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LOGICAL CONSEQUENCE IN MODAL LOGIC: NATURAL DEDUCTION IN S5

JOHN CORCORAN and GEORGE WEAVER

This paper presents a (modal, sentential) logic $\mathcal{L}\Box\Box$ which may be thought of as a partial systematization of the semantic and deductive properties of a sentence operator ($\Box$) which expresses certain kinds of necessity. The language of $\mathcal{L}\Box\Box$ is $\mathsf{L}\Box\Box$, the smallest set of formulas containing a countably infinite set of sentential constants and closed under forming $\Box\varphi$, $\sim\varphi$, and $(\varphi \supset \psi)$ from any formulas φ and ψ already in the set. This is essentially the language of S5. The semantics of $\mathcal{L}\Box\Box$ is essentially Kripke's semantics for S5. In the semantics for $\mathcal{L}\Box\Box$, however, an interpretation of $\mathsf{L}\Box\Box$ is defined as an ordered pair $\langle a, P\rangle$ where a is an ordinary truth-value interpretation of the sentential constants and P is a set of such interpretations with $a \varepsilon P$. $\langle a, P\rangle$ is a model of φ if φ is true under $\langle a, P\rangle$ and $\langle a, P\rangle$ is a model of $S \subseteq \mathsf{L}\Box\Box$ if each ψ in S is true under $\langle a, P\rangle$. This permits logical consequence to be defined: for $S \subseteq \mathsf{L}\Box\Box$, $\varphi \varepsilon \mathsf{L}\Box\Box$, φ is a logical consequence of $S(S \models \varphi)$ iff every model of S is a model of φ. The deductive system consists of "natural" rules permitting proofs from arbitrary sets of premises and we let $S \vdash \varphi$ mean that there is a proof of φ from the premises S.

Let $\mathsf{L}\Box$ be the sublanguage of $\mathsf{L}\Box\Box$ containing all formulas of $\mathsf{L}\Box\Box$ devoid of iterated or nested $\Box$. Let $\mathcal{L}\Box$ be the restriction of $\mathcal{L}\Box\Box$ to $\mathsf{L}\Box$, i.e. $\mathcal{L}\Box$ is the logic with language $\mathsf{L}\Box$ such that, for $S \subseteq \mathsf{L}\Box$ and $\varphi \varepsilon \mathsf{L}\Box$, $S \models \varphi$ relative to $\mathcal{L}\Box$ iff $S \models \varphi$ in $\mathcal{L}\Box\Box$ and $S \vdash \varphi$ relative to $\mathcal{L}\Box$ iff there is a proof in $\mathcal{L}\Box\Box$ of φ from S containing only formulas of $\mathsf{L}\Box$. Strong completeness ($S \models \varphi$ implies $S \vdash \varphi$) for $\mathcal{L}\Box\Box$ is proved from the following three lemmas: (1) strong soundness ($S \vdash \varphi$ implies $S \models \varphi$) for $\mathcal{L}\Box\Box$, (2) strong completeness for $\mathcal{L}\Box$ and (3), a reduction theorem to the effect that every formula in $\mathsf{L}\Box\Box$ is provably equivalent in $\mathcal{L}\Box\Box$ to a formula in $\mathsf{L}\Box$. From these results, together with Kripke's weak soundness and weak completeness results for S5, it follows that φ is a theorem of S5 iff φ is a theorem of $\mathcal{L}\Box\Box$.

1 *The Logic of* $\mathcal{L}\Box\Box$. Let $\mathcal{C}$ be a countably infinite set of symbols called *sentential constants* and let $\mathsf{L}\Box\Box$ be the smallest set of formulas containing

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$\mathcal{C}$ and closed under forming $\Box\varphi$, $\sim\varphi$ and $(\varphi \supset \psi)$ from any formulas φ and ψ already in the set. The set of (purely) *modal formulas* is the smallest containing all formulas in $\mathsf{L}\Box\Box$ of the form $\Box\varphi$ and closed under forming $\sim\psi$ and $(\psi \supset \eta)$. Let L be the sentential language included in $\mathsf{L}\Box\Box$, i.e. L is the smallest set of formulas containing $\mathcal{C}$ and closed under forming $\sim\psi$ and $(\psi \supset \eta)$.

1.1 The interpretations of $\mathsf{L}\Box\Box$ are to be ordered pairs $\langle a, P\rangle$ where a is an ordinary truth-value interpretation of L and P is a set of such interpretations with $a \varepsilon P$. In a particular interpretation $\langle a, P\rangle$, a is to be thought of as an "actual world" and P is to be thought of as a set of "possible worlds". We use the notation aP to indicate an ordered pair, $\langle a, P\rangle$. Accordingly, formulas in L are true in aP iff they are true in a: formulas not involving necessity ($\Box$) have their truth-values determined by the actual world a. $\Box\varphi$ will be defined as true in aP iff φ is true in all bP, $b \varepsilon P$. Thus, in effect we will have, in addition to the actual world a and the set of possible worlds P (both of the interpretation aP), the "actual" interpretation aP and *its corresponding set of possible interpretations*: $\{bP : b \varepsilon P\}$. The corresponding set of possible interpretations for a possible interpretation is the same as that of the actual interpretation. The above remarks might aid the reader in seeing beyond the formal development which follows.

Below a, b, a', b', etc. are functions from $\mathcal{C}$ to $\{t, f\}$ and P, P', etc. are sets of such functions. aP is an *interpretation* (of $\mathsf{L}\Box\Box$) iff $a \varepsilon P$. For each interpretation aP we define a unique function V^{aP} from $\mathsf{L}\Box\Box$ to $\{t, f\}$ in such a way that for each $\varphi \varepsilon \mathsf{L}\Box\Box$, $V^{aP}(\varphi) = t(= f)$ when and only when φ is to be regarded as true in aP (false in aP). Let N and C be the usual truth-functions corresponding to $\sim$ and $\supset$.

Definition: For $\varphi \varepsilon \mathcal{C}$, $V^{aP}(\varphi) = a(\varphi)$. If $\varphi = \sim\psi$ then $V^{aP}(\varphi) = N(V^{aP}(\psi))$. If $\varphi = (\psi \supset \eta)$ then $V^{aP}(\varphi) = C(V^{aP}(\psi), V^{aP}(\eta))$. If $\varphi = \Box\psi$ then $V^{aP}(\varphi) = t$ iff $V^{bP}(\psi) = t$, for all $b \varepsilon P$. V^{aP} is called the (truth) *valuation* in aP.

If $V^{aP}(\varphi) = t$ then aP is a *model of* φ. If aP is a model of every formula in S then aP is a model of S. If every interpretation is a model of φ then we say that φ is *logically true* and write $\vDash\varphi$. If every model of S is a model of φ, we say that φ *is a logical consequence of* S and write $S \vDash \varphi$. Thus, letting Λ denote the empty set, $\Lambda \vDash \varphi$ iff $\vDash\varphi$. For conciseness we write 'S, φ' instead of '$S \cup \{\varphi\}$' and '$\varphi \vDash \psi$' instead of '$\{\varphi\} \vDash \psi$' and accordingly, the following are consequences of the above definitions.

S0.0 If $S \vDash \varphi$, for all $\varphi \varepsilon S'$, and $S' \vDash \psi$, then $S \vDash \psi$.
S0.1 If $S \subseteq S'$ and $S \vDash \varphi$ then $S' \vDash \varphi$.
S1 $S, \varphi \vDash \varphi$.
S2 If $S \vDash \varphi$ and $S \vDash (\varphi \supset \psi)$ then $S \vDash \psi$.
S3 If $S, \varphi \vDash \psi$ then $S \vDash (\varphi \supset \psi)$.
S4 If $S, \sim\varphi \vDash \psi$ and $S, \sim\varphi \vDash \sim\psi$ then $S \vDash \varphi$.
S5 If $S \vDash \Box\varphi$ then $S \vDash \varphi$.
S6 If all formulas in S are modal and $S \vDash \varphi$ then $S \vDash \Box\varphi$.

The first three of these principles follow from the definitions of "$\models$" and "model of S" without regard for the particular character of the semantics. The next three hold because the valuations preserve the normal meanings of $\sim$ and $\supset$. *S5* follows since $V^{aP}(\Box\varphi) = t$ implies $V^{aP}(\varphi) = t$.

