarrow (\exists n)\,[n \in \alpha \text{ and } \psi(T_n) = T_m] \\ &\Rightarrow m \in (\alpha)\sigma. \end{aligned}$$

Hence σ is a permutation of α. Let x_1, x_2, x_3 be three distinct elements of α. Suppose any number $n \in \alpha$ is given. Then at least two of the three numbers x_1, x_2, x_3 are different from n. We may assume without loss of generality that $n \neq x_1$ and $n \neq x_2$. Since ψ has an effectively computable extension, namely ψ_0, we can compute

$$\psi[(n, x_1)] = (m, y_1),\ \psi[(n, x_2)] = (m, y_2),$$

where m, y_1, y_2 are distinct elements of α. Then

$$(n)\sigma = m = \text{common element of } (m, y_1) \text{ and } (m, y_2),$$

can be effectively found. It is readily proved that σ has a partial recursive extension. Note that ψ^{-1} also has an effectively computable extension, namely ψ_0^{-1}. Thus given any number $m \in \alpha$ we can (assuming without loss of generality that $m \neq x_1$, $m \neq x_2$) compute

$$\psi^{-1}[(m, x_1)] = (n, y_3),\ \psi^{-1}[(m, x_2)] = (n, y_4),$$
$$(m)\sigma^{-1} = n = \text{the common element of } (n, y_3) \text{ and } (n, y_4).$$

It can be proved (in the same way as for σ) that σ^{-1} has a partial recursive extension. We conclude by (1) that the permutation σ of α has a one-to-one partial recursive extension, i.e., that σ is an ω-permutation of α.

It remains to be shown that

$$\psi^*(f^*) = (\sigma^{-1} f \sigma)^*, \text{ for } f^* \in P(\alpha),$$

or equivalently that

(3) $$\psi(f) = \sigma^{-1} f \sigma, \text{ for } f \in \boldsymbol{P}(\alpha).$$

Define $\theta(f) = \sigma\psi(f)\sigma^{-1}$, for $f \in \boldsymbol{P}(\alpha)$. Let f be a transposition of α, say $f = (n, x)$. Suppose that $\psi(f) = (m, y)$, then $(n)\sigma = m$, $(x)\sigma = y$ or vice versa. We will assume $(n)\sigma = m$, $(x)\sigma = y$. Then

$$(n)\theta(f) = (n)[\sigma\psi(f)\sigma^{-1}] = (m)[\psi(f)\sigma^{-1}] = (y)\sigma^{-1} = x,$$
$$(x)\theta(f) = (x)[\sigma\psi(f)\sigma^{-1}] = (y)[\psi(f)\sigma^{-1}] = (m)\sigma^{-1} = n.$$

Moreover, if $z \in \alpha$, but $z \notin \{x, n\}$,

$$(z)\theta(f) = (z)[\sigma\psi(f)\sigma^{-1}] = (z)\sigma\sigma^{-1} = z.$$

We have therefore proved that $\theta(f) = f$, for every transposition f of α. The definition of θ implies that θ is an automorphism of $\boldsymbol{P}(\alpha)$. Thus, since every element of $\boldsymbol{P}(\alpha)$ can be expressed as a product of finitely many transpositions of α, θ must be the identity mapping. Hence $\sigma\psi(f)\sigma^{-1} = f$, for $f \in \boldsymbol{P}(\alpha)$ and $\psi(f) = \sigma^{-1} f \sigma$, for $f \in \boldsymbol{P}(\alpha)$. Thus (3) holds and this completes the proof.

Proposition P3. *Let α be a non-empty isolated set. Then every ω-automorphism of $P(\alpha)$ is strong.*

Proof: Let α be finite. Then every automorphism of $P(\alpha)$ is a recursive automorphism, hence we are through. If, on the other hand, α is immune, the desired result follows from the two parts of P2.

