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LEŚNIEWSKI'S TERMINOLOGICAL EXPLANATIONS AS RECURSIVE CONCEPTS

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In 1929 Leśniewski published terminological explanations for his system of logic [5] where he used certain concepts from his system of mereology along with others such as equiformity. In [1] Peano's axioms for arithmetic are shown to be derivable in Leśniewski's system of ontology extended by an axiom of infinity. In that exposition use is made of a numerical epsilon, first defined in [2], in order to provide a characteristically onotlogical model for the natural numbers. It is shown there that analogues for the axiom, rule of extensionality, and rule of definition for the primitive epsilon (ε) of ontology are derivable for the numerical epsilon (ε_∞). Thus, one has available for the numerical epsilon analogues of every thesis of ontology involving the primitive epsilon.

The numerical epsilon serves in this paper to reduce Leśniewski's terminological explanations to numerical concepts. That is, each terminological concept is shown to be definable as a numerical concept within ontology extended by an axiom of infinity. Since the definitions to be given are recursive, the incompleteness of this extension of onotlogy is readily established.

1. *Preliminary definitions* In [5] Leśniewski defined whatever notions he needed for his terminological explanations as name forming functors. Here, we shall define only numerical name forming functors which are primitive recursive in the sense that if Φ is the numerical name formed by the functor from arguments $x_1, \ldots, x_n$, then there is a primitive recursive function Ψ such that

$$[Ax_1, \ldots, x_n]: A\varepsilon_\infty \Phi < x_1, \ldots, x_n > .\equiv. \Psi < A, x_1, \ldots, x_n > = _\infty 0$$

is a thesis of ontology extended by the axiom of infinity. This will be achieved by limiting definiens for the numerical name forming functor to those propositional functions obtainable by primitive recursive methods. In particular only the following methods will be employed: the limited quantifiers, the effective minimal operator, proposition forming functors for

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propostional arguments, composition of functions, and recursive schemata for defining primitive recursive functions.

In [1] it is explained how the recursive schemata for defining functions can be represented in ontology. First these implicit definitions are reduced to explicit definitions using Frege's method employing "impredicative" definitions. From these explicit definitions which are protothetical in nature, one next obtains theses which are analogous to ontological definitions, and with these given it is then possible to obtain the two theses which represent the recursive definition of the concept in question. For example, if addition were to be given in this way, one eventually obtains as theses:

$$[\Phi]: \Phi \varepsilon_\infty \mathbf{Fin} . \supset . \Phi =_\infty \Phi + 0$$
$$[\Phi\Psi]: \Phi \varepsilon_\infty \mathbf{Fin} . \Psi \varepsilon_\infty \mathbf{Fin} . \supset . \Phi + \mathbf{S}\langle\Psi\rangle =_\infty \mathbf{S}\langle\Phi + \Psi\rangle$$

For brevity only these last two theses shall concern us: whichever definitions they rely on being presupposed. Moreover, the hypotheses of such theses will be omitted: throughout the paper it is assumed that all relevant variables have as their values finite numerals. Under this stipulation, if addition were to be introduced, it would be given by exhibiting only the following theses.

$$[\Phi] . \Phi =_\infty \Phi + 0$$
$$[\Phi\Psi] . \Phi + \mathbf{S}\langle\Psi\rangle =_\infty \mathbf{S}\langle\Phi + \Psi\rangle$$

Given the above procedure, the method of defining a concept recursively is readily available. But in order to further simplify the exposition other conventions will be adopted. In particular, it is desirable to have the use of the "limited quantifiers". To this end, *numerically less than or identical with* is defined

$$[AB] .\because. A =_\infty B . \mathrm{v} . [\exists C] . A + C =_\infty B : \equiv . A \leqq B$$

and the following thesis is obtained:

$$[AB] .\because. A \leqq B . \equiv : A =_\infty B . \mathrm{v} . [\exists C] . C \leqq B . A + C =_\infty B$$

showing the relation to be primitive recursive. With the availability of the above, it is possible to introduce the use of limited quantifiers. Of course, such quantifiers are mere abbreviations—as is the use of the particular quantifier. Thus propositions of the form:

$$[x_1, \ldots, x_n] : x_1 \leqq y, \ldots, x_n \leqq y . \supset . \Phi$$
$$[\exists x_1, \ldots, x_n] : x_1 \leqq y, \ldots, x_n \leqq y : \Phi$$

will be abbreviated as:

$$[x_1, \ldots, x_n \leqq y] : \Phi$$
$$[\exists x_1, \ldots, x_n \leqq y] : \Phi$$

The (effective) minimal operator can be given in ontology as a numerical name forming functor. This is done by the following definitional thesis:

$[AB\varphi p]::A\ \varepsilon_\infty\ B.B\ \varepsilon_\infty\ \mathbf{Fin}:[C]:\varphi(C).\supset.C\ \varepsilon_\infty\ C\ .\therefore B\varepsilon_\infty 0.\sim([\exists C].\varphi(C)).\mathrm{v}:$
$p:[C]:\varphi(C).\supset.B\leqq C.\cdot.\equiv.A\varepsilon_\infty\ \mu\dagger B\varphi\dagger[p]$

Here the 'μ' is the constant being defined[1] and the definidendum may be read as "A is the least B (for φ) such that p". On the basis of the above, the following is immediately derivable.

