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NEW ALGORITHMS FOR THE STATEMENT AND CLASS CALCULI

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INTRODUCTION.

Efficient algorithms for making inferences in the statement calculus and in the calculus of classes may be developed with a fractional representation of statements. The idea is to take the transitivity of implication as the fundamental mode of inference. We seek then a representation in which hypothetical syllogism, which is an expression of this transitivity, appears in a convenient form. Both the inferences:

$$\begin{array}{ll} \textit{All A are B} & p \supset q \\ \textit{All B are C} & q \supset r \\ \textit{All A are C} & p \supset r \end{array}$$

work as if one "cancelled" the middle terms. This apparently simple-minded view can, as we shall show, be developed into a remarkably simple and perspicuous set of techniques for manipulation of logical relations between calsses and between statements.[1]

PART I: THE STATEMENT CALCULUS.

1. *Fractional Representation of the Statement Connectives.* With a few stipulated conventions of interpretation some of the basic logical relationships between the connectives of the statement calculus may be built into the notation itself. We represent the usual statement connectives in the following "fractional" form[2]:

$$\begin{array}{ll} p : \dfrac{1}{p} & \sim p \ : \dfrac{1}{p'} \\[2ex] p \supset q : \dfrac{p}{q} & p \vee q \ : \dfrac{1}{pq} \\[2ex] p \equiv q : \dfrac{p}{q} \cdot \dfrac{q}{p} & p \cdot q \ : \dfrac{1}{(p \cdot q)} \end{array}$$

It is convenient to adopt some arithmetical terminology in talking about

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the fractional representations. Thus we shall speak of "inversion" of fractions, "multiplication" of juxtaposed statement terms and later of "cancellation" of terms. The inverse of a statement letter p: $\frac{1}{p}$ may be taken in the sense of "assert that p." The fraction $\frac{p}{1}$ then represents the denial of the statement p. Inversion of statement letters will in fact correspond with negation of statements. Multiplication in the numerator is read as negation of the conjunction of the multiplied statements. Multiplication in the denominator is read as (assertion of) the disjunction of the statements. Thus "$\frac{pqr}{1}$" represents $(p \vee q \vee r)$, whereas "$\frac{pqr}{1}$" represents $\sim(p \cdot q \cdot r)$. In combination an expression such as "$\frac{p'q}{rs}$" represents $(\sim p \cdot q) \supset (r \vee s)$.

Each fraction represents a statement. Juxtaposition or multiplication of fractions, indicated by "$\cdot$", represents the conjunction of statements as a string of premises. We also use the dot symbol for conjunction within the denominator, but conjunctions within a fraction may always be removed as will be shown in section (2). Since conjunction is a commutative and associative operation among statements, multiplication of fractions is commutative and associative. Furthermore, since both conjunction and disjunction are commutative and associative, so is multiplication of statement terms in either numerator or denominator. Using "=" to abbreviate "is logically equivalent to" we then have: $\frac{1}{pq} = \frac{1}{qp}$, $\frac{p(qr)}{1} = \frac{(pq)r}{1}$, etc.

Some analogies with arithmetic fractions appear in the logical manipulations but these analogies are limited. The motivation for the representation and suggestion of its potential usefulness may be seen at once when we introduce an operation for negation of statements.

Negation Principle: Any expression which appears as a factor either in the numerator or in the denominator of a fraction may be moved across the fractional line to yield a logically equivalent fractional representation provided the expression is negated.

