Notre Dame Journal of Formal Logic
Volume XI, Number 2, April 1970

AN UNSOLVABLE PROBLEM CONCERNING IMPLICATIONAL CALCULI

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Propositional calculi are assumed to be defined as in Harrop [2]. A propositional calculus is called implicational if it has exactly one connective and that a binary one. Algebraic structures called finite models of a propositional calculus **P** in Harrop [2] will be called here *finite rule-models* of **P**. An algebraic structure of the appropriate kind is called a *model* of **P** if every theorem of **P** is valid in it. We note that every finite rule-model of **P** is a finite model of **P**, but the converse need not hold. It is proved in Harrop [2] (lemma 3.1, pp. 5-6) that *there is an effective method for deciding whether or not a finite algebraic structure is a finite rule-model of a propositional calculus* **P**. We prove here the following:

Theorem. *There is no effective method for deciding whether or not a finite algebraic structure is a model of an arbitrary subcalculus (with modus ponens as its only rule of inference) of the classical implicational calculus.*

We first prove a lemma which is of some independent interest.

Lemma. *For every wff α there is a finite algebraic structure $\mathbf{M}_\alpha$ such that for any wff β, β is invalid in $\mathbf{M}_\alpha$ if and only if α is a substitution instance of β.*

Proof. If δ is a wff all of whose propositional variables are given without repetition in the list $v_1, v_2, \ldots, v_k$, then δ is also written as $\delta(v_1, v_2, \ldots, v_k)$. Let $\alpha(v_1, v_2, \ldots, v_s)$ be given, and let the set of all subformulas of α be listed in the sequence $\gamma_1, \gamma_2, \ldots, \gamma_s, \gamma_{s+1}, \ldots, \gamma_{s+m}$ such that $\gamma_i = v_i$, $i = 1, 2, \ldots, s$ and no γ_j is a subformula of any of its predecessors in the above sequence. For example, if α is *CCpqCNqNp* then a desired sequence of subformulas of α is: *p, q, Np, Nq, Cpq, CNqNp, CCpqCNqNp.*

Based on the construction of α we define the finite algebraic structure $\mathbf{M}_\alpha$. Let $S = \{0, 1, 2, \ldots, s + m\}$, and $D = S - \{s + m\}$ be the set of designated elements of $\mathbf{M}_\alpha$. If Ω^* is the set of all propositional connectives, then corresponding to each k-ary $\omega^* \varepsilon \Omega^*$ we define a distinct k-ary operation ω on S as follows: For any k-tuple $i_1, i_2, \ldots, i_k$ of elements of S, $\omega(i_1, i_2, \ldots, i_k) = j$, if there is a subformula γ_j of α such that

Received March 4, 1969

$$\gamma_j = \omega^*(\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_k}); \text{ otherwise } \omega(i_1, i_2, \ldots, i_k) = 0.$$

For the example $CCpqCNqNp$, $s = 2$, $m = 5$ and the operations N and C defined on the set $\{0, 1, 2, \ldots, 7\}$ are:

$$N(1) = 3,\ N(2) = 4,\ N(x) = 0 \text{ otherwise;}$$
$$C(1, 2) = 5,\ C(4, 3) = 6,\ C(5, 6) = 7,\ C(x, y) = 0 \text{ otherwise.}$$

Clearly, $\alpha(v_1, v_2, \ldots, v_s)$ is invalid in $\mathbf{M}_\alpha$, because α takes the value $s + m$ when $v_i = i$, $i = 1, 2, \ldots, s$. Hence, any wff β of which α is a substitution instance is also invalid in $\mathbf{M}_\alpha$. Before proving that if β is invalid in $\mathbf{M}_\alpha$, then α is a substitution instance of β we note the following properties of $\mathbf{M}_\alpha$.

(1). The elements $1, 2, \ldots, s$ are not in the range of any operation $\omega \varepsilon \Omega$.

(2). For any $j \varepsilon S$ and $s < j$, there is a unique k-ary operation $\omega \varepsilon \Omega$ and a unique k-tuple of elements from S such that ω takes the value j for this k-tuple.

(3). For each $j \varepsilon S$ and $s < j$, j has a unique factorization using only elements from the set $\{1, 2, \ldots, s\}$ and operations from Ω.

(4). For each $\omega \varepsilon \Omega$, $\omega(\ldots, 0, \ldots) = 0$.

Let $\beta(u_1, u_2, \ldots, u_n)$ be invalid in $\mathbf{M}_\alpha$. Then, there is an invalidating assignment to the variables of β from the set $\{1, 2, \ldots, s + m\}$ (see property (4) of $\mathbf{M}_\alpha$) such that β takes the value $s + m$ for this assignment. Let u_i take the value j_i for this assignment, $i = 1, 2, \ldots, n$. Let $\delta = \beta(u_1/\gamma_{j_1}, u_2/\gamma_{j_2}, \ldots, u_n/\gamma_{j_n})$. Since each γ_i is a subformula of α and γ_i takes the value i when v_t takes the value t, $t = 1, 2, \ldots, s$, δ takes the value $s + m$ when v_t takes the value t, $t = 1, 2, \ldots, s$. In view of property (3) of $\mathbf{M}_\alpha$, δ must be α. For our example $CCpqCNqNp$, if β were $Cu_1CNu_2u_3$ then the invalidating assignment gives 5, 2, 3 respectively to u_1, u_2, and u_3. Substituting the fifth, the second and the third subformulas of $CCpqCNqNp$ in the previously given list for u_1, u_2, and u_3 respectively in β we get $CCpqCNqNp$.

Remark. A rule of inference is called non-trivial if it has some application for which the conclusion is not a substitution instance of any of the premises of that application. No non-trivial rule of inference is satisfied by every $\mathbf{M}_\alpha$.

Proof of the Theorem. Assume it is false. Let $\mathbf{P}$ be an implicational calculus (all of whose theorems are classical tautologies) with modus ponens as the only rule of inference. Let α be any wff with implication as its only connective. Consider $\mathbf{M}_\alpha$. By assumption we can decide whether or not $\mathbf{M}_\alpha$ is a model of $\mathbf{P}$. If $\mathbf{M}_\alpha$ is a model of $\mathbf{P}$ then α is not a theorem of $\mathbf{P}$ by the construction of $\mathbf{M}_\alpha$. If $\mathbf{M}_\alpha$ is not a model of $\mathbf{P}$, then some theorem of $\mathbf{P}$ must be invalid in $\mathbf{M}_\alpha$. Thus, by the lemma, α is a substitution instance of a theorem of $\mathbf{P}$ and hence α is itself a theorem of $\mathbf{P}$. Therefore, $\mathbf{P}$ is decidable. This contradicts theorem 1 of Gladstone [1].

REFERENCES

[1] Gladstone, M. D., "Some ways of constructing a propositional calculus of any required degree of unsolvability," *Transactions of the American Mathematical Society*, vol. 124 (1965), pp. 192-210.

[2] Harrop, R., "On the existence of finite models and decision procedures for propositional calculi," *Proceedings of the Cambridge Philosophical Society*, vol. 54 (1958), pp. 1-13.

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