$[AB\varphi]::A\ \varepsilon_\infty\ \mu\dagger B\varphi\dagger[\varphi(B)].\equiv.\cdot.A\ \varepsilon_\infty\ B.B\ \varepsilon_\infty\ \mathbf{Fin}:[C]:\varphi(C).\supset.C\ \varepsilon_\infty C\ .\therefore$
$B\ \varepsilon_\infty 0.\sim([\exists C].\varphi(C)).\mathrm{v}:\varphi(B):[C]:\varphi(C).\supset.B\leqq C$

That is, whenever A is the least B (for φ) such that $\varphi(B)$, either A is zero, if φ is not satisfied, or A is the least numeral satisfying φ.

Finally, it is necessary to establish a one-to-one correspondence between (some subset of) the positive integers and the expressions of ontology. Given such a correspondence, the relevant variables in the numerical terminological explanations that are given in the next section can be considered as relativized to this subset. That is, under the correspondence the explanations given here refer to some given expression (of ontology) as do Leśniewski's original terminological explanations.

Ontology takes as basic semantical categories propositions and names.[2] All other semantical categories are understood ultimately in terms of these. In order to determine the semantical category of any term it is only necessary to specify the number and types of its arguments and the functor produced by it. For example: "it is not the case that . . ." is completely determined by indicating that it is a propositional functor formed from one propositional argument, while ". . . is unempty" is determined by indicating that it is a propositional functor formed from one nominal argument.

It is thus possible to establish a one-to-one correspondence between the semantical categories and a subset of the positive integers. Propositions and names are assigned the numbers one and two respectively. If n is the number of arguments used to form a function, $n + 2$ is associated with that number of arguments. In this way prime factorizations of numbers can be used to code the semantical categories and the number of the semantical category can be used as a subscript to determine unique parentheses for each semantical category. For example, if one writes "it is not the case that p" as:

$$\sim({}_\alpha p)_\alpha$$

where $\alpha = 2^3 \times 3^1 \times 5^1$ then the sign of negation is determined to be a propositional functor formed from one propositional argument: the first exponent indicating the number of arguments of the functor, the second the category of the single argument, and the last the category of the functor formed by the negation sign. Whereas, if one writes "the A is b" as:

$$\varepsilon({}_\beta Ab)_\beta$$

where $\beta = 2^4 \times 3^2 \times 5^2 \times 7^1$ then the epsilon is determined to be a propositional functor formed from two nominal arguments: the first exponent indicating the number of arguments of the epsilon to be two, the second and third exponents indicating that each argument is nominal, and the last indicating that the epsilon forms a proposition.

And in general, a type-t functor forming functor of n-arguments is represented by a prime factorization of the form:

$$2^{n+2} \times C_1 \times C_2 \times \ldots \times C_n \times \tau$$

where each C_i (and τ) is a prime factorization beginning with the next prime in order of magnitude not yet appearing in the representation and indicating the category of the argument (of t, that is, the cateogry of the functor formed by the term in question). Thus, if one writes

$$\varepsilon({}_{\delta}a)_{\delta}({}_{\gamma}b)_{\gamma}$$

where $\delta = 2^3 \times 3^2 \times 5^3 \times 7^2 \times 11^1$ and $\gamma = 2^3 \times 3^2 \times 5^1$, this epsilon forms a γ-functor from one propositional argument and its semantical category is δ.

Thus, prime factorizations (with primes in order of magnitude) represent semantical categories in the following way: exponents one or two indicate propositions and names, exponents greater than two indicate the number of arguments used to form a functor—similar representations of categories are to be found in the literature, see for instance, Curry [3] on grammatical categories.