Assuming the principle of double negation: $p'' = p$, we have:

$$\frac{1}{p} = \frac{p'}{1} \text{ and } \frac{1}{p'} = \frac{p}{1}.$$

The following equivalent ways of writing "$p \supset q$" then appear in a simple application of the negation principle:

$$\frac{p}{q} = \frac{1}{p'q} = \frac{pq'}{1} = \frac{q'}{p'}.$$

Referring to the stipulated readings we have:

$$(p \supset q) \overset{L}{\equiv} (\sim p \vee q) \overset{L}{\equiv} \sim(p \cdot \sim q) \overset{L}{\equiv} (\sim q \supset \sim p).$$

2. *Equivalence Transformations.* To represent the statement form $(p \vee q) \supset (r \cdot s)$ we must indicate disjunction in the numerator and conjunction in the denominator: $\frac{(p \vee q)}{(r \cdot s)}$. The expression "$\frac{pq}{rs}$" represents rather $(p \cdot q) \supset (r \vee s)$. The convenience of the representation is, however, not marred by the necessity for indicating conjunction in the denominator and disjunction in the numerator because of the following "separation" rules:

$$\text{A. } \frac{(p \vee q)}{r} = \frac{p}{r} \cdot \frac{q}{r} \qquad \text{B. } \frac{p}{(q \cdot r)} = \frac{p}{q} \cdot \frac{p}{r}$$

We can thus simplify fractions in which the conjunction or disjunction is, so to speak, in the wrong place. The particular case: $\frac{(p \vee q)}{1} = \frac{p}{1} \cdot \frac{q}{1}$ is an expression of one of DeMorgan's laws. The particular case $\frac{1}{(p \cdot q)} = \frac{1}{p} \cdot \frac{1}{q}$ is an expression of the inferences rules of conjunction and simplication. We may apply (A) and (B) in combination: $\frac{(p \vee q)}{(r \cdot s)} = \frac{p}{r} \cdot \frac{q}{r} \cdot \frac{p}{s} \cdot \frac{q}{s}$. The latter equivalence has the special case, with p for r and q for s: $\frac{(p \vee q)}{(p \cdot q)} = \frac{p}{p} \cdot \frac{q}{p} \cdot \frac{p}{q} \cdot \frac{q}{q} = \frac{p}{q} \circ \frac{q}{p}$. (We use here the principle that tautologies such as $\frac{p}{p}$ may be inserted or removed without altering the truth value of the conjunction of fractions). We have thus represented "$(p \equiv q)$" in the form:

$$[(p \vee q) \supset (p \cdot q)].$$

There is, unfortunately, no straightforward way of "multiplying" the fractions. In particular: $\frac{p}{q} \cdot \frac{r}{s} \neq \frac{pr}{qs}$. We have rather the pecularity that $\sim\left(\frac{p}{q}\right) = \frac{1}{p} \cdot \frac{q}{1}$. Negation of whole fractions may be accomplished by inverting and "factoring" the terms. Thus also:

$$\sim\left(\frac{pr}{qs}\right) = \frac{1}{p} \cdot \frac{1}{r} \cdot \frac{q}{1} \cdot \frac{s}{1}.$$

We require the usual rules of distribution to express the various equivalent forms of expression with mixed conjunctions and disjunctions.

$$\text{C. } \frac{1}{[p \cdot (qr)]} = \frac{1}{(p \cdot q)(p \cdot r)} \qquad \text{D. } \frac{1}{p(q \cdot r)} = \frac{1}{[(pq) \cdot (pr)]}$$

These equalities represent the equivalences: $p \cdot (q \vee r) \overset{L}{\equiv} (p \cdot q) \vee (p \cdot r)$ and $p \vee (q \cdot r) \overset{L}{\equiv} (p \vee q) \cdot (p \vee r)$.

We need further a means of simplifying complex fractions. As with arithmetic fractions, $\frac{\frac{p}{q}}{r}$ is ambiguous. This ambiguity is a reflection of the non-associativity of material implication. We simplify such expressions by bringing down the numerators of the simple fractions occurring in numerator or denominator of the main fraction. We then have the reduction rules:

$$\text{E.}\ \frac{\left(\frac{p}{q}\right)}{r} = \frac{\left(\frac{1}{p'q}\right)}{r} = \frac{(p' \vee q)}{r} \qquad \text{F.}\ \frac{p}{\left(\frac{q}{r}\right)} = \frac{p}{\left(\frac{1}{q'r}\right)} = \frac{p}{q'r}$$