S6 follows from the following proposition: If φ is a modal sentence then for all a, $b \varepsilon P$, $V^{aP}(\varphi) = V^{bP}(\varphi)$. To see this, assume the proposition and let S be a set of modal formulas such that $S \models \varphi$. Let aP be any model of S. If S is empty then φ is true in all interpretations, in particular in all interpretations bP with $b \varepsilon P$. Thus $V^{aP}(\Box\varphi) = t$. If S is not empty then, by the proposition, bP is also a model of S for any $b \varepsilon P$. Since $S \models \varphi$ each such interpretation must be a model of φ. Thus $V^{aP}(\Box\varphi) = t$. Since aP is any model of S, $S \models \Box\varphi$.

The proposition mentioned above is proved as follows. Let P and a be arbitrary except that $a \varepsilon P$. Consider the property of formulas φ: for all $b \varepsilon P$, $V^{aP}(\varphi) = V^{bP}(\varphi)$. From the definition of valuation it follows that the property holds for all formulas of the form $\Box\psi$ and also that it holds for $\sim\psi$ and $(\psi \supset \eta)$ whenever it holds for ψ and η. Thus it holds for all modal formulas.

1.2 The deductive system for $\mathcal{L}\Box\Box$ parallels the semantic principles (*S0.0* through *S6*) so closely that the (strong) soundness proof is quite simple. The sentential (or tautologous) part of the system is essentially Jaskowski's system (referred to by Leblanc [7], p. 16). This consists of the familiar rules: assumption (A), repetition (R), detachment or *modus ponens* ($\supset$E), conditionalization ($\supset$I) and a form of *reductio ad absurdum* ($\sim$E). The modal rules are $\Box$ introduction ($\Box$I) and $\Box$ elimination ($\Box$E).

The proofs are written left-to-right on one line rather than top-to-bottom on several lines. Brackets,] and [, are used as punctuation in the proofs. Thus proofs will be strings over $\mathbf{L}\Box\Box \cup \{], [\}$; i.e. certain strings of formulas of $\mathbf{L}\Box\Box$ having] and [interposed as punctuation will be proofs. An occurrence of $[\varphi$ at a place in a proof Π indicates that φ is assumed in Π at that place. An occurrence of $]\varphi$ at a place in a proof Π indicates (a) that the right-most assumption (not already discharged) is discharged and (b) that φ is derived from the remaining (undischarged) assumptions. An occurrence of φ (not preceded by] or [) at a place in a proof indicates that φ is derived from the undischarged assumptions preceeding it. In order to give the reader an orientation to the new linear notation, we give several familiar natural-deduction proofs using it.

Examples: The string (a) below is a proof of $(\varphi \supset \varphi)$ from Λ, (b) is a proof of $(\varphi \supset (\psi \supset \varphi))$ from Λ, (c) is a proof of $((\varphi \supset \psi) \supset (\varphi \supset \eta))$ from $(\varphi \supset (\psi \supset \eta))$, (d) is a proof of φ from $(\sim\varphi \supset \sim\psi)$ and $(\varphi \supset \psi)$ and (e) is a proof of $\sim\varphi$ from $(\varphi \supset \psi)$ and $(\varphi \supset \sim\psi)$.

(a) $[\varphi\varphi]\ (\varphi \supset \varphi)$
(b) $[\varphi[\psi\varphi](\psi \supset \varphi)]\ (\varphi \supset (\psi \supset \varphi))$
(c) $[(\varphi \supset (\psi \supset \eta))\,[(\varphi \supset \psi)\,[\varphi\psi(\psi \supset \eta)\ \eta]\ (\varphi \supset \eta)]\ ((\varphi \supset \psi) \supset (\varphi \supset \eta))$
(d) $[(\sim\varphi \supset \psi)\ [(\sim\varphi \supset \sim\psi)\ [\sim\varphi\psi\sim\psi]\varphi$
(e) $[(\varphi \supset \psi)[(\varphi \supset \sim\psi)[\sim\sim\varphi\ [\sim\varphi\sim\varphi\sim\sim\varphi]\varphi\psi\ \sim\psi]\sim\varphi$

Proof (e) has more lines than necessary but the superfluous lines make the application of ($\sim$E) more transparent.

We say that an occurrence of a formula in a proof is open if it is derived from undischarged assumptions and that it is closed if it is derived from discharged assumptions. Because the concepts of open and closed are crucial in discussing proofs we will give fairly rigorous definitions of them despite the fact that "closed" amounts to being enclosed in paired brackets and "open" is simply "not closed". It is also perhaps desirable from the point of view of exposition to first define a fairly "small" set of strings which will include all proofs.

Proof expression (p.e.) is defined recursively as follows: $[\varphi$ is a proof expression, if Π is a proof expression so are $\Pi[\varphi$, $\Pi\varphi$ and $\Pi]\varphi$ where φ is any formula of L□□. No string (over L□□ $\{], [\}$ is a p.e. except by the above two rules. Proofs will be defined as certain of the proof expressions. E.g., $[\varphi\varphi$ will be a proof and $[\varphi\sim\varphi$ will not.

Now we will define "closed in a p.e." (enclosed between paired brackets) and "open in a p.e." recursively by reference to substrings of proof expressions called closed sections. *Closed section* is defined as follows: $[\varphi_1\varphi_2, \ldots, \varphi_n]$ is a closed section. If β is a closed section so is $[\varphi_1\varphi_2, \ldots, \varphi_n\beta\psi_1\psi_2, \ldots, \psi_n]$. No string (over L□□ $\{], [\}$) is a closed section except by the above two rules. Let β be a p.e. An occurrence of φ in β is *closed in* β iff it is within a closed section which is a substring of β. An occurrence of φ is *open in* β iff it is not closed in β. An occurrence of a formula in β which is immediately preceded by a $[$ is an *assumption in* β. A formula (not an occurrence) is an *assumption of* β iff it occurs as an open assumption in β.

Definition of Proof: In the following rules φ, ψ and η are formulas of L□□; Π, Π_1, and σ, σ' are strings over L□□$\cup\{], [\}$. Π is a *proof* iff Π is constructed by a finite number of applications of the following rules.

(A) $[\varphi$ is a proof and if Π is a proof then $\Pi[\varphi$ is a proof

(R) If Π is a proof and φ occurs open in Π, then $\Pi\varphi$ is a proof.

($\supset$E) If Π is a proof, φ occurs open in Π and $(\varphi \supset \psi)$ occurs open in Π then $\Pi\psi$ is a proof.

($\supset$I) If $\Pi = \sigma[\varphi\sigma'\psi$ is a proof where the indicated occurrence of φ is the right-most open assumption in Π then $\Pi](\varphi \supset \psi)$ is a proof.

($\sim$E) If $\Pi = \sigma[\sim\varphi\sigma'$ is a proof where (1) the indicated occurrence of $\sim\varphi$ is the right-most open assumption in Π and (2) both ψ and $\sim\psi$ occur open in Π, for some ψ, then $\Pi]\varphi$ is a proof

(□E) If Π is a proof and $\Box\varphi$ is open in Π then $\Pi\varphi$ is a proof.

(□I) If Π is a proof, φ occurs open in Π and all open assumptions in Π are modal formulas then $\Pi\Box\varphi$ is a proof.

Notice that in ($\supset$I) and ($\sim$E), if σ is not null then σ is a proof. Moreover, every proof Π ends with an open formula. The last formula in Π is called the conclusion of Π and is denoted by $C(\Pi)$. The set of (open) assumptions *of* Π is denoted by $A(\Pi)$. We define: $S \vdash \varphi$ iff there exists a Π

such that $A(\Pi) \subseteq S$ and $C(\Pi) = \varphi$. Instead of $\Lambda \vdash \varphi$ we write $\vdash \varphi$. If $\vdash \varphi$, φ is said to be a theorem (of $\mathcal{L}\Box\Box$).