In order to attain a one-to-one correspondence between expressions of ontology and some subset of the positive integers we shall standardize the exposition of onotlogy. The formulas of ontology will employ, following Leśniewski, square corners for quantification, but, unlike Leśniewski, for semantical categories only one kind of parentheses (say '('and')') shall be used—they will, however, always be given with some subscript appended. The parentheses will continue to determine the semantical categories of non-parenthetical expressions, but the determination shall now be formal rather than lexicographical. That is, instead of introducing a new style of parentheses whenever they are needed we shall introduce a new subscript which codes the desired semantical category. For variables and constants one may use any continuous symbol other than those selected to serve as parentheses and corners. Thus the shortest possible axiom of onotlogy given in [9], which would be written as

$$[a,b]: \varepsilon\{a,b\} .\equiv. [\exists c] . \varepsilon\{a,c\} . \varepsilon\{c,b\}$$

in an informal manner, and as

$$\llcorner ab \lrcorner \ulcorner \phi(\varepsilon\{a,b\} \vdash (\llcorner c \lrcorner \ulcorner \vdash (\text{ϙ}(\varepsilon\{a,c\}\varepsilon\{c,b\}))\urcorner))\urcorner$$

by Leśniewski, where '('and')' are used for proposition forming functors all of whose arguments are propositional and '{'and'}' are used for propositional functors all of whose arguments are nominal, shall here be rendered with $\eta = 2^4 \times 3^1 \times 5^1 \times 7^1$ as

$$\llcorner ab \lrcorner \ulcorner \phi({}_{\eta}\,\varepsilon({}_{\beta}ab)_{\beta} \vdash ({}_{\alpha}\llcorner c \lrcorner \ulcorner \vdash ({}_{\alpha}\text{ϙ}({}_{\eta}\,\varepsilon({}_{\beta}ac)_{\beta}\,\varepsilon({}_{\beta}cb)_{\beta})_{\eta})_{\alpha}\urcorner)_{\alpha})_{\eta}\urcorner$$

The square corners are used exclusively in association with quantifiers, and the subscripted parentheses in the above example indicate that 'ϕ' and 'φ' are propositional functors for two propositional arguments, 'ε' is a propositional functor for two nominal arguments, and '$\vdash$' is a propositional functor for one propositional argument. In effect then, the standardization envisaged is the adoption of Leśniewski's formalization with the sole exception of employing subscripted parentheses as single symbols instead of a variety of kinds of parentheses.

We now set up a one-to-one correspondence of (some subset) of the positive integers with the symbols of ontology in the following way:

$\llcorner$	$\lrcorner$	$\ulcorner$	$\urcorner$	$(_m$	$)_m$
1	3	5	7	5^m	7^m

where m is the number of a semantical category; variables and constants being assigned prime numbers greater than seven.

Expressions shall correspond to numbers whose prime factorizations have only the above kinds of exponents. Thus there is a one-to-one correspondence between expressions and a subset of the positive integers. Once definite prime numbers greater than seven have been assigned to the functors and variables occurring in the above axiom a unique number becomes associated with the axiom. This number shall be designated by 'Axo' which may be read as "the Gödel-number associated with the axiom of ontology".

Starting now with the concepts of addition, multiplication, exponentiation, less than or numerically identical with, numerical identity, and numerical difference—each of which is primitive recursive, we define a group of numerical name forming functors by the methods indicated above: each of which will thus be primitive recursive.

D1.1 $[AB]: < (AB) .\equiv. A \leqq B . A \neq_{\infty} B$

A is *strictly less* than B. This functor will usually be written as "$A < B$".

D1.2 $[AB]: A \,\varepsilon_{\infty}\, \mathsf{dis} \langle B \rangle .\equiv. [\exists C \leqq A] . A =_{\infty} B \times C$

A is *divisible* by B.

D1.3 $1 =_{\infty} \mathbf{S} \langle 0 \rangle$

One is identical to the successor of zero. Any other particular constants that are needed later shall be considered as defined, for instance, two, three, etc.

D1.4 $[A]: A \,\varepsilon_{\infty}\, \mathsf{prim} .\equiv. 1 < A . \sim ([\exists B \leqq A] . B \neq_{\infty} 1 . B \neq_{\infty} A . A \,\varepsilon_{\infty}\, \mathsf{dis} \langle B \rangle)$

A is a *prime* number.

D1.5a $[ABC]: \Phi_1 \langle AC \rangle (B) .\equiv. B \,\varepsilon_{\infty}\, \mathsf{prim} . A \,\varepsilon_{\infty}\, \mathsf{dis} \langle B \rangle . C < B . B \leqq A .$

This definition is given merely to facilitate the following. Whenever the minimal operator is employed there will be occasion to have such an auxiliary definition.

D1.5b $[A] . \text{pr} < 0A > =_\infty 0$

D1.5c $[ABn] . \text{pr} < n+1A > =_\infty \mu \dagger B \Phi_1 \dagger A \ \text{pr} < nA > \dagger \dagger [B \ \varepsilon_\infty \ \text{prim} \ .$
$A \ \varepsilon_\infty \ \text{dis} < B > . \ \text{pr} < nA > < B . B \leqq A]$

The above two theses indicate the value of the *nth prime factor* of A presented in order of magnitude. With this single example given, we shall for brevity ommit such auxiliary definitions and shorten expressions using the minimal operator by ommitting its parameters. Under this convention the nth prime factor is given by asserting:

D1.5a $[A] . \text{pr} < 0A > =_\infty 0$

D1.5b $[ABn] . \text{pr} < n+1A > =_\infty \mu [B \ \varepsilon_\infty \ \text{prim} \ . A \ \varepsilon_\infty \ \text{dis} < B > . \ \text{pr} < nA > < B . B \leqq A]$

as will be the case for other definitions requiring the minimal operator.