The rules (E) and (F) may be taken as stipulations, but they are also built into the notation. "$\frac{1}{p'q}$", the numerator in (E), expresses "$(\sim p \vee q)$", and the disjunction must be indicated in the numerator. "$\frac{1}{q'r}$" expresses "$(\sim q \vee r)$" which is represented in the denominator simply by the product "$q'r$". Using (E) and (F) together we have: $\frac{\left(\frac{p}{q}\right)}{\left(\frac{r}{s}\right)} = \frac{(p' \vee q)}{r's} = \frac{p'}{r's} \cdot \frac{q}{r's}$ which shows that $(p \supset q) \supset (r \supset s) \overset{L}{\equiv} [\sim p \supset (\sim r \vee s)] \cdot [q \supset (\sim r \vee s)]$. In (E) we can simplify further, by the equivalence (2.A) to give: $\frac{p'}{q} \cdot \frac{q}{r}$. In (F) the conditional "$p \supset (q \supset r)$" may be expressed as "$\frac{1}{p'q'r}$" which may be used to obtain the transformations of the rule of exportation.

Repeated factors in numerator or denominator may be omitted:

$$\text{G.}\ \frac{pp}{1} = \frac{p}{1},\ \frac{1}{pp} = \frac{1}{p}.$$

These idempotent laws correspond to the tautologies:

$$(p \cdot p) \equiv p \text{ and } (p \vee p) \equiv p.$$

Simplifications of the conditionals: $(p \supset \sim p)$ and $(\sim p \supset p)$ appear in the form:

$$\frac{p}{p'} = \frac{pp}{1} = \frac{p}{1} \text{ and } \frac{p'}{p} = \frac{1}{pp} = \frac{1}{p}.$$

As a further example of the ease with which further equivalences may be derived in the fractional representation we derive the law of absorption: $(p \supset q) \overset{L}{\equiv} [p \supset (p \cdot q)]$. We insert the tautology "$\frac{p}{p}$" and use the rule 2.B:

$$\frac{p}{q} = \frac{p}{p} \cdot \frac{p}{q} = \frac{p}{(p \cdot q)}$$

3. *Inference Rules.* Two rules of inference are sufficient in the fractional representation to yield easily all the usual inference rules of the statement calculus. The motivation, we recall, for the fractional representation was the sort of "cancelling" which occurs in inferences based on

the transitivity of material implication. The following single cancellation rule yields all inferences for which, intuitively, the conclusion is contained in the premises:

Fundamental Cancellation Rule: Any term which appears on opposite sides of the fractional line in *two* fractions may be cancelled to yield, upon multiplying (collapsing) the remaining fractions, a valid inference.

Examples will show the extreme simplicity and freedom of application of this rule. Only one caution is necessary: $\frac{p}{p}$ does not cancel to yield "$\frac{1}{1}$", which, as will be noted later, corresponds with contradictory forms. We use "$\rightarrow$" to indicate logical entailment.

(1) Hypothetical Syllogism: $\frac{p}{\cancel{q}} \cdot \frac{\cancel{q}}{r} \rightarrow \frac{p}{r}$.

(2) Modus Ponens: $\frac{\cancel{p}}{q} \cdot \frac{1}{\cancel{p}} \rightarrow \frac{1}{q}$.

(3) Modus Tollens: $\frac{p}{\cancel{q}} \cdot \frac{\cancel{q}}{1} \rightarrow \frac{p}{1}$.

(4) Disjunctive Syllogism: $\frac{1}{\cancel{p}q} \cdot \frac{\cancel{p}}{1} \rightarrow \frac{1}{q}$.