Before considering the soundness of $\mathcal{L}\Box\Box$ it is appropriate to demonstrate some facts about the deductive system alone. By suitable extensions of proofs given above we have:

D0.1 $(\varphi \supset (\psi \supset \varphi))$
D0.2 $((\varphi \supset (\psi \supset \eta)) \supset ((\varphi \supset \psi) \supset (\varphi \supset \eta))$
D0.3 $((\sim\varphi \supset \psi) \supset ((\sim\varphi \supset \sim\psi) \supset \varphi))$
D0.4 $(\varphi \supset \varphi)$
D0.5 $((\varphi \supset \psi) \supset ((\varphi \supset \sim\psi) \supset \sim\varphi))$

As meta-proofs below we either exhibit an object-proof or indicate how an object-proof may be constructed.

D0.6 $(\varphi \supset \sim\sim\varphi)$
$[\varphi[\sim\sim\sim\varphi[\sim\sim\varphi]\sim\varphi]\sim\sim\varphi](\varphi \supset \sim\sim\varphi)$
D0.7 $(\sim\sim\varphi \supset \varphi)$
$[\sim\sim\varphi[\sim\varphi]\varphi](\sim\sim\varphi \supset \varphi)$
D0.8 $((\varphi \supset \psi) \supset ((\sim\varphi \supset \psi) \supset \psi))$
$[(\varphi \supset \psi)[(\sim\varphi \supset \psi)[\sim\psi[\sim\varphi\psi]\varphi\psi]\psi]((\sim\varphi \supset \psi) \supset \psi)]((\varphi \supset \psi)$
$\supset ((\sim\varphi \supset \psi) \supset \psi))$

Let us verify the obvious fact that

DRO If $S \vdash \varphi$ and $S \vdash (\varphi \supset \psi)$ then $S \vdash \psi$

Proof: Assume $S \vdash \varphi$ and $S \vdash (\varphi \supset \psi)$. Thus there are two proofs, say Π and Π', such that $A(\Pi) \cup A(\Pi') \subseteq S$, $C(\Pi) = \varphi$ and $C(\Pi') = (\varphi \supset \psi)$. $\Pi\ \Pi'$ is also a proof with $A(\Pi\ \Pi') \subseteq S$. Moreover, φ and $(\varphi \supset \psi)$ are both open in $\Pi\ \Pi'$. Thus $\Pi\ \Pi'\psi$ is a proof with $A(\Pi\ \Pi'\psi) \subseteq S$ and $C(\Pi\ \Pi'\psi) = \psi$.

Now we consider some facts involving $\Box$.

D1.1 $\vdash(\Box\varphi \supset \varphi)$
$[\Box\varphi\varphi](\Box\varphi \supset \varphi)$
D1.2 $\vdash(\sim\Box\varphi \supset \Box\sim\Box\varphi)$
$[\sim\Box\varphi\,\Box\sim\Box\varphi](\sim\Box\varphi \supset \Box\sim\Box\varphi)$
D1.3 $\vdash(\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi))$
$[\Box(\varphi \supset \psi)[\Box\varphi\varphi(\varphi \supset \psi)\psi\Box\psi](\Box\varphi \supset \Box\psi)](\Box(\varphi \supset \psi) \supset (\Box\varphi \supset \Box\psi))$
D1.4 $\vdash(\Box\varphi \supset \Box\Box\varphi)$
$[\Box\varphi\,\Box\Box\varphi](\Box\varphi \supset \Box\Box\varphi)$
D1.5 $\vdash(\varphi \supset \sim\Box\sim\varphi)$
$[\varphi[\sim\sim\Box\sim\varphi[\sim\Box\sim\varphi]\Box\sim\varphi\sim\varphi]\sim\Box\sim\varphi](\varphi \supset \sim\Box\sim\varphi)$
D1.6 $\vdash(\varphi \supset \Box\sim\Box\sim\varphi)$

To the end of the previous proof adjoin the following.

$[\sim\Box\sim\varphi\,\Box\sim\Box\sim\varphi](\sim\Box\sim\varphi \supset \Box\sim\Box\sim\varphi)\ [\varphi\sim\Box\sim\varphi\,\Box\sim\Box\sim\varphi](\varphi \supset \Box\sim\Box\sim\varphi)$

The indirectness of the proof of *D1.6* indicates a quite far-reaching troublesome situation. Specifically: there are infinitely many non-modal formulas φ such that $\varphi \models \Box\psi$ for some ψ, but $\Box\psi$ can not be proved directly from φ by proving ψ from φ and then boxing. ($\Box$I) can only be used when all open assumptions are modal. This restriction is obviously necessary. Thus if $\mathcal{L}\Box\Box$ is to be strongly complete, there must always be an "indirect" way of getting from such a φ to such a $\Box\psi$.

DR1 If $\vdash\varphi$ *and* $\vdash(\varphi\supset\psi)$ *then* $\vdash\psi$

Proof: Special case of *DR0* when $S = \Lambda$.

DR2 If $\vdash\varphi$ *then* $\vdash\Box\varphi$

Proof: Let Π be a proof of φ from Λ. All open assumptions are modal formulas (vacuously). Thus $\Pi\Box\varphi$ is a proof of $\Box\varphi$ from Λ.

1.3 In this section we compare the deductive system of $\mathcal{L}\Box\Box$ both with a formulation of sentential logic (Mendelson [8], pp. 30-37) and with a formulation of S5 as a logistic system (Kripke [6], pp. 67-68).

Let SC_1 be the following sentential logic. L, the language of SC_1, is the set of formulas of $\mathsf{L}\Box\Box$ devoid of $\Box$. An interpretation, a, of L is a function from $\mathcal{O}$ to $\{t, f\}$. As usual, the truth-valuation in a is the function V^a from L to $\{t, f\}$ such that $V^a(\varphi) = a(\varphi)$ for $\varphi\,\varepsilon\,\mathcal{O}$, $V^a(\sim\varphi) = NV^a(\varphi)$, $V^a((\varphi\supset\psi)) = C(V^a(\varphi), V^a(\psi))$. a is a model of φ iff $V^a(\varphi) = t$. a is a model of $S \subseteq \mathsf{L}$ iff $V^a(\varphi) = t$ for all $\varphi\,\varepsilon S$. $S\,\mathrm{T}\vDash\varphi$ (S tautologically implies φ) iff every model of S is a model of φ. If $\Lambda\,\mathrm{T}\vDash\varphi$ then we write $\mathrm{T}\vDash\varphi$ and call φ a tautology as usual. The deductive system of SC_1 is defined as follows: Π is a proof in SC_1 iff Π is a proof in $\mathcal{L}\Box\Box$ and every formula in Π is in L. $S\mathrm{T}\vdash\varphi$ iff there is a proof Π in SC_1 such that $A(\Pi) \subseteq S$ and $C(\Pi) = \varphi$. If $\Lambda\,\mathrm{T}\vdash\varphi$ then we write $\mathrm{T}\vdash\varphi$ and say that φ is tautologically provable.

By considering the proofs of *D.01*, *D0.2*, *D0.3* and *DR0* it is clear that they still hold when $\vdash$ is replaced by $\mathrm{T}\vdash$. It follows by the reasoning in Mendelson already cited that SC_1 is complete, i.e., that $\mathrm{T}\vDash\varphi$ implies $\mathrm{T}\vdash\varphi$. Thus the sentential rules of $\mathcal{L}\Box\Box$ form a complete sentential logic.

$S \subseteq \mathsf{L}$ is *consistent* (in the sense of SC_1) iff there is no $\psi\,\varepsilon\,\mathsf{L}$ such that $S\,\mathrm{T}\vdash\psi$ and $S\,\mathrm{T}\vdash\sim\psi$. $S \subseteq \mathsf{L}$ is *maximally consistent* (in the sense SC_1) iff S is consistent and, for $\varphi\,\varepsilon\,\mathsf{L}$, if $\varphi\not\varepsilon S$ then S, φ is inconsistent. By the reasoning in Davis [3], pp. 7-9, every consistent set is a subset of a maximally consistent set and every maximally consistent set S has a unique model (viz. for each $\varphi\,\varepsilon\,\mathcal{O}$, $a(\varphi) = t$ if $\varphi\,\varepsilon\,S$ and $a(\varphi) = f$ if $\varphi\not\varepsilon\,S$). The latter fact will be used below. It follows that SC_1 is strongly complete ($S\,\mathrm{T}\vDash\varphi$ implies $S\,\mathrm{T}\vdash\varphi$): suppose $S\,\mathrm{T}\vDash\varphi$; then S, $\sim\varphi$ has no model; by the above result, S, $\sim\varphi$ is inconsistent; S, $\sim\varphi$ $\mathrm{T}\vdash\psi$ and S, $\sim\varphi$ $\mathrm{T}\vdash\sim\psi$; using ($\sim$E), $S\,\mathrm{T}\vdash\varphi$.