4. *Regular ω-isomorphisms*

Proposition P4. *Let α and β be non-empty isolated sets. Then every ω-isomorphism from $P(\alpha)$ onto $P(\beta)$ is regular.*

Proof: Let ψ^* be an ω-isomorphism from $P(\alpha)$ onto $P(\beta)$, then ψ^* has a one-to-one partial recursive extension, say ψ_0^*. We may assume without loss of generality that both $\delta\psi_0^*$ and $\rho\psi_0^*$ consist of Gödel numbers of finite permutations. Define $\delta = \delta\psi_0^*$, $\rho = \rho\psi_0^*$, $\mathfrak{D} = \{f \in \boldsymbol{P}(\varepsilon) \mid f^* \in \delta\}$, and

$$\text{for } f \in \boldsymbol{P}(\alpha),\ \psi(f) = g \text{ means: } \psi^*(f^*) = g^*,$$
$$\text{for } f \in \mathfrak{D},\ \psi_0(f) = g \text{ means: } \psi_0^*(f^*) = g^*.$$

Then ψ is an isomorphism from $\boldsymbol{P}(\alpha)$ onto $\boldsymbol{P}(\beta)$ which has an effectively computable one-to-one extension, namely ψ_0. Let $A = \mathrm{Req}(\alpha)$, $B = \mathrm{Req}(\beta)$, then $o(P(\alpha)) = A!$, $o(P(\beta)) = B!$. The fact that ψ_0^* maps $P(\alpha)$ onto $P(\beta)$ implies that $A! = B!$. Since $A, B > 0$, it follows that $A = B$, i.e., $\alpha \simeq \beta$, say by p. Define $\mathcal{X}(g) = pgp^{-1}$, for $g \in \boldsymbol{P}(\beta)$, and $\mathcal{X}_0(g) = pgp^{-1}$, for $g \in \boldsymbol{P}(\rho p)$. Note that for $g \in \boldsymbol{P}(\beta)$,

$$n \in \alpha \Rightarrow (n)p \in \beta \Rightarrow (n)pg \in \beta \Rightarrow (n)pgp^{-1} \in \alpha \Rightarrow (n)[\mathcal{X}(g)] \in \alpha.$$

Since g changes at most finitely many elements of β, $\mathcal{X}(g)$ changes at most finitely many elements of α. Thus $g \in \boldsymbol{P}(\beta)$ implies that $\mathcal{X}(g) \in \boldsymbol{P}(\alpha)$. It is readily proved that $\mathcal{X}$ is an isomorphism from $\boldsymbol{P}(\beta)$ onto $\boldsymbol{P}(\alpha)$. Similarly, one can prove that $\mathcal{X}_0$ is an isomorphism from $\boldsymbol{P}(\rho p)$ onto $\boldsymbol{P}(\delta p)$. Moreover, $\mathcal{X}_0$ is an effectively computable extension of $\mathcal{X}$. Put $\theta = \mathcal{X}\psi$,

$$\mathfrak{D}_0 = \{f \in \mathfrak{D} \mid \psi_0(f) \in \boldsymbol{P}(\rho p)\},$$
$$\theta_0(f) = \mathcal{X}_0\psi_0(f), \text{ for } f \in \mathfrak{D}_0.$$

Then θ is an automorphism of $\boldsymbol{P}(\alpha)$ which has an effectively computable one-to-one extension, namely θ_0. Define

$$\text{for } f^* \in P(\alpha),\ \theta^*(f^*) = g^* \text{ means: } \theta(f) = g,$$
$$\text{for } f^* \in \mathfrak{D}_0^*,\ \theta_0^*(f^*) = g^* \text{ means: } \theta_0(f) = g.$$

It follows that θ^* is an automorphism of $P(\alpha)$ which has a partial recursive one-to-one extension, namely θ_0^*. Hence θ^* is an ω-automorphism of $P(\alpha)$. Applying P2 we conclude that there exists an ω-permutation σ of α such that $\theta(f) = \sigma^{-1}f\sigma$, for $f \in \boldsymbol{P}(\alpha)$. Substituting $\mathcal{X}\psi$ for θ and using the definition of $\mathcal{X}$, we obtain

$$\mathcal{X}\psi(f) = \sigma^{-1}f\sigma, \text{ for } f \in \boldsymbol{P}(\alpha),$$
$$p\psi(f)p^{-1} = \sigma^{-1}f\sigma, \text{ for } f \in \boldsymbol{P}(\alpha),$$
$$\psi(f) = (\sigma p)^{-1}f(\sigma p), \text{ for } f \in \boldsymbol{P}(\alpha).$$