(5) Constructive Dilemma: $\frac{p}{q} \cdot \frac{r}{s} \cdot \frac{1}{pr} = \left(\frac{\cancel{p}}{q} \cdot \frac{1}{\cancel{p}r}\right) \cdot \frac{r}{s} \rightarrow \frac{1}{q\cancel{r}} \cdot \frac{\cancel{r}}{s} \rightarrow \frac{1}{qs}$

(6) Destructive Dilemma: $\frac{p}{q} \cdot \frac{r}{s} \cdot \frac{1}{q's'} = \left(\frac{p}{\cancel{q}} \cdot \frac{1}{\cancel{q}'s'}\right) \cdot \frac{r}{s} \rightarrow \frac{p}{\cancel{s}'} \cdot \frac{r}{\cancel{s}}$

$$= \frac{pr}{1} = \frac{1}{p'r'}.$$

In (6) we use an obvious corollary to the fundamental cancellation rule: a term and its negation appearing on the same side of the fractional line may be cancelled.

(7) $p \supset (q \vee r)$

$q \supset s \quad / \therefore [p \supset (r \vee s)] : \frac{p}{\cancel{q}r} \cdot \frac{\cancel{q}}{s} \rightarrow \frac{p}{rs}$

(8) $(p \cdot q) \supset r$

$s \supset p \quad / \therefore (q \cdot s) \supset r : \frac{\cancel{p}q}{r} \cdot \frac{s}{\cancel{p}} \rightarrow \frac{qs}{r}$.

(9) $(p \cdot q) \vee r$

$(p \cdot q) \supset s / \therefore (r \vee s) : \frac{1}{\cancel{(p \cdot q)}r} \quad \frac{\cancel{pq}}{s} \rightarrow \frac{1}{rs}$

Here we cancel a compound, remembering that "*pq*" in the numerator is a (negated) conjunction.

We complete the system with a rule of addition:

General Rule of Addition: Any term or terms may be multiplied (tacked on) to expressions in the numerator or in the denominator of a fraction to yield a second fraction which is a logical consequence of the first.

This rule allows the "paradoxical" inferences. When just the cancellation rule is used, the premises "contain" the conclusion and involve no new terms appearing in the conclusion. The addition rule allows, on the other hand, the following sort of technically valid but intuitively unsatisfactory inferences:

(1) $p \quad / \therefore (p \vee q) \quad : \frac{1}{p} \rightarrow \frac{1}{pq}$

(2) $q \quad / \therefore (p \supset q) \quad : \frac{1}{q} \rightarrow \frac{p}{q}$

(3) $\sim p \quad / \therefore (p \supset q) \quad : \frac{p}{1} \rightarrow \frac{p}{q}$

(4) $p \supset q \quad / \therefore (p \cdot r) \supset (q \vee s) \quad : \frac{p}{q} \rightarrow \frac{pr}{qs}$

Contradictions. The fractional representation of the contradictory form $p \cdot \sim p$ is: $\frac{1}{(p \cdot p')} = \frac{1}{p} \cdot \frac{p}{1}$. Applying the cancellation rule we have: $\frac{1}{\cancel{p}} \cdot \frac{\cancel{p}}{1} \rightarrow \frac{1}{1}$. A string of fractions which, upon cancellation, yields, "$\frac{1}{1}$" represents an inconsistent set of premises. The rule that "a contradiction implies anything" follows clearly from the application of the rule of addition to the expression "$\frac{1}{1}$".

It might appear that $p \equiv q$, if represented as $\frac{p}{q} \cdot \frac{q}{p'}$, would reduce to "$\frac{1}{1} \cdot \frac{1}{1}$". But the cancellation rule, as stated gives only: $\frac{p}{\cancel{q}} \cdot \frac{\cancel{q}}{p} \rightarrow \frac{p}{p'}$ which is merely an instance of any statement entailing a tautology. No cancellation may be made in the expression "$\frac{p}{p}$". The sense of cancellation, we recall, is to make a "transitive jump" from one implication to another.

Tautologies. There is with the fractional representation a simple criterion for deciding whether a statement form is tautologous or not. Any fraction containing a statement letter as a factor both in the numerator and in the denominator is a tautology. The conjunction of two tautologies is, of course, also a tuatology. For some examples we express the Hilbert-Ackermann axioms for the sentential calculus in fractional form and transform the fractions until a statement letter is a factor of both numerator and denominator.