Let SC_2 be exactly like SC_1 except that the language is now all of $\mathsf{L}\Box\Box$. Interpretations will be functions from $\mathcal{O}\cup\{\Box\varphi{:}\varphi\,\varepsilon\,\mathsf{L}\Box\Box\}$ to $\{t, f\}$ and the proofs are those of $\mathcal{L}\Box\Box$ *not* involving use of ($\Box$E) or ($\Box$I). All of the above results for SC_1 hold *mutatis mutandis* for SC_2. Thus all of the tautologies in $\mathsf{L}\Box\Box$ are theorems of SC_2.

Except for a change in notation, the set of theorems of S5 as formulated by Kripke (loc. cit.) is the smallest set containing all tautologies of $\mathsf{L}\Box\Box$ and all formulas of the forms *D1.1*, *D1.2* and *D1.3* and satisfying *DR1* and *DR2*. Thus if φ is a theorem of S5 then φ is a theorem of $\mathcal{L}\Box\Box$. Kripke showed that S5 is complete relative to the semantics of $\mathcal{L}\Box\Box$, i.e., if $\vDash\varphi$ then φ is a theorem of S5. It follows then that

Theorem 1.3: $\mathcal{L}\Box\Box$ is (weakly) complete in the sense that $\vDash\varphi$ implies $\vdash\varphi$.

1.4 The completeness theorem for $\mathcal{L}\Box\Box$ and other provability results, as well, are without significance unless $\mathcal{L}\Box\Box$ is at least shown to be consistent (i.e. there is no φ such that $\vdash\varphi$ and $\vdash\sim\varphi$). Moreover, the deductive system of $\mathcal{L}\Box\Box$ is inadequate unless every one of its proofs Π can be relied on to establish logical consequence between premises and conclusion ($A(\Pi)\vDash C(\Pi)$). In this section we demonstrate that $\mathcal{L}\Box\Box$ has this property. It will follow that $\mathcal{L}\Box\Box$ is strongly sound, i.e. that $S\vdash\varphi$ implies $S\vDash\varphi$. From strong soundness it follows that $\mathcal{L}\Box\Box$ is (weakly) sound ($\vdash\varphi$ implies $\vDash\varphi$) and from (weak) soundness consistency follows.

Lemma 1.4: For every proof Π in $\mathcal{L}\Box\Box$, if φ occurs open in Π then $A(\Pi)\vDash\varphi$.

Proof: The proof is by a "course-of-values" induction on the proofs of $\mathcal{L}\Box\Box$. The lemma is obvious for proofs of the form $[\psi$. Now let Π be an arbitrary proof and assume that the lemma holds for all proper subproofs of Π, i.e. assume that for all Π' such that $\Pi = \Pi'\sigma$, σ non-null, $A(\Pi')\vDash\varphi$ for all φ open in Π'. This is the induction hypothesis. We show that the lemma holds for Π. There are seven cases according to which of the seven rules of inference was used to form the last "step" of Π. In carrying out these seven cases we had found it convenient to define for each Π the set of formulas occurring open in Π. Since we merely sketch the proof here, we will not introduce any new notation for this concept.

In case (A) was the last rule used, the induction step follows from the induction hypothesis, *S0.1* and *S1*. For (R) as the last rule used, the induction step follows from the induction hypotheses. For ($\supset$E), ($\supset$I), ($\sim$E), ($\Box$E) and ($\Box$I) the induction step follows by the induction hypothesis, *S0.0* and, respectively, *S2*, *S3*, *S4*, *S5* and *S6*. The induction step follows in general. therefore, and the proof is complete.

Incidentally, since the deductive systems of SC_1, SC_2 and the sentential part of $\mathcal{L}\Box\Box$ are all included in that of $\mathcal{L}\Box\Box$ and since *S0.0, S0.1, S1, S2, S3, S4* and *S5* all hold for the semantics of sentential logic; it follows by the above reasoning that the theorem holds for these logics as well.

Theorem 1.4: $\mathcal{L}\Box\Box$ is strongly sound, i.e. $S\vdash\varphi$ implies $S\vDash\varphi$.

Proof: Assume $S\vdash\varphi$. Then there exists a proof Π with $A(\Pi)\subseteq S$ and $C(\Pi) = \varphi$. Since φ is open in Π, by the lemma $A(\Pi)\vDash\varphi$. By *S0.1*, $S\vDash\varphi$.

In the special case where S is empty we have the weak soundness result for $\mathcal{L}\Box\Box$, i.e. $\vdash\varphi$ implies $\vDash\varphi$. Putting this together with Theorem *1.3* (weak completeness) we have what is called by some logicians a "weak bridge result" for $\mathcal{L}\Box\Box$, viz. $\vdash\varphi$ iff $\vDash\varphi$, thereby partially bridging the semantics ($\vDash$) and the deductive system ($\vdash$). Since S5 (as formulated by Kripke) also has the weak bridge property we have that φ is a theorem of S5 iff φ is a theorem of $\mathcal{L}\Box\Box$.

Note: The rule (R) is superfluous in the sense that if $S \vdash \varphi$ in $\mathcal{L}\Box\Box$ then there is a proof in $\mathcal{L}\Box\Box$ of φ from S which does not make use of (R). This is proved using the following proposition: if Π is a proof in $\mathcal{L}\Box\Box$ and Π' is the result of deleting from Π the first occurrence of a formula obtained by use of (R) then Π' is a proof in $\mathcal{L}\Box\Box$ and every φ open in Π is open in Π'. Now assume $S \vdash \varphi$. It follows that there is a proof Π_0 of φ from S such that every other proof of φ from S contains at least as many formulas as Π, i.e. it follows that there is a "shortest" proof Π_0 of φ from S. By the above-mentioned proposition Π_0 cannot contain any formulas obtained by use of (R).

2 *The Logic* $\mathcal{L}\Box$ Here we consider a proper subsystem $\mathcal{L}\Box$ of $\mathcal{L}\Box\Box$. The language of $\mathcal{L}\Box$ is $\mathsf{L}\Box$, the set of formulas of $\mathsf{L}\Box\Box$ without iterated $\Box$. More precisely, $\mathsf{L}\Box$ is the smallest set of formulas (1) containing all formulas in L, (2) containing all formulas $\Box\varphi$ with φ in L and (3) closed under forming $\sim\varphi$ and $(\varphi \supset \psi)$ from φ and ψ already in the set. The semantics of $\mathcal{L}\Box$ is exactly the same as that of $\mathcal{L}\Box\Box$ except all valuations V^{aP} are restricted to $\mathsf{L}\Box$. The proofs of $\mathcal{L}\Box$ are the proofs of $\mathcal{L}\Box\Box$ which contain only sentences in $\mathsf{L}\Box$—i.e. the only changes in the rules come (1) for ($\Box$I) where one cannot form $\Box\varphi$ unless φ has no $\Box$ occurring in it and (2) for (A) where one cannot write $[\varphi$ unless φ is in $\mathsf{L}\Box$. Here we use $\vDash$ and $\vdash$ relative to $\mathcal{L}\Box$. If there is any chance of confusion, we will add "in $\mathcal{L}\Box$" or "in $\mathcal{L}\Box\Box$".

Theorem 2.1: $\mathcal{L}\Box$ is strongly sound.

Proof: Let $S, \varphi \subseteq \mathsf{L}\Box$. Suppose $S \vdash \varphi$ in $\mathcal{L}\Box$. Then $S \vdash \varphi$ in $\mathcal{L}\Box\Box$. By Theorem *1.4* (strong soundness), $S \vDash \varphi$ in $\mathcal{L}\Box\Box$. But since $S, \varphi \subseteq \mathsf{L}\Box$, it follows by the remarks of the last paragraph that $S \vDash \varphi$ in $\mathcal{L}\Box$.