D1.6a $! < 0 > =_\infty 1$

D1.6b $[n] . ! < n+1 > =_\infty < n+1 > \times ! < n >$

The above two theses give the value of the *nth factorial* in order of magnitude.

D1.7a $\text{pr} < 0 > =_\infty 1$

D1.7b $[An] . \text{pr} < n+1 > =_\infty \mu [A \ \varepsilon_\infty \ \text{prim} \ . \ \text{pr} < n > + 1 \leqq A . A \leqq ! < \text{pr} < n >> + 1]$

The above two theses give the value of the *nth prime number* according to magnitude.

D1.8 $[ABn] . \text{gl} < nA > =_\infty \mu [B \leqq A . A \ \varepsilon_\infty \ \text{dis} < \text{pr} < nA >^B > .$
$\sim (A \ \varepsilon_\infty \ \text{dis} < \text{pr} < nA >^{B+1} >)]$

This thesis gives the value of the *nth term* of the sequence of numbers corresponding to A.

D1.9 $[AB] . \text{L} < A > =_\infty \mu [B \leqq A . 1 \leqq \text{gl} < BA > . 0 =_\infty \text{gl} < B + 1A >]$

This thesis gives the value of the *length* of the sequence of numbers corresponding to A—it is, in effect, the number of prime factors occurring in the prime factorization of A.

D1.10 $[ABC] . A * B =_\infty \mu [C \leqq \text{pr} << \text{L} < A > + \text{L} < B >>^{A+B} > . [n \leqq \text{L} < A >] .$
$\text{gl} < nC > =_\infty \text{gl} < nA > : [n \leqq \text{L} < B >] : 1 \leqq n . \supset .$
$\text{gl} < n + \text{L} < A > B > =_\infty \text{gl} < nB >]$

This thesis gives the value of the *concatenation* of A with B.

D1.11 $[A] . \text{R} < A > =_\infty 2^A$

This thesis gives the value of the *sequence* corresponding to A.

The above definitions are all given by Gödel [4]. The remainder of the introductory definitions are developed in order to produce the desired terminological explanations. In each case, some concept that Leśniewski took as undefined in [5] is defined here. Once these definitions are given it is possible to reproduce Leśniewski's terminological explanations (which is taken up in the following section).

D1.12 $[AB]:A\ \varepsilon_\infty\ \mathbf{ingr}<B>.\equiv.[\exists CD].C\leqq B.D\leqq B.B=_\infty C*<A*D>$

A is an *ingredient* of B. Since the only concern here is with sequences of symbols it is enough to define a numerical name which holds only for such items even though Leśniewski uses the broader concept given in mereology. However, from the above definition it is clear that A is an ingredient of B if and only if A is identical to B (let C and D in the above definition be identical with one) or A is a proper (and continuous) part of B. Thus, this numerical '**ingr**' is just the mereological '**ingr**' restricted in application to prime factorizations of numbers. In general, Leśniewski's defined names used in his terminological explanations will be reproduced in the sense that they will be numerical names restricted in this way. Under the Gödel-numbering this amounts to restricting the application of the numerical concepts to expressions.

D1.13 $[AB].A\ \varepsilon_\infty\ \mathbf{cnf}<B>.\equiv.[n\leqq A+B].\mathsf{gl}<nA>=_\infty \mathsf{gl}<nB>$

A is *equiform* to B.

D1.14 $[A]\therefore A\ \varepsilon_\infty\ \mathbf{expr}.\equiv::1\leqq \mathsf{L}<A>::[\exists B\leqq A]::B\ \varepsilon_\infty\ \mathbf{cnf}<A>.\because$
$[1\leqq n\leqq \mathsf{L}<A>]\because \mathsf{pr}<nB>=_\infty \mathsf{pr}<n>.\because \mathsf{gl}<nB>=_\infty 1.\mathsf{v}.$
$\mathsf{gl}<nB>=_\infty 3.\mathsf{v}.8\leqq \mathsf{gl}<nB>.\mathsf{gl}<nB>\ \varepsilon_\infty\ \mathbf{prim}.\mathsf{v}:[m\leqq \mathsf{L}<A>]:$
$1\leqq m:\mathsf{gl}<nB>=_\infty 5^m.\mathsf{v}..\mathsf{gl}<nB>=_\infty 7^m$

A is an *expression.* Assuming that m is one or the number of some semantical category, this numerical name indicates that subset of prime factorizations which is the most interesting for the purposes at hand. For, as will be seen in the next few definitions, expressions are just those prime factorizations whose exponents correspond to the primitive symbols of the system.