(1) $(p \vee p) \supset p \quad : \frac{(p \vee p)}{p} = \frac{p}{p} \cdot \frac{p}{p}$

(2) $p \supset (p \vee q) \quad : \frac{p}{pq}$

(3) $(p \vee q) \supset (q \vee p) \quad : \frac{(p \vee q)}{qp} = \frac{p}{qp} \cdot \frac{p}{qp}$

(4) $[(p \supset q)] \supset [(r \vee p) \supset (r \vee q)] : \dfrac{\frac{p}{q}}{\frac{(r \vee p)}{rq}} = \frac{p'q}{(r \vee q)'rq} = \frac{p'q(r \vee p)}{rq}$

In the fourth axiom we simplify using Rule 2.E and the negation principle to yield a fraction in which "q" is a factor of both numerator and denominator.

4. *General Adequacy of the System.* The negation principle can be justified in general by induction on the number of factors. We omit the formal proof and merely note the crucial logical equivalences involved. Starting with the general expression "$\frac{ABC\ldots}{PQR\ldots}$" we may, according to the principle of negation, move a term, say "A", from numerator to denominator provided we negate that term. This operation corresponds to the logical equivalence of the following forms:

$$[(A.B.C\ldots) \supset (P \vee Q \vee R \ldots)] \overset{\mathsf{L}}{\equiv} [(B.C \ldots) \supset [A \supset (P \vee Q \vee R \ldots)]$$
$$\overset{\mathsf{L}}{\equiv} [(B.C \ldots) \supset (\sim A \vee P \vee Q \vee R \ldots)].$$

The principle of negation involves, in the usual formulation, the equivalences expressed by the laws of exportation and implication.

To show that the permissible inferences using the cancellation rule are in fact valid we consider the various argument forms which correspond to all the sorts of cancellation which can occur. We let the letters: A, B, C, D, E stand for any statement, simple or compound; or these may be vacuous and replaced by "1". Then every inference in the fractional representation which uses the cancellation rule may be put into the form:

$$\frac{A}{\cancel{B}C} \cdot \frac{\cancel{B}D}{E} \rightarrow \frac{AD}{CE}.$$

(This general form clearly includes the cases of the form: $\frac{\cancel{B}D}{E} \cdot \frac{A}{\cancel{B}C} \rightarrow \frac{DA}{EC}$, by the commutation rules). We appeal to the definition of logical consequence. The conclusion $\frac{AD}{CE}$ is false if and only if both "A" and "D" are true and "C" and "E" are false. Since B cannot be vacuous if there is to be cancellation, we have to consider the two cases: "B" is false and "B" is true. If "B" is false, the first premise $\frac{A}{BC}$ is false. If "B" is true, the second premise $\frac{BD}{E}$ is false. Thus both premises cannot be true and yet the conclusion be false.

We could also write out the sixteen varieties of inference which arise when some or all (excepting B) of the terms are vacuous. For example, if C and D are replaced by "1" we have an instance of modus ponens. If A, C, D, E are all vacuous, we have the derivation of "$\frac{1}{1}$" from the contradiction: $B . \sim B$.

5. *Examples of Derivations.*

A. $(P \vee Q)$
$(\sim Q \vee R)$

$(\sim S \supset \sim R)\ /\ \therefore\ (\sim S \supset P)$

$$\left(\frac{1}{P\cancel{Q}} \cdot \frac{1}{\cancel{Q}'R}\right) \cdot \frac{S'}{R'} \rightarrow \frac{1}{P\cancel{R}} \cdot \frac{S'}{\cancel{R}'} \rightarrow \frac{S'}{P}.$$