Our object now is to show strong completeness for $\mathcal{L}\Box$, i.e. that $S \vDash \varphi$ implies $S \vdash \varphi$. The proof parallels the ideas of Henkin [5]. Consistent and maximally consistent, both relative to $\mathcal{L}\Box$, are defined as above *mutatis mutandis*. By Lindenbaum's argument (Davis [3], p. 7) every consistent set of sentences is a subset of some maximally consistent set. We show that a model aP can be formed for each maximally consistent set. Thus every consistent set of sentences has a model. From this fact, strong completeness follows (See section **1.3** above):

First we need some elementary facts about maximally consistent sets of formulas of $\mathcal{L}\Box$. Let $\Box(S) = \{\varphi\colon\ \Box\varphi\, \varepsilon S\}$

Theorem 2.2 If S is maximally consistent (relative to $\mathcal{L}\Box$) then

(1) *If* $\varphi \not\varepsilon S$ *then* $S \vdash \sim\varphi$,
(2) *If* $S \vdash \varphi$ *then* $\varphi \varepsilon S$, (*Hence* $\varphi \varepsilon S$ *iff* $S \vdash \varphi$),
(3) $\varphi \varepsilon S$ *or* $\sim\varphi \varepsilon S$,
(4) $\sim\varphi \varepsilon S$ *iff* $\varphi \not\varepsilon S$,
(5) $(\varphi \supset \psi)\, \varepsilon S$ *iff* $\varphi \not\varepsilon S$ *or* $\psi \varepsilon S$,
(6) $S \cap \mathsf{L}$ *is maximally consistent in the sense of sentential logic*,
(7) $\Box(S)$ *is consistent in the sense of sentential logic, and*
(8) $\Box(S) \vdash \varphi$ *implies* $\varphi\, \varepsilon \Box(S)$.

Proof: Parts (1) through (6) are easily proved using *D0.1*, *D0.5* and the definitions. (7) follows from (6) using the observation that $\Box(S) \subseteq S \cap \mathsf{L}$. To see (8), assume that $\Box(S) \vdash \varphi$. Let Π be a proof of φ from $\Box(S)$ and let $\varphi_1, \varphi_2, \ldots$, and φ_n be the (open) assumptions of Π. Then $[\Box\varphi_1\, [\Box\varphi_2\, [\Box\varphi_3, \ldots, [\Box\varphi_n\, \Pi'$ is also a proof where Π' is obtained from Π by deleting the ['s before $\varphi_1, \varphi_2, \ldots$, and φ_n in Π—the occurrence of φ_i in the new proof is justified by ($\Box$E). Since each φ_i is in $\Box(S)$, each $\Box\varphi_i$ is in S. Since all assumptions of the new proof are modal and φ is open in the new proof, $\Box\varphi$ can be added on by ($\Box$I). Thus $S \vdash \Box\varphi$ and by 2, $\Box\varphi \varepsilon S$. Thus $\varphi \varepsilon S$.

Next we form a model of any given maximally consistent set. But first consider any consistent $S \subseteq \mathsf{L}\Box$. $S \cap \mathsf{L}$ is consistent in the sense of sentential logic and thus has a model a. Thus for any P with $a \varepsilon P$, aP is a model (in the sense of $\mathcal{L}\Box$) of $S \cap \mathsf{L}$. $\Box(S)$ is also consistent in the sense of sentential logic and thus has a model, b. Let P' be the set of all models of $\Box(S)$. For any $a' \varepsilon P'$, $a'P'$ is a model of $\{\Box\varphi : \varphi \varepsilon \Box(S)\}$. But aP' need not be a model of S. We will show that in case S is maximally consistent aP' where a is a model of $S \cap \mathsf{L}$ and P' is the set of models of $\Box(S)$ is a model of S.

Theorem 2.3 If S is maximally consistent and a is the (unique) model of $S \cap \mathsf{L}$ *and P is the set of all models (in the sense of sentential logic) of* $\Box\ (S)$ *then*

(1) *for all* $\varphi \varepsilon \mathsf{L}\Box$, $\varphi \varepsilon S$ *iff* $\mathrm{V}^{aP}(\varphi) = \mathrm{t}$ *and, therefore,*
(2) aP *is a model of* S

Proof: $\mathsf{L}\Box$ is the smallest set containing $\mathcal{C}$ and $\{\Box\varphi : \varphi \varepsilon \mathsf{L}\}$ and closed under forming $\sim\varphi$ and $(\varphi \supset \psi)$ for φ, ψ already in the set.

(1) can easily be seen to hold for all formulas φ in $\mathcal{C}$. It also holds for all formulas $\Box\varphi$ for $\varphi \varepsilon \mathsf{L}$. This is seen as follows. Let $\varphi \varepsilon \mathsf{L}$. Then $\mathrm{V}^{aP}(\Box\varphi) = \mathrm{t}$ iff $\mathrm{V}^{aP}(\varphi) = \mathrm{t}$ for all $b \varepsilon P$. Since $\varphi \varepsilon \mathsf{L}$, $\mathrm{V}^{bP}(\varphi) = b(\varphi)$. Thus $\mathrm{V}^{aP}(\Box\varphi) = \mathrm{t}$ iff $b(\varphi) = \mathrm{t}$, for all $b \varepsilon P$. Now we show 'only if': Suppose $\Box\varphi \varepsilon S$. Then $\varphi \varepsilon \Box(S)$. Thus $b(\varphi) = \mathrm{t}$ for all $b \varepsilon P$ (by hypothesis). Thus $\mathrm{V}^{aP}(\Box\varphi) = \mathrm{t}$. For 'if' we need part (8) of Theorem *2.2*. Assume $\mathrm{V}^{aP}(\Box\varphi) = \mathrm{t}$. Then $b(\varphi) = \mathrm{t}$ for all $b \varepsilon P$. Since P is the set of all models (in the sense of sentential logic) of $\Box(S)$, $\Box(S)\ \mathrm{T} \vDash \varphi$. By strong completeness of sentential logic $\Box\ (S) \vdash \varphi$. By part (8) of *2.2*, $\varphi \varepsilon \Box S$. By the definition of $\Box(S)$, $\Box\varphi \varepsilon S$.

It can easily be seen that the property of (1) holds of $\sim\varphi$ and $(\varphi \supset \psi)$ if it holds for φ and ψ. Thus it holds for all formulas in $\mathsf{L}\Box$.

Theorem 2.4: $\mathcal{L}\Box$ *is strongly complete.*

Corollary 2.4: The semantics of $\mathcal{L}\Box$ *is compact,* i.e.
(1) *If* $S \vDash \varphi$ *then there is a finite* $F \subseteq S$ *such that* $F \vDash \varphi$ *and (equivalently)*
(2) *if S has no model then there is a finite* $F \subseteq S$ *such that F has no model.*

Proof: Suppose $S \vDash \varphi$. By strong completeness $S \vdash \varphi$. Then there is a proof Π of φ from S. Π can only have a finite set F of assumptions. Thus $F \vdash \varphi$. By soundness $F \vDash \varphi$.

Putting Theorems *2.4* and *1.4* (strong soundness) together we have

$S \vDash \varphi$ iff $S \vdash \varphi$, the strong bridge result for $\mathcal{L}\Box$. This completely bridges the semantics and the deductive system of $\mathcal{L}\Box$. In addition, we have the following.