D1.15 $[A]:A\ \varepsilon_\infty\ \mathbf{lst}.\equiv.[\exists B\leqq A].B\ \varepsilon_\infty\ \mathbf{cnf}<A>.[n\leqq \mathsf{L}<A>].$
$\mathsf{pr}<nB>=_\infty \mathsf{pr}<n>.\mathsf{gl}<nB>\ \varepsilon_\infty\ \mathbf{expr}.$

A is a *list* of expressions. Lists of expressions are sequences of primes whose exponents represent expressions. This concept was not given by Leśniewski, but is useful in this exposition.

D1.16 $[A]:A\ \varepsilon_\infty\ \mathbf{vrb}.\equiv.A\ \varepsilon_\infty\ \mathbf{expr}.\mathsf{L}<A>=_\infty 1$

A is a *word.* Thus words are represented by single prime factors whose exponents corresponds to one of the symbols of the system.

D1.17 $[A]:A\ \varepsilon_\infty\ \mathbf{prntl}.\equiv.[\exists n\leqq A].A=_\infty \mathsf{R}<5^n>$

A is a *left parenthesis*. Thus, assuming that n is the number of some semantical category, a left parenthesis is the unique word whose exponent indicates the particular semantical category with which the parenthesis is associated.

D1.18 $[A]:A\ \varepsilon_\infty\ \mathbf{prntr}.\equiv.[\exists n\leqq A].A=_\infty \mathsf{R}<7^n>$

A is a *right parenthesis,* assuming that n is the number of some semantical category (not given by Leśniewski).

D1.19 $[AB]\therefore A \,\varepsilon_\infty \mathbf{prntsym}<B>.\equiv:[\exists n\leqq B]:B =_\infty \mathsf{R}<5^n>.A =_\infty \mathsf{R}<7^n>.\vee.$
$[\exists n\leqq B].B =_\infty \mathsf{R}<7^n>.A =_\infty \mathsf{R}<5^n>$

A is a *symmetric parenthesis* with B. Thus, assuming that n is the number of some semantical category, symmetric parentheses are always associated with the same semantical category.

D1.20 $[A]\therefore A \,\varepsilon_\infty \mathbf{prnt}.\equiv: A\,\varepsilon_\infty \mathbf{prntl}.\vee.A\,\varepsilon_\infty \mathbf{prntr}.$

A is a *parenthesis*.

D1.21 $[ABC]: A\,\varepsilon_\infty \mathbf{prcd}<BC>.\equiv.[\exists nm\leqq C].\mathsf{pr}<\mathsf{L}<A>A> =_\infty \mathsf{pr}<nC>.$
$\mathsf{pr}<1B> =_\infty \mathsf{pr}<mC>.n<m$

A is a predecessor of B in C. Thus, assuming that A and B are ingredients of C, A precedes B if all of A comes before the beginning of B. This name differs from Lesniewski's '$A\,\varepsilon\,\mathrm{prcd}(B)$' in that the particular context in which A precedes B has been made explicit. This is only a minor difference—though necessary: the context is made explicit in order to retain the limited quantification. And, for this purpose, it is very often necessary to make explicit contexts which are only implicit in Leśniewski's terminological explanations. See for instance the next concept.

D1.22 $[ABC]:A\,\varepsilon_\infty \mathbf{scd}<BC>.\equiv.B\,\varepsilon_\infty \mathbf{prcd}<AC>$

A is a *succeeder* of B in C. Thus, A follows B in C if all of B precedes A in C.

D1.23 $[AB].\mathbf{Uprcd}<AB> =_\infty \mu[C\leqq B.C\,\varepsilon_\infty \mathbf{prcd}<AB>.B\,\varepsilon_\infty \mathbf{expr}\therefore$
$[D\leqq B]:.D\,\varepsilon_\infty \mathbf{prcd}<AB>.\supset:D\,\varepsilon_\infty \mathbf{prcd}<CB>.\vee.D =_\infty C]$

This thesis gives the value of the *last word preceding* A in B. Thus, the last word preceding A in B is the single word which comes immediately before the beginning of A in an expression B.

D1.24 $[An].\mathbf{Ingr}<nA> =_\infty \mu[B\leqq A.B =_\infty \mathsf{pr}<nA>^{\mathsf{gl}<nA>}.A\,\varepsilon_\infty \mathbf{expr}]$

This thesis gives the value of the *nth word* in A. Thus, the nth word in A is the unique word represented by the nth term in an expression A.