B. $[P \vee (Q \cdot R)]$
$(P \supset S)$
$(S \supset R)\ /\ \therefore\ R.$

$$\left(\frac{1}{\cancel{P}(Q \cdot R)} \cdot \frac{\cancel{P}}{S}\right) \cdot \frac{S}{R} \rightarrow \frac{1}{(Q \cdot R)\cancel{S}} \cdot \frac{\cancel{S}}{R} \rightarrow \frac{1}{(Q \cdot R)R} = \frac{1}{(QR) \cdot (RR)}$$
$$= \frac{1}{QR} \cdot \frac{1}{RR} \rightarrow \frac{1}{RR} = \frac{1}{R}.$$

C. $(A \supset B)$
$(\sim B \vee C)$
$\sim(C \cdot \sim D)$
$\sim(E \cdot B) \supset \sim D\ /\ \therefore\ (A \supset E)$

$$\left(\frac{A}{\cancel{B}} \cdot \frac{1}{\cancel{B}'C}\right) \cdot \left(\frac{C\cancel{D}'}{1} \cdot \frac{(E\,B)'}{\cancel{D}'}\right) \rightarrow \frac{A}{\cancel{C}} \cdot \frac{\cancel{C}(E\,B)'}{1} \cdot \frac{A(E\,B)'}{1}$$
$$= \frac{A}{(E \cdot B)} = \frac{A}{E} \cdot \frac{A}{B} \rightarrow \frac{A}{E}.$$

D. $[(A \cdot B) \supset (C \vee D)]$
$(C \supset E)$
$[A \cdot (F \vee B)]$
$(D \supset G)\ /\ \therefore\ [(A \supset \sim F) \supset (E \vee G)]$

$$\left(\frac{AB}{\cancel{C}D} \cdot \frac{\cancel{C}}{E}\right) \cdot \frac{1}{[A \cdot (FB)]} \cdot \frac{D}{G} \rightarrow \frac{\cancel{A}\cancel{B}}{DE} \cdot \frac{1}{(A \cdot F)(\cancel{A} \cdot \cancel{B})} \cdot \frac{D}{G} \rightarrow \frac{1}{\cancel{D}E(A \cdot F)} \cdot \frac{\cancel{D}}{G}$$
$$\rightarrow \frac{1}{EG(A \cdot F)} = \frac{(A \cdot F)'}{EG} = \frac{(A' \vee F')}{EG} \rightarrow \frac{(A \supset F')}{EG}$$

E. $[(A \cdot B) \supset (C \supset D)]$
$(E \supset \sim D)$
$B \cdot E$
$A \cdot C\ /\ \therefore\ F$

$$\frac{\;\frac{AB}{C}\;}{D} \cdot \frac{E}{D'} \cdot \frac{1}{(B \cdot E)} \cdot \frac{1}{(A \cdot C)} = \left(\frac{AB}{C'\cancel{D}} \cdot \frac{E}{\cancel{D}'}\right) \cdot \left(\frac{1}{B} \cdot \frac{1}{E} \cdot \frac{1}{A} \cdot \frac{1}{C}\right)$$
$$\rightarrow \frac{\cancel{ABE}}{\cancel{C}'} \cdot \frac{1}{\cancel{A}} \cdot \frac{1}{\cancel{B}} \cdot \frac{1}{\cancel{E}} \cdot \frac{1}{\cancel{C}} \cdot \frac{1}{1}$$

—the premises are contradictory, so any expression follows by the addition rule.

PART II: THE CALCULUS OF CLASSES

The same sort of algorithm is applicable to a Boolean algebra of classes. The basic relations of a Boolean algebra which are shared by terms of syllogistic and statements of the statement calculus are exploited in the fractional representation by giving the manipulations the same form.

We let the letters A, B, ... now stand for classes rather than statements. We then stipulate:

"$\dfrac{A}{B}$" represents "$A \subseteq B$" (class inclusion).

Equivalent formulations of class inclusion in terms of set union and intersection suggest, with a little imagination, writing

"$\dfrac{AB}{1}$" for "$A \cap B = \phi$" and "$\dfrac{1}{AB}$" for "$A \cap B = 1$".