Theorem 2.5: *Let* N *and* N_0, $\varphi \subseteq \mathsf{L}$ *and let* $M_0 = \{\Box\varphi : \varphi \varepsilon N_0\}$. *Then*

(1) $N \vdash \varphi$ *iff* $N\mathrm{T} \vdash \varphi$

(2) $\vdash \Box\varphi$ *iff* $\mathrm{T} \vDash \varphi$

(3) *if* N *is consistent then* $N \vdash \Box\varphi$ *iff* $\vdash \Box\varphi$

(4) *if* $M_0 \cup N$ *is consistent then* $M_0 \cup N \vdash \varphi$ *iff* $M_0 \vdash \Box\varphi$

(5) $M_0 \vdash \varphi$ *iff* $N_0 \vdash \varphi$

Proof: The 'if' parts of (1) through (4) are immediate. We will show the 'only if' parts of (1) through (4) first and then show (5). (1): Suppose $N \vdash \varphi$. Then $N \vDash \varphi$. Now we show $N\mathrm{T} \vDash \varphi$ which implies $N\mathrm{T} \vdash \varphi$ by strong completeness of sentential logic. Let a be any model of N. Then $a\{a\}$ is a model (in $\mathcal{L}\Box\Box$ of N. Thus $a\{a\}$ is a model of φ. Since $\varphi \varepsilon \mathsf{L}$, a is a model of φ. (2): Suppose $\vdash \Box\varphi$. Then $\vdash \varphi$. Hence $\vDash \varphi$. Let a be any interpretation. $\mathrm{V}^{a\{a\}}(\Box\varphi) = \mathrm{t}$ iff $a(\varphi) = \mathrm{t}$. But $\vDash \varphi$. Thus a is a model of φ. So $\mathrm{T} \vDash \varphi$. (3): Suppose $N \vdash \Box\varphi$ where N is consistent. Then $N \vDash \Box\varphi$ and N has a model a. Hence aP is a model of N where P is the set of *all* interpretations of L. Thus $\mathrm{V}^{aP}(\Box\varphi) = \mathrm{t}$. But $\mathrm{V}^{aP}(\Box\varphi) = \mathrm{t}$ iff $b(\varphi) = \mathrm{t}$ for all b. Thus $\mathrm{T} \vDash \varphi$. Thus $\mathrm{T} \vdash \varphi$. So $\vdash \varphi$ and $\vdash \Box\varphi$. (4): Suppose that $M_0 \cup N$ is consistent and $M_0 \cup N \vdash \Box\varphi$. By soundness $M_0 \cup N \vDash \Box\varphi$ and there exists a model, aP say, of $M_0 \cup N$. We will show $M_0 \vDash \Box\varphi$ from which $M_0 \vdash \Box\varphi$ follows by strong completeness. Let $a'P'$ be any model of M_0. Then $a(P \cup P')$ is a model of $M_0 \cup N$. Why? Suppose $\eta \varepsilon N$ then $a(P \cup P')$ is a model of η iff $a(\eta) = \mathrm{t}$. But aP is a model of N. Suppose $\eta \varepsilon M_0$. Then $\eta = \Box\psi$ and $a(P \cup P')$ is a model of $\Box\psi$ iff $b(\psi) = \mathrm{t}$ for all $b \varepsilon P \cup P'$. But aP is a model of M_0, hence $b(\psi) = \mathrm{t}$ for all $b \varepsilon P$, and $a'P'$ is a model of M_0, hence $b(\psi) = \mathrm{t}$ for all $b \varepsilon P'$. Hence $a(P \cup P')$ is a model of $M_0 \cup N$. Since $M_0 \cup N \vDash \Box\varphi$, $a(F \cup P')$ is a model of $\Box\varphi$. Thus $b(\varphi) = \mathrm{t}$ for all $b \varepsilon (P \cup P')$, in particular, for all $b \varepsilon P'$. Thus $a'P'$ is a model of $\Box\varphi$. But $a'P'$ was an arbitrary model of M_0; hence $M_0 \vDash \Box\varphi$. (5) Since $\Box\psi \vdash \psi$, if $N_0 \vdash \varphi$ then $M_0 \vdash \varphi$. Now assume $M_0 \vdash \varphi$. Let a be any model of N_0. $a\{a\}$ is, therefore, a model of M_0 and hence of φ. But $\varphi \varepsilon \mathsf{L}$ so $\mathrm{V}^{a\{a\}}(\varphi) = \mathrm{V}^{a}(\varphi)$. Thus $N_0\mathrm{T} \vDash \varphi$. Thus $N_0\mathrm{T} \vdash \varphi$. By (1), $N_0 \vdash \varphi$.

The significance of these results is as follows. (1) tells us that, if we can prove a formula of sentential logic from a set of sentential logic assumptions using the system $\mathcal{L}\Box$, then we can prove the same thing using only the sentential logic rules. Thus ($\Box$E) and ($\Box$I) provide a "conservative" extension of sentential logic. (2) tells us that, $\Box\varphi$ is a theorem of $\mathcal{L}\Box$ iff φ is a tautology. (3) tells us that even in the case of a theory, all of whose assumptions are devoid of $\Box$, $\Box\varphi$ is a theorem of the system iff φ is a tautology. Thus extending SC_1 to $\mathcal{L}\Box$ enables one to distinguish from among the consequences of a set of sentential logic assumptions those which do not depend on the particular assumptions. (4) implies that we can use $\mathcal{L}\Box$ to systematize the distinction between assumptions which we might want to regard as necessary and other assumptions which we would consider merely "factual" or merely true. If we put boxes before the former and list the later (simply) then: if we can prove $\Box\varphi$ then φ is a consequence of the necessary assumptions. However, in such a case if we can prove φ, but not $\Box\varphi$, it does not follow that φ is a consequence of the "factual" assumptions. Example: $\Box P, P \supset R \vdash R$ but R does not follow from $P \supset R$ alone. For further discussion along these lines see Curry [2], pp. 359-360 (5) tells us that we do not get any "truths" by assuming that our axioms are necessary that we could not already obtain by taking the assumptions as merely true.

Note: Notice that, in the proof of strong completeness of $\mathcal{L}\Box$, the rule ($\Box$I) was used only once, viz. in the proof of clause (8) of Theorem *2.2*. Here a weakened rule, ($\Box$I'), would do just as well.

($\Box$I'): If Π is a proof, all of whose assumptions are of the form $\Box\psi$ and in which φ is open, then $\Pi\Box\varphi$ is a proof.

The resulting system is precisely the one called PN in [1]. ($\Box$I') is analogous to Curry's sequent rule, [2], p. 361.

It is also of interest to point out that every theorem of $\mathcal{L}\Box$ is also a theorem of S4, S5 and the Brouwerian System and that every formula in $\mathsf{L}\Box$ which is a theorem of S4, S5 *or* the Brouwerian System is a theorem of $\mathcal{L}\Box$. Thus S4, S5 and the Brouwerian System agree on $\mathsf{L}\Box$. This follows from Kripke's soundness and completeness proofs for these systems with the help of an easy theorem to the effect that for every S4 or Brouwerian interpretation, i, there exists an interpretation aP such that, for all $\varphi \varepsilon \mathsf{L}\Box$, $V^{aP}(\varphi) = V^{i}(\varphi)$.

3 *Strong Completeness of* $\mathcal{L}\Box\Box$; The strong completeness of $\mathcal{L}\Box\Box$ is proved by reducing the question to that of the strong completeness and (strong) soundness of $\mathcal{L}\Box$. Notice that $\mathcal{L}\Box\Box$ is an extension of $\mathcal{L}\Box$ in two senses. Semantic extension: every interpretation aP of $\mathsf{L}\Box$ is an interpretation of $\mathsf{L}\Box\Box$ and vice versa. Moreover, for every formula $\varphi \varepsilon \mathsf{L}\Box$, aP is a model of φ in the sense of $\mathcal{L}\Box$ iff aP is a model of φ in the sense of $\mathcal{L}\Box\Box$. Thus, for S, $\varphi \subseteq \mathsf{L}\Box$, $S \vDash \varphi$ in the sense of $\mathcal{L}\Box$ iff $S \vDash \varphi$ in the sense of $\mathcal{L}\Box\Box$. Consequently, the phrase "in the sense of . . ." is redundant—if S, $\varphi \subseteq \mathsf{L}\Box$, then the ambiguity of '$S \vDash \varphi$' is inconsequential; if S, $\varphi \nsubseteq \mathsf{L}\Box$, then '$S \vDash \varphi$' can only mean '$S \vDash \varphi$ in the sense of $\mathcal{L}\Box\Box$'. Deductive extension: If Π is a proof in $\mathcal{L}\Box$ then Π is a proof in $\mathcal{L}\Box\Box$. Hence, for S, $\varphi \subseteq \mathsf{L}\Box$, if $S \vdash \varphi$ in the sense of $\mathcal{L}\Box$ then $S \vdash \varphi$ in the sense of $\mathcal{L}\Box\Box$. Now suppose S, $\varphi \subseteq \mathsf{L}\Box$ and $S \vdash \varphi$ in the sense of $\mathcal{L}\Box\Box$. By soundness $S \vDash \varphi$. By strong completeness of $\mathcal{L}\Box$, $S \vdash \varphi$ in the sense of $\mathcal{L}\Box$. Thus the phrase "in the sense of . . ." is redundant here as above.