D1.25 $[A].\mathbf{Uingr}<A> = \mu[B\leqq A.B =_\infty \mathbf{Ingr}<\mathsf{L}<A>A>]$

This thesis gives the value of the *last word* in A.

D1.26a $[AB].\mathsf{Occ}<0AB> =_\infty \mu[C\leqq B.\ C\,\varepsilon_\infty \mathbf{cnf}<A>.\ C\,\varepsilon_\infty \mathbf{ingr}<B>]$

This thesis gives the value of the *first occurrence* of A in B.

D1.26b $[ABn].\mathsf{Occ}<n+1AB> = \mu[C\,\varepsilon_\infty \mathbf{cnf}<A>.\ C\leqq B.$
$\mathbf{Ingr}<1C>\varepsilon_\infty \mathbf{scd}<\mathbf{Ungr}<\mathsf{Occ}<nAB>>B>]$

This thesis gives the value of the $n+1$ *occurence* of A in B, in terms of the nth occurrence of A in B. Thus the value of the nth occurrence of A in B has been determined. Notice that this concept is applicable to the nth word in an expression or to the nth expression in a list. This concept and the remaining ones in this section were not given by Leśniewski, but are useful in later developments.

D1.27 $[AB].\mathsf{N}\langle AB\rangle =_\infty \mu[m \leqq \mathsf{L}\langle B\rangle . \mathsf{Occ}\langle mAB\rangle =_\infty 0]$

This thesis gives the value of the *number of occurrences* of A in B (which is to be taken as zero if there is no occurrence of A in B). Notice that the concept is applicable to the number of times a word occurs in an expression or the number of times an expression occurs in a list.

The last concept to be considered in this introductory section, is given in order to describe the rule of substitution in a manner similar to Leśniewski's original description. In that description, Leśniewski compares a given expression with the expression that reaults from it by making some substitution in the expression. Thus what is needed is a symbol by symbol association of two expressions except in those places at which a substitution may have occurred. Hence, it is indicated when a word A in a given expression E is to be associated with a word B in a given expression F where the association does not apply to certain segments, C of E and D of F—these segments being the places for possible substitution.

D1.28 $[ABCDEF] :: A\, \varepsilon_\infty \mathsf{assoc}\langle BCDEF\rangle \because$

1) $E\, \varepsilon_\infty\, \mathbf{expr}.$
2) $F\, \varepsilon_\infty\, \mathbf{expr}.$
3) $[\exists G \leqq E].G =_\infty \mathsf{Occ}\langle 0CE\rangle .$
4) $\mathsf{N}\langle CE\rangle =_\infty \mathsf{N}\langle DF\rangle .$
5) $A\, \varepsilon_\infty \mathbf{vrb}.$
6) $B\, \varepsilon_\infty\, \mathbf{vrb}.$
7) $A\, \varepsilon_\infty\, \mathbf{ingr}\langle E\rangle .$
8) $B\, \varepsilon_\infty\, \mathbf{ingr}\langle F\rangle :$
9) $[GHIJ \leqq E+F]: E =_\infty G * H . G =_\infty \mathsf{Occ}\langle 0CE\rangle .\equiv. F =_\infty I * J.$
   $I =_\infty \mathsf{Occ}\langle 0DF\rangle .\therefore$
10) $[GHIJ \leqq E+F] \therefore L =_\infty G * H . F =_\infty I * J.$
   $\mathbf{Ingr}\langle 1H\rangle =_\infty \mathbf{Ingr}\langle 1\mathsf{Occ}\langle 0CE\rangle\rangle .$
   $\mathbf{Ingr}\langle 1J\rangle =_\infty \mathbf{Ingr}\langle 1\mathsf{Occ}\langle 0DF\rangle\rangle .\supset:$
   $[n \leqq G+H]: A =_\infty \mathbf{Ingr}\langle nG\rangle .\equiv. B =_\infty \mathbf{Ingr}\langle nJ\rangle \because$
11) $2 \leqq \mathsf{N}\langle CE\rangle .\supset:: [GHIJKM \leqq E+F] :: E =_\infty G * H * I . F =_\infty J * K * M .\therefore$
   $[n \leqq \mathsf{N}\langle CE\rangle] \therefore \mathbf{Uingr}\langle G\rangle =_\infty \mathbf{Uingr}\langle \mathsf{Occ}\langle nCE\rangle\rangle .$
   $\mathbf{Ingr}\langle 1I\rangle =_\infty \mathbf{Ingr}\langle 1\mathsf{Occ}\langle n+1CE\rangle\rangle .$
   $\mathbf{Uingr}\langle J\rangle =_\infty \mathbf{Uingr}\langle \mathsf{Occ}\langle nDF\rangle\rangle .$
   $\mathbf{Ingr}\langle 1M\rangle =_\infty \mathbf{Ingr}\langle 1\mathsf{Occ}\langle n+1DF\rangle\rangle .\supset.$
   $[m \leqq H+K]: A =_\infty \mathbf{Ingr}\langle mH\rangle .\equiv. B =_\infty \mathbf{Ingr}\langle mK\rangle \because$
12) $[GHIJ \leqq E+F] \therefore E =_\infty G * H . F =_\infty I * J . [\exists nm \leqq L+F].$
   $m+1 =_\infty \mathsf{N}\langle CD\rangle . n+1 =_\infty \mathsf{N}\langle DF\rangle .$
   $\mathbf{Uingr}\langle \mathsf{Occ}\langle mCE\rangle\rangle =_\infty \mathbf{Uingr}\langle G\rangle .$
   $\mathbf{Uingr}\langle I\rangle =_\infty \mathbf{Uingr}\langle \mathsf{Occ}\langle nDF\rangle\rangle .\supset: [n \leqq H+J]:$
   $A =_\infty \mathbf{Ingr}\langle nH\rangle .\equiv. B =_\infty \mathbf{Ingr}\langle nJ\rangle$