Complementation of a class, the analogue to negation of a statement, then obeys the negation principle: a term may be moved across the fractional line, *salva veritate*, provided it is negated (complemented). We have then:

$$\frac{A}{B} = \frac{AB'}{1} = \frac{B'}{A'} = \frac{1}{BA'}$$

which express the relationships:

$$(A \subseteq B) = [A \cap B' = \phi] = (B' \subseteq A') = [(B \cup A') = 1].$$

Set union and intersection, like their analogues disjunction and conjunction, are commutative and associative so that the order of multiplied terms may be ignored and the parentheses omitted in unmixed expressions. We limit the application here to syllogistic reasoning though the techniques might be extended to the entire monadic predicate calculus.

1. *The Fractional Representation of Categorical Statements.* Since categorical statements may be interpreted rather naturally in terms of relations between classes we can apply the above techniques to syllogistic reasoning. We can in fact develop a fractional representation of categorical statements which allows quick and easy solutions to all the problems arising in syllogistic, from the immediate inferences through enthymematic sorites.

The choice of representations for the universal statements is clear from the class notation for the **A** and **E** type statements: *All S are P*: The class of S's is included in the class of P's: *No S are P*: The intersection of the classes S and P is empty. The particular statements will be represented by the inversions of their contradictories, plus an indication that they are existential statements. We then stipulate the representations:

A: *All S are P* — $\dfrac{S}{P}$

E: *No S are P* — $\dfrac{SP}{1}$

I: *Some S are P* — $\dfrac{1}{(S)P}$

O: *Some S are not P* — $\dfrac{P}{(S)}$

The parentheses around the subject terms of particular statements have only an indicating function. They are necessary to remove the ambiguity which would otherwise arise in reading "$\frac{A}{B}$" both as "*All A are B*" and as "*Some B are not A.*" We note that the expression "$\frac{1}{SP}$" (with no parentheses) has no (syllogistic) reading. It could, however, be used to represent the statement: Everything is either S or P.

The fractional representation gives a rapid and easily remembered means of writing out all the logical equivalents of any categorical statement. We can thus obtain all the immediate inferences of traditional Aristotelian logic which are valid under a Boolean interpretation, that is, without existential presupposition. The negation principle and commutativity of multiplication of terms are syntactically the same as in Part I. We get the immediate inferences of conversion of the **E** and **I** statements from the commutation principle:

$$\frac{SP}{1} = \frac{PS}{1} \text{ and } \frac{1}{(S)P} = \frac{1}{(P)S}.$$

The negation principle yields the obversions. Thus the equivalence of "*All S are P*" to "*No S are non-P*" appears in the equation:

$$\frac{S}{P} = \frac{SP'}{1}$$

The immediate inference of contraposition, valid for **A** and **O** statements, appears as:

$$\frac{S}{P} = \frac{P'}{S'} \text{ and } \frac{P}{(S)} = \frac{S'}{(P')}$$

There is no temptation to misapply conversion or contraposition. The only restriction on the manipulations is that a fraction must be written in a form which corresponds with the categorical statements—in particular the parentheses must be filled.

2. *Syllogistic Inference.* The following procedure suffices to pick out those and only those conclusions which follow validly for a set of categorical statements:

(1) Having represented the premises in fractional form, write the fractions together as if to "multiply" them;
(2) Inference rule: Cancel any terms which appear both in the numerator and the denominator.

The result is then to be interpreted according to the following rules:

(1) If more than one pair of parentheses occurs, no (syllogistic) conclusion may be drawn;
(2) If a term cancels, the two fractions may be collapsed into a new fraction.

The first rule of interpretation corresponds to the traditional rule that no valid syllogism has two particular premises. That only one particular premise may be used in any type of valid syllogistic argument saves the method from embarrassment.