The ω-permutation σ of α has a partial recursive one-to-one extension, say σ_0. Define

$$\alpha_0 = \{x \in \delta\sigma_0 \mid (x)\sigma_0 \in \delta p\},$$
$$(x)q = (x)\sigma_0 p, \text{ for } x \in \alpha_0,$$
$$\beta_0 = (\alpha_0)q.$$

Then $\alpha \subset \alpha_0 \subset \delta p$, where α_0 is r.e. Since q is the composition of two partial recursive one-to-one functions, it is itself a partial recursive one-to-one function. Moreover, q is an extension of σp. While σp maps α onto β, q maps the r.e. superset α_0 of α onto the r.e. superset β_0 of β. Define $\psi_1(f) = q^{-1}fq$, for $f \in \boldsymbol{P}(\alpha_0)$. Then ψ_1 is an extension of ψ, an isomorphism from $\boldsymbol{P}(\alpha_0)$ onto $\boldsymbol{P}(\beta_0)$ and an effectively computable mapping. Let for $f^* \in P(\alpha_0)$, $\psi_1^*(f^*) = g^*$ mean: $\psi_1(f) = g$. It follows that ψ_1^* is a recursive isomorphism from the r.e. supergroup $P(\alpha_0)$ of $P(\alpha)$ onto the r.e. supergroup

$P(\beta_0)$ of $P(\beta)$. Since ψ_1^* is also an extension of ψ^*, the ω-isomorphism ψ^* is regular.

5. *Inner ω-automorphisms*

Definition. Let ϕ be an automorphism of the ω-group G. Then ϕ is an *inner* (or *outer*) ω-automorphism of G, if ϕ is both an inner (respectively outer) automorphism of G and an ω-automorphism of G.

Notations. For an ω-group G,

$\mathrm{Aut}(G)$ = the group of all automorphisms of G,
$\mathrm{In}(G)$ = the group of all inner automorphisms of G,
$\mathrm{Aut}_\omega(G)$ = the group of all ω-automorphisms of G,
$\mathrm{In}_\omega(G)$ = the group of all inner ω-automorphisms of G.

Notation. We write $H \leq G$ for H is a subgroup of the group G.

Remark. We immediately see that for an ω-group G, we have $\mathrm{Aut}_\omega(G) \leq \mathrm{Aut}(G)$ and $\mathrm{In}_\omega(G) \leq \mathrm{In}(G)$. The second relation can be strengthened.

Proposition P5. *Let G be an ω-group. Then every inner automorphism of G is an ω-automorphism of G, i.e.,* $\mathrm{In}(G) = \mathrm{In}_\omega(G)$.

Proof: Left to reader.

Proposition P6. *Let α be an infinite set. Then the ω-group $P(\alpha)$ has exactly c automorphisms. Among them exactly $\aleph_0$ are inner automorphisms and exactly c are outer automorphisms.*

Proof: For every permutation σ of α, the mapping

$$\phi_\sigma: f \to \sigma^{-1} f \sigma, \text{ for } f \in \boldsymbol{P}(\alpha)$$

is an automorphism of $\boldsymbol{P}(\alpha)$. In our proof of P2 we showed (only using the fact that α is infinite) that for permutations σ, τ of α,

$$\sigma \neq \tau \Rightarrow \phi_\sigma \neq \phi_\tau.$$

Taking into account that α has exactly c permutations, since α is denumerable, it follows that $\boldsymbol{P}(\alpha)$ has at least c automorphisms. However, there are only c mappings from $\boldsymbol{P}(\alpha)$ into itself, hence $\boldsymbol{P}(\alpha)$ has exactly c automorphisms. If σ ranges without repetition over the denumerable family of all finite permutations of α, ϕ_σ ranges without repetition over the family of all inner automorphisms of $\boldsymbol{P}(\alpha)$. Hence there are exactly $\aleph_0$ inner automorphisms of $\boldsymbol{P}(\alpha)$. The remaining c automorphisms of $\boldsymbol{P}(\alpha)$ must be outer ones.

Remark. Let α be an infinite set and $G = P(\alpha)$. In view of P6, $\mathrm{In}_\omega(G) \leq \mathrm{Aut}_\omega(G) < \mathrm{Aut}(G)$. We wish to find out for which immune sets α, $\mathrm{In}_\omega(G) < \mathrm{Aut}_\omega(G)$. This clearly depends only on $\mathrm{Req}(\alpha)$.