The above is the last of the introductory concepts needed in order to give Leśniewski's terminological explanations. Of the concepts that Leśniewski originally took as primitive, we have defined the following:

A ε_∞ **ingr** $<B>$	A ε ingr (B)
A ε_∞ **cnf** $<B>$	A ε cnf (B)
A ε_∞ **expr**	A ε expr
A ε_∞ **vrb**	A ε vrb
A ε_∞ **prntl**	A ε prntl
A ε_∞ **prntsym** $<B>$	A ε prntsym (B)
A ε_∞ **prnt**	A ε prnt
A ε_∞ **prcd** $<BC>$	A ε prcd (B)
A ε_∞ **scd** $<BC>$	A ε scd (B)
A ε_∞ **Uprcd** $<BC>$	A ε Uprcd (B)
A ε_∞ **Uingr** $<B>$	A ε Uingr (B)
A ε_∞ **Ingr** $<nB>$	A ε 1ingr(B)
	A ε 2ingr (B)
	etc.

As has been noted some of the above make explicit contexts only implicit in the originals. With these terms at hand, Leśniewski's terminological explanations can now be reproduced exactly with only occassional differences each of which will be noted.

2. *Protothethetical terminological explanations* In this section the first forty-three terminological explanations of Leśniewski are given. In each case Leśniewski's terminological explanations define name forming functors according to the methods of ontology. Here the terminological explanations define numerical name forming functors according to the methods of recursion. However, the concepts given below parallel as exactly as possible those given by Leśniewski—commentaries following definitions will note any important differences in exposition. The immediate advantage of this procedure is that it makes clear that Leśniewski's terminological explanations can be given by the methods of recursion. The subsequent advantages of the procedure allow one to establish the incompleteness of ontology (extended by the axiom of infinity) in a manner exactly similar to Gödel's original work.

In order to facilitate comparison of the following terminological explanations with Leśniewski's, his numbering of terminological explanations has been employed. And in order to make the following explanations definite, reference will be made to particular axioms. The axiom for ontology is given above and will be referred to by its Gödel-number: Axo. The axiom for protothetic will be that of Sobociński [8], which in the informal notation of this exposition, is given as:

$$[pq]{:}{:}\ p\,.{\equiv}.\,q\ {:}{\equiv}{:}{:}\ [f]{:}{:}\,f(pf(p[u]\,.\,u))\ .{\equiv}.\dot{}.\,[r]\,.\dot{}.\,f(qr)\ .{\equiv}{:}\ q\,.{\equiv}.\ p$$

The Gödel-number of this axiom will be designated by 'Axp', and the axiom will be referred to by means of its Gödel-number. Thus, in the formal notation, Axp is given as:

$$\llcorner pq \lrcorner \ulcorner \phi({}_\eta \phi({}_\eta pq)_\eta \llcorner f \lrcorner \ulcorner \phi({}_\eta f({}_\eta pf({}_\eta p \llcorner u \lrcorner \ulcorner u \urcorner)_\eta)_\eta \llcorner r \lrcorner \ulcorner \phi({}_\eta f({}_\eta qr)_\eta \phi({}_\eta qp)_\eta \urcorner)_\eta \urcorner)_\eta \urcorner$$