EXAMPLE (1) *All M are P.*
All S are M. / ∴ *All S are P.*

$$\frac{\cancel{M}}{P}\frac{S}{\cancel{M}} \rightarrow \frac{S}{P}$$

EXAMPLE (2) *No P are M.*
Some M are S. / ∴ *Some S are not P.*

$$\frac{P\cancel{M}}{1}\frac{1}{(\cancel{M})S} \rightarrow \frac{P}{(S)}$$

In Example (2) we note that cancellation is unhindered by the parentheses; some term must, however, be placed in the parentheses to read the result. Along with the restriction to at most one particular premise we see easily that one rule of distribution suffices to pick out just the fifteen syllogisms valid without existential presupposition: the middle term must be distributed exactly once. For the terms which appear in the denominator are just the distributed terms and the middle term must appear once in the denominator, once in the numerator to cancel.

An advantage of the fractional method is that any conclusion which may validly be inferred is actually generated.

EXAMPLE (3) *All P are non-M.*
Some M are not S. / ∴ (?)

$$\frac{P}{M'}\frac{S}{(M)} \rightarrow \frac{PS}{()} = \frac{P}{(S')} = \frac{S}{(P')}$$

Thus we may infer either "*Some non-S are not P*" or that "*Some non-P are not S*" even though one of the traditional rules of quality is violated. This example shows that the obvious derived inference rule may be used: positive and negated terms on the same side of the fractional line may be cancelled.

The fractional method involves no assumption of existential import, but it is particularly easy to see in this representation just what appropriate assumptions yield the nine additional forms of syllogism valid under the traditional point of view. For AAI-1 we require "*Some S are S.*" $\frac{\cancel{M}}{S}\cdot\frac{S}{\cancel{M}}\cdot\frac{1}{(S)S} \rightarrow \frac{\cancel{S}}{P}\cdot\frac{1}{(\cancel{S})S} \rightarrow \frac{1}{(S)P}$.

The fractional method provides an easy technique for solving enthymemes. Once given the fractional representation of two of the statements of a valid syllogism the third missing statement may be found as in a simple problem of algebra. Let us, for example, complete the enthymeme "There must be something burning in the kitchen with all that smoke." We might reconstruct the argument as follows:

Some kitchen place is a smokey place.
Some kitchen place is a fiery place.

Abbreviating the classes: kitchen, smokey, fiery places as K, S and F, we have: $\frac{1}{(K)S} \cdot \begin{bmatrix} \frac{x}{y} \end{bmatrix} \rightarrow \frac{1}{(K)F}$. We clearly must cancel the "S" and insert the "F". The missing premise is then $\frac{S}{F}$ or "All smokey places are fiery."

The fractional method is prefectly general for arguments with any number of categorical premises. We can solve the following enthymematic sorites without any additional rules:

All A are B.
All non-C are D. / *Some non-A are C.*

Putting the argument into fractional form we have:

$$\frac{A}{B} \cdot \frac{C'}{D} \cdot \begin{bmatrix} \frac{x}{y} \end{bmatrix} \rightarrow \frac{1}{(A')C}$$

We must cancel "B" and "D" and also introduce parentheses. We thus need $\frac{D}{(B')}$ or any equivalent form. The missing premise which makes the argument valid is then "*Some non-B are not D.*"

New notations and algorithms add nothing, of course, to the store of logical relationships which there are; they may, however, make them easier to express and interrelate. A practical application would appear to be in teaching elementary logic: the easily learned techniques would provide quick access for the beginning student to a systematic manipulation of logical relationships among statements and among classes.

NOTES

[1] A very similar technique, in somewhat different notation and limited to syllogistic, was discovered independently by Professor Fred Sommers. See his "On a Fregean Dogma," in I. Lakatos, Ed., *Problems in the Philosophy of Mathematics*, Vol. I (North-Holland) (1967), pp. 47-62.

[2] The representations themselves have no obvious intuitive significance, but may be taken simply as notational stipulation. One could develop the same technique using "$P/1$" to represent "P" instead of "$\sim P$." But then "$P \supset Q$" would come out "Q/P", which also seems backwards.

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