Definition. An isol A is *multiple-free* if for every isol B, $2B \leq A \Rightarrow B \in \varepsilon$.

It is readily seen that c isols are multiple-free while c are not. For if

X is an infinite indecomposable isol. X is multiple-free, but $2X$ is not. Moreover, there are exactly c infinite indecomposable isols.

Notation. (1) The cardinality of the set α will be denoted by $\mathrm{card}(\alpha)$.
(2) For two sets α and β, $\alpha|\beta$ means α is separable from β.

The following proposition is due to B. Cole.

Proposition P7. *Let $T \in \Lambda - \varepsilon$ and $\tau \in T$. Then there is an ω-permutation of τ which moves infinitely many elements of τ if and only if T is not multiple-free.*

Proof: Let $T \in \Lambda - \varepsilon$ and $\tau \in T$. Suppose that f is an ω-permutation of τ which moves infinitely many elements of τ. Define

$$\begin{aligned}
\gamma_x &= \{x, f(x), f^2(x), \ldots\}, \text{ for } x \in \delta f,\\
D &= \{\gamma_x \mid \gamma_x \text{ is finite}\},\\
\delta &= \text{union of all sets in } D,\\
\mathrm{c}(x) &= \min \gamma_x,\ \mathrm{d}(x) = \max \gamma_x, \text{ for } x \in \delta.
\end{aligned}$$

Then $\tau \subset \delta \subset \delta f$, where δ is an r.e. set. Also $\mathrm{c}(x)$ and $\mathrm{d}(x)$ are partial recursive functions. Put

$$\sigma' = \{x \in \delta \mid \mathrm{card}\ \gamma_x \geq 2\},\ \sigma = \{x \in \tau \mid \mathrm{card}\ \gamma_x \geq 2\},$$

then $\sigma \subset \sigma'$ where σ' is r.e. Define

$$\alpha' = \mathrm{c}(\sigma'),\ \beta' = \mathrm{d}(\sigma'),\ \alpha = \mathrm{c}(\sigma),\ \beta = \mathrm{d}(\sigma), A = \mathrm{Req}(\alpha).$$

Note that $\alpha \subset \alpha'$, $\beta \subset \beta'$, where α' and β' are disjoint r.e. sets. Hence $\alpha|\beta$. Moreover

$$\mathrm{c}(x) \to \mathrm{d}(x), \text{ for } x \in \sigma'$$

is a partial recursive one-to-one function which maps α onto β. It follows that $\alpha \simeq \beta$ and

$$\mathrm{Req}(\alpha \cup \beta) = \mathrm{Req}(\alpha) + \mathrm{Req}(\beta) = A + A = 2A. \tag{4}$$

Observe that $x \in \sigma'$ if and only if $\mathrm{c}(x) \neq \mathrm{d}(x)$, for $x \in \delta$, and $x \in \alpha' \cup \beta'$ if and only if $x = \mathrm{c}(x)$ or $x = \mathrm{d}(x)$, for $x \in \sigma'$. We conclude that for $x \in \delta$,

$$\begin{aligned}
x \in \alpha' \cup \beta' &\Leftrightarrow \mathrm{c}(x) \neq \mathrm{d}(x) \text{ and } [x = \mathrm{c}(x) \text{ or } x = \mathrm{d}(x)],\\
x \notin \alpha' \cup \beta' &\Leftrightarrow \mathrm{c}(x) = \mathrm{d}(x) \text{ or } [x \neq \mathrm{c}(x) \text{ and } x \neq \mathrm{d}(x)].
\end{aligned}$$

Since $\mathrm{c}(x)$ and $\mathrm{d}(x)$ are defined on δ, the sets $\alpha' \cup \beta'$ and $\delta - (\alpha' \cup \beta')$ are disjoint and r.e. Thus $\alpha \cup \beta \subset \alpha' \cup \beta'$, $\tau - (\alpha \cup \beta) \subset \delta - (\alpha' \cup \beta')$, and

$$\alpha \cup \beta | \tau - (\alpha \cup \beta),\ \mathrm{Req}(\alpha \cup \beta) \leq T. \tag{5}$$

Combining (4) and (5) we obtain $2A \leq T$. The set σ is infinite because f moves infinitely many elements of τ. This implies that the set α and the isol A are infinite. Hence T is not multiple-free.