Let us introduce the usual definitions of &, v and $\equiv$. For all φ and all ψ, $(\varphi \& \psi) = \sim(\varphi \supset \sim\psi)$, $(\varphi \vee \psi) = (\sim\varphi \supset \psi)$ and $(\varphi \equiv \psi) = ((\varphi \supset \psi) \& (\psi \supset \varphi))$. The usual rules of inference for &, v and $\equiv$ are derived rules in SC_1 and hence in $\mathcal{L}\Box$ and $\mathcal{L}\Box\Box$. The same holds for theorems involving the defined symbols. In addition, we have

D3.0 $\vdash \Box(\varphi \& \psi) \equiv (\Box\varphi \& \Box\psi)$
$\vdash \Box(\varphi_1 \& \varphi_2 \& \ldots \& \varphi_n) \equiv (\Box\varphi_1 \& \Box\varphi_2 \& \ldots \& \Box\varphi_n)$

D3.1 If ψ and $\psi_1, \psi_2, \ldots, \psi_n$ are modal formulas then
$\vdash\Box(\varphi \vee \psi) \equiv (\Box\varphi \vee \psi)$
$\vdash\Box(\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_n \vee \psi_1 \vee \psi_2 \vee \ldots \vee \psi_n) \equiv (\Box(\varphi_1 \vee \varphi_2 \vee \ldots \vee \varphi_n) \vee \psi_1 \vee \psi_2 \vee \ldots \vee \psi_n)$

D3.0 and *D3.1* will not be used immediately.

Lemma 3.1: If for every $\varphi \varepsilon \mathbf{L}\Box\Box$ *there exists* $\varphi' \varepsilon \mathbf{L}\Box$ *such that* $\vdash\varphi \equiv \varphi'$, *then* $\mathcal{L}\Box\Box$ *is strongly complete.*

Proof: Assume the hypothesis and let $S \models \varphi$. For each $\psi \varepsilon S$ choose a $\psi' \varepsilon \mathbf{L}\Box$ such that $\vdash\psi \equiv \psi'$ and let S' be the set of such formulas. Let φ' be a member of $\mathbf{L}\Box$ such that $\vdash\varphi \equiv \varphi'$. By soundness of $\mathcal{L}\Box\Box$, if $\vdash\eta \equiv \eta'$ then $\models\eta \equiv \eta'$. Thus $S' \models \varphi'$. By strong completeness of $\mathcal{L}\Box$, $S' \vdash \varphi'$. Let Π be a proof in $\mathcal{L}\Box$ of φ' from S' and let $\psi_1', \psi_2', \ldots, \psi_n'$ be the assumptions of Π. By n applications of ($\supset$I) we have a proof Π_0 of $\vdash(\psi_1' \supset (\psi_2' \supset \ldots \supset (\psi_n' \supset \varphi'))$. Let $\psi_1, \psi_2, \ldots, \psi_n'$ be members of S provably equivalent to ψ_1', $\psi_2', \ldots, \psi_n'$ and let $\Pi_1, \Pi_2, \ldots, \Pi_n$ and Π_{n+1} be respectively proofs of $(\psi_1 \supset \psi_1'), (\psi_2 \supset \psi_2'), \ldots, (\psi_n \supset \psi_n')$ and $(\varphi' \supset \varphi)$. These exist because of the "derived rule": $(\eta \equiv \eta')$ implies $(\eta \supset \eta')$ and $(\eta' \supset \eta)$.

$$\Pi_0\Pi_1\Pi_2, \ldots, \Pi_{n+1}[\psi_1[\psi_2, \ldots, [\psi_n\psi_1'\psi_2', \ldots, \psi_n'(\psi_2' \supset (\psi_3' \supset, \ldots, (\psi_n' \supset \varphi'), \ldots,), \ldots, (\psi_n' \supset \varphi')\ \varphi'\ \varphi$$

The above is a proof of φ from $\psi_1, \psi_2 \ldots \psi_n$. Thus $S \vdash \varphi$. The above proof is constructed as follows. The concatenation of $\Pi_0, \Pi_1, \Pi_2, \ldots,$ and Π_{n+1} is a proof having no open assumptions and having $(\psi_1 \supset \psi_1'), (\psi_2 \supset \psi_2'), \ldots, (\psi_n \supset \psi_n'), (\psi_1' \supset (\psi_2' \supset, \ldots, (\psi_n' \supset \varphi'), \ldots,)$ and $(\varphi' \supset \varphi)$ as open formulas. Thus after $\psi_1, \psi_2, \ldots,$ and ψ_n have been introduced as assumptions, $\psi_1', \psi_2', \ldots,$ and ψ_n' can be added as open formulas by detachment ($\supset$E). Now $(\psi_2' \supset (\psi_3' \supset, \ldots, (\psi_n' \supset \varphi'), \ldots,)$ can be added by detachment. By continuing to detach the consequent of the last formula added we ultimiately add φ'. Now we detach φ from $(\varphi' \supset \varphi)$.

It remains only to show that every formula in $\mathbf{L}\Box\Box$ is provably equivalent to a formula in $\mathbf{L}\Box$. For this we need "substitutivity of equivalence". For this, in turn, we need the following.

D3.2 $\vdash((\varphi \equiv \psi) \supset (\sim\varphi \equiv \sim\psi))$
D3.3 $\vdash((\varphi \equiv \psi) \supset ((\eta \supset \varphi) \equiv (\eta \supset \psi))$
D3.4 $\vdash((\varphi \equiv \psi) \supset ((\varphi \supset \eta) \equiv (\psi \supset \eta))$
D3.5 If $\vdash(\varphi \equiv \psi)$ then $\vdash(\Box\varphi \equiv \Box\psi)$

The first three of these are tautologies. *D3.5* is essentially two applications of *D1.3*. Incidentally, it is not true that $\models((\varphi \equiv \psi) \supset (\Box\varphi \equiv \Box\psi))$.

Theorem 3.2 (*"Substitutivity of Equivalence"*). *If* φ *occurs at least once in* η *and* $\eta\begin{bmatrix}\varphi\\\psi\end{bmatrix}$ *is the result of replacing one particular occurrence of* φ *in* η *by* ψ *then* $\vdash(\varphi \equiv \psi)$ *implies* $\vdash\left(\eta \equiv \eta\begin{bmatrix}\varphi\\\psi\end{bmatrix}\right)$.

The proof of *3.2* involves a tedious induction on the subformulas of η in which φ occurs. The smallest of these is φ itself and $\vdash(\varphi \equiv \psi)$ implies $\vdash(\varphi \equiv \psi)$. Suppose that the theorem has been established for the nth largest such subformula. Then one of *D3.2*, *D3.3*, *D3.4* or *D3.5* will apply to establish the theorem for the $n + 1$st largest. Since η is finite, the argument terminates.

In order to proceed, we define the rank of a formula as the depth of iterated boxes in it. Specifically, $r(\varphi) = 0$ for $\varphi \varepsilon \mathsf{L}$, $r(\Box\varphi) = r(\varphi) + 1$, $r(\sim\varphi) = r(\varphi)$, $r(\varphi \supset \psi) = \max(r(\varphi), r(\psi))$. Notice that $\mathsf{L}\Box$ contains all and only formulas of rank ≤ 1. Let us call φ an atom of ψ iff φ occurs in ψ and either (1) $\varphi \varepsilon \mathcal{C}$ and at least one occurrence of φ in ψ is *not* inside of a formula of the form $\Box\eta$ in ψ or (2) φ is itself a formula of the form $\Box\eta$ and at least one occurrence of φ is not (properly) inside of a formula of the form $\Box\eta'$ in ψ. E.g. if P, Q and R are in $\mathcal{C}$ then the only atom of $\sim\Box((\Box P \supset Q) \supset R)$ is $\Box((\Box P \supset Q) \supset R)$ and the atoms of $((P \supset Q) \supset (\Box(R \supset \Box P) \supset \Box P))$ are P, Q, $\Box(R \supset \Box P)$ and $\Box P$. Obviously, the rank of φ is the maximum of the ranks of the atoms of φ.

Since SC_2 is complete we have the normal form theorems. In particular, we have that every formula φ is provably equivalent to a formula φ' in conjunctive normal form. If we adopt the convention that a conjunctive normal form of a tautology is the disjunction of the atoms and the negations of the atoms of the tautology, then we have that every formula φ of $\mathsf{L}\Box\Box$ is provably equivalent to a conjunction of one or more disjunctions of its atoms and (or) their negations. Every such conjunctive normal form of φ has the same rank as φ.

Theorem 3.3 If $r(\varphi) = n$, $n \geq 1$ *then there is a formula* φ' *such that* $r(\varphi') = n$ *and* $\vdash \Box\varphi \equiv \varphi'$.

Proof: Let $r(\varphi) = n$, $n \geq 1$. Then $\vdash \varphi \equiv \psi_1 \,\&\, \psi_2 \,\&\, \ldots \,\&\, \psi_n$ where ψ_1, $\psi_2, \ldots,$ and ψ_n are disjunctions of the atoms and (or) negations of atoms of φ. By *D3.0*. $\vdash \Box\varphi \equiv \Box\psi_1 \,\&\, \Box\psi_2 \,\&\, \ldots \,\&\, \Box\psi_n$. Without loss of generality, we may assume that each $\psi_i = \alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n \vee \beta_1 \vee \beta_2 \vee \ldots \vee \beta_m$, where each α_j has the form α or $\sim\alpha$ for $\alpha \varepsilon \mathcal{C}$ and where each β_j has the form $\Box\beta$ or $\sim\Box\beta$, $\Box\beta$ being an atom of φ. By *D3.1* $\vdash \Box\psi \equiv (\Box(\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n) \vee (\beta_1 \vee \beta_2 \vee \ldots \vee \beta_n))$. Putting these together using substitutivity of equivalence we have $\vdash \Box\varphi \equiv \varphi'$, where φ' is a truth-functional combination of (1) the modal atoms of φ and (2) some new modal atoms of rank 1. Since φ has rank $n \geq 1$, φ' has the same rank.

Theorem 3.4 (*Reduction Theorem*): *For every formula* $\varphi \varepsilon \mathsf{L}\Box\Box$ *there exists a formula* $\varphi' \varepsilon \mathsf{L}\Box$ *such that* $\vdash \varphi \equiv \varphi'$.

Proof: By induction on rank. For $r(\varphi) = 0$ and $r(\varphi) = 1$ use φ itself. Now assume that for every formula φ of rank n there exists $\varphi' \varepsilon \mathsf{L}\Box$ such that $\vdash \varphi \equiv \varphi'$. Let $r(\varphi) = n + 1$. Then one or more atoms of φ are of rank $n + 1$ and the rest are of rank less than $n + 1$. Using substitutivity of equivalence and *3.3*, replace every atom of rank $n + 1$ in φ by an equivalent formula of rank n obtaining ψ of rank n such that $\vdash \varphi \equiv \psi$. By induction hypothesis $\vdash \psi \equiv \psi'$ where $r(\psi') \leq 1$. Thus $\vdash \varphi \equiv \psi'$ where $\psi' \varepsilon \mathsf{L}\Box$.

Theorem 3.5: $\mathcal{L}\Box\Box$ is strongly complete.

Corollary 3.5: Kripke's semantics for S5 *is compact.*

Putting *3.5* together with *1.4* we have the strong bridge result for $\mathcal{L}\Box\Box$.

4 *Some further conclusions*. $\mathcal{L}\Box\Box$ is an extension of Kripke's system S5. The primary difference between the two systems stems from the fact that Kripke's system handles only "logical truth" whereas $\mathcal{L}\Box\Box$ handles both "logical truth" and "valid arguments". Deductively, this means that $\vdash\varphi$ is defined in Kripke's system whereas $\vdash\varphi$ and $S\vdash\varphi$ are both defined in $\mathcal{L}\Box\Box$. Semantically, this means that $\models\varphi$ is defined in Kripke's system, whereas both $\models\varphi$ and $S\models\varphi$ are defined in $\mathcal{L}\Box\Box$. In order to define $\models\varphi$ it is sufficient to define interpretations of formulas. To define $S\models\varphi$ it is much more natural to define interpretations of the entire language. Our definitions of "interpretation of $\mathsf{L}\Box\Box$" is obtained by replacing the words 'a formula' in Kripke's definition of "interpretation of a formula" by the word '$\mathsf{L}\Box\Box$'. Since $\mathcal{L}\Box\Box$ is strongly sound and strongly complete, every other deductive system which has these two properties relative to the above semantics is essentially the same as the deductive system of $\mathcal{L}\Box\Box$ in the sense that for every S, $\varphi\subseteq\mathsf{L}\Box\Box$, φ is provable from S in the other system iff φ is provable from S in $\mathcal{L}\Box\Box$.

Some logicians (e.g. Feys, [6] p. 6) have expressed a special interest in "superposed" or iterated modalistics. As a result of the above investigations we can assert that iterated modalities are completely dispensable within $\mathcal{L}\Box\Box$. The reduction theorem (*3.5*) and the strong soundness theorem (*1.4*) imply that iterated modalities are dispensable in regard to expressive power in the sense that for every formula φ containing iterated modalities there is another sentence φ having no iterated modalities and having exactly the same models as φ, i.e. $\models\varphi\equiv\varphi'$. Moreover, the method employed in the proof of strong completeness also yields the result that iterated modalities are dispensable in regard to deduction in the sense that if S, φ contains iterated modalities and S', φ' is a corresponding semantically equivalent set of formulas not containing iterated modalities, then $S\vdash\varphi$ implies that there is a proof of φ' from S' *none* of whose formulas contain iterated modalities.

As Theorem *2.5* is stated it holds for $\mathcal{L}\Box\Box$ as well for $\mathcal{L}\Box$. As a result of this theorem we have that (1) $\Box\varphi$ is provable in $\mathcal{L}\Box$ iff φ is a tautology. Naturally, this latter result does not hold in $\mathcal{L}\Box\Box$, e.g. if P and Q are in $\mathcal{O}$ then in $\mathcal{L}\Box\Box$ we have $\vdash\Box(\Box\mathrm{P}\supset\Box(\mathrm{Q}\supset\mathrm{P}))$. Noticing that fact (1) compares provability of a rank 1 expression to a semantical property of a related rank 0 expression, we can generalize it as follows. Let $\mathsf{L}\Box_n$ be the sent of formulas of rank n. Thus $\mathsf{L}\Box_0=\mathsf{L}$, $\mathsf{L}\Box_1=\mathsf{L}\Box$, and $\bigcup_n\mathsf{L}\Box_n=\mathsf{L}\Box\Box$. Let $\mathcal{L}\Box_n$ be the restriction of $\mathcal{L}\Box\Box$ to $\mathsf{L}\Box_n$ in the same sense that $\mathcal{L}\Box$ is a restriction of $\mathcal{L}\Box\Box$ to $\mathsf{L}\Box$. The method used in proving strong completeness for $\mathcal{L}\Box\Box$ from the strong completeness of $\mathcal{L}\Box$ can also be used to prove, for each $n\geq 1$, the strong completeness of $\mathcal{L}\Box_{n+1}$ from the strong completeness of $\mathcal{L}\Box_n$. Strong soundness of $\mathcal{L}\Box_n$, for all n, is a corollary of strong soundness for $\mathcal{L}\Box\Box$. Thus for all φ and all n, $\Box\varphi$ is provable in $\mathcal{L}\Box_{n+1}$ iff φ is logically true in the semantics of $\mathcal{L}\Box_n$. In case $n = 1$ we have fact (1) above.

The novelty of our approach to modal logic has been two-fold. In the first place, we emphasize the relation of logical consequence, whereas all previous writers have restricted themselves to the consideration of logical truth. In our opinion, this restriction is unnecessary and, moreover, it effectively blocks utilization of insights which have been achieved from the broader viewpoint. In the second place, our deductive system was designed to embody semantical insights, i.e. the intuitions which would lead one to guess that a given argument is valid will also lead one to discover a proof of it in our system. This advantage saves an enormous amount of essentially useless manipulation.

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University of Pennsylvania
Philadelphia, Pennsylvania