To prove the converse we suppose that the infinite isol T is not multiple-free. Let A be an infinite isol such that $2A \leq T$. Suppose that $\alpha_1, \alpha_2 \in A$ and $\alpha_1 | \alpha_2$. Let $\alpha_1{}'$ and $\alpha_2{}'$ be disjoint r.e. sets and $h(x)$ a partial recursive one-to-one function such that

$$\alpha_1 \subset \alpha_1', \ \alpha_2 \subset \alpha_2', \ \alpha_1 \subset \delta h, \ h(\alpha_1) = \alpha_2.$$

We may assume that $\delta h = \alpha_1'$, $\rho h = \alpha_2'$. In view of the fact that $2A \leq T$, we may also suppose that

$$\alpha_1 \cup \alpha_2 \subset \tau, \ \alpha_1 \cup \alpha_2 \,|\, \tau - (\alpha_1 \cup \alpha_2).$$

Let γ and δ be disjoint r.e. sets such that $\alpha_1 \cup \alpha_2 \subset \gamma$ and $\tau - (\alpha_1 \cup \alpha_2) \subset \delta$. Define the functions $f(x)$, $f_0(x)$ by

$$\delta f = \tau, \ \delta f_0 = \delta \cup (\alpha_1' \cup \alpha_2') \cdot \gamma,$$

$$f(x) = \begin{cases} x & , \text{ if } x \notin \alpha_1 \cup \alpha_2, \\ h(x) & , \text{ if } x \in \alpha_1, \\ h^{-1}(x), & \text{ if } x \in \alpha_2, \end{cases}$$

$$f_0(x) = \begin{cases} x & , \text{ if } x \in \delta, \\ h(x) & , \text{ if } x \in \alpha_1' \cap \gamma, \\ h^{-1}(x), & \text{ if } x \in \alpha_2' \cap \gamma. \end{cases}$$

First of all, $f(x)$ is a permutation of τ and $f_0(x)$ a partial recursive one-to-one extension of $f(x)$. Thus $f(x)$ is an ω-permutation of τ. Moreover, f moves all elements of the subset α_1 of τ, while α_1 is infinite, since $\alpha_1 \in A$. It follows that f moves infinitely many elements of τ.

Remark. Proposition P7 enables us to characterize all immune sets α for which the ω-group $P(\alpha)$ has $\aleph_0$ outer ω-automorphisms.

Proposition P8. *Let α be an immune set. Then $P(\alpha)$ has $\aleph_0$ outer ω-automorphisms if and only if the isol* $\mathrm{Req}(\alpha)$ *is not multiple-free.*

Proof: Let for every ω-permutation σ of α, the ω-automorphism ϕ_σ^* be defined by

$$\phi_\sigma^* : f^* \to (\sigma^{-1} f \sigma)^*, \text{ for } f^* \in P(\alpha).$$

According to P2 every ω-automorphism of $P(\alpha)$ is of the form ϕ_σ^* for some σ and the mapping $\sigma \to \phi_\sigma^*$ is one-to-one. Clearly, ϕ_σ^* is an outer ω-automorphism of $P(\alpha)$ if and only if σ moves infinitely many elements of α. Thus by P7, $P(\alpha)$ has an outer ω-automorphism if and only if the isol $\mathrm{Req}(\alpha)$ is not multiple-free. But $P(\alpha)$ has at least one outer ω-automorphism implies $P(\alpha)$ has $\aleph_0$ outer ω-automorphisms since there are $\aleph_0$ distinct inner ω-automorphisms of $P(\alpha)$ and the composition of an outer ω-automorphism with an inner ω-automorphism is an outer ω-automorphism. This completes the proof.

Remark. Proposition P8 can also be phrased as follows: for an immune set α, $\mathrm{Aut}_\omega P(\alpha) = \mathrm{In}_\omega P(\alpha)$ if and only if $\mathrm{Req}(\alpha)$ is multiple-free and $\mathrm{card}(\mathrm{Aut}_\omega P(\alpha) - \mathrm{In}_\omega P(\alpha)) = \aleph_0$ if and only if $\mathrm{Req}(\alpha)$ is not multiple-free.

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