The only other axiom to be considered, is the axiom of infinity. The

Gödel-number of this axiom will be designated by 'Axinf', and the axiom will also be referred to by means of its Gödel-number. Thus, in the formal notation, where $\theta = 2^4 \times 3^2 \times 5^2 \times 7^2$, Axinf is given as:

$$\llcorner a \lrcorner^{\ulcorner} \text{⟜} ({}_{\eta} \llcorner \varphi \lrcorner^{\ulcorner} \text{⟜} ({}_{\eta} \text{⫯} ({}_{\eta} \varphi({}_{\gamma} \wedge)_{\gamma} \llcorner Bb \lrcorner^{\ulcorner} \text{⟜} ({}_{\eta} \text{⫯} ({}_{\eta}\, \varepsilon({}_{\beta} Ba)_{\beta}\, \varphi({}_{\gamma} b)_{\gamma})_{\eta}$$
$$\varphi({}_{\gamma} \cup ({}_{\theta} bB)_{\theta})_{\gamma})_{\eta}{}^{\urcorner})_{\eta} \varphi({}_{\gamma} a)_{\gamma})_{\eta} \vdash ({}_{\alpha} \llcorner A \lrcorner^{\ulcorner} \vdash ({}_{\alpha} \text{⫯} ({}_{\eta}\, \varepsilon({}_{\beta} AA)_{\beta} \vdash$$
$$({}_{\alpha}\, \varepsilon({}_{\beta} Aa)_{\beta})_{\alpha})_{\eta})_{\alpha}{}^{\urcorner})_{\alpha})_{\eta}{}^{\urcorner}$$

Given the Gödel-numbers for the three axioms, it is now possible to state Leśniewski's terminological explanations.

D2.1 $[A]: A\ \varepsilon_{\infty} \mathsf{vrb1} .\equiv. A\ \varepsilon_{\infty} \mathbf{cnf} < \mathbf{Ingr} < 1\, \mathsf{Axp} >>$

This thesis defines a *left lower corner* as any word equiform to the first word in the axiom of protothetic. Although left lower corners could have been defined as words equiform to $\mathsf{R} < 1 >$ in this exposition, in order to parallel Leśniewski's:

T.E.I $[A]: A\ \varepsilon \,\mathrm{vrb1} .=. A\ \varepsilon\ \mathrm{cnf}(1\mathrm{ingr}(\mathrm{A1}))$

that is, the left lower corner is any word equiform to the first word in A1, where 'A1' is the name he gives to the relevant axiom, the more complicated definition is used. In the terminological explanations that follow, if there is nothing to the contrary, it may be assumed that the only differerences in notation from Leśniewski's terminological explanations are as minor as those that occur here.

D2.2 $[A]: A\ \varepsilon_{\infty} \mathsf{vrb2} .\equiv. A\, \varepsilon_{\infty} \mathbf{cnf} < \mathbf{Ingr} < 4\, \mathsf{Axp} >>$

This thesis defines a *right lower corner* as any word equiform to the fourth word in the axiom of protothetic, that is, as equiform to $\mathsf{R} < 3 >$.

D2.3 $[A]: A\ \varepsilon_{\infty} \mathsf{vrb3} .\equiv. A\ \varepsilon_{\infty} \mathbf{cnf} < \mathbf{Ingr} < 5\, \mathsf{Axp} >>$

This thesis defines a *left upper corner*, which are words equiform to $\mathsf{R} < 5 >$.

D2.4 $[A]: A\ \varepsilon_{\infty} \mathsf{vrb4} .\equiv. A\, \varepsilon_{\infty} \mathbf{cnf} < \mathbf{Uingr} < \mathsf{Axp} >>$

This thesis defines a *right upper corner*, which are words equiform to $\mathsf{R} < 7 >$.

D2.5 $[A]: A\, \varepsilon_{\infty} \mathsf{trm} .\equiv. A\, \varepsilon_{\infty} \mathbf{vrb} . \sim(A\, \varepsilon_{\infty} \mathbf{prnt}) . \sim(A\, \varepsilon_{\infty} \mathsf{vrb1}) . \sim(A\ \varepsilon_{\infty} \mathsf{vrb2}) .$
$\sim(A\, \varepsilon_{\infty} \mathsf{vrb3}) . \sim(A\, \varepsilon_{\infty} \mathsf{vrb4})$

This thesis defines a *term* as a word which is neither a punctuator for quantification nor a parenthesis for semantical categories.

D2.6 $[AB]: A\ \varepsilon_{\infty} \mathsf{int} < B > .\equiv. B\, \varepsilon_{\infty} \mathbf{expr} . A\ \varepsilon_{\infty} \mathbf{vrb} . A\ \varepsilon_{\infty} \mathbf{ingr} < B > .$
$\sim(A\ \varepsilon_{\infty} \mathbf{Ingr} < 1B >) . \sim(A\ \varepsilon_{\infty} \mathbf{Uingr} < B >)$

A is a *word inside* of B. Leśniewski's *T.E.VII* is given as:

T.E.VII: $[A,a] :: A\ \varepsilon\, \mathrm{Cmpl}(a) .\equiv.\!\cdot\!\cdot\, A\, \varepsilon\, \mathrm{expr} .\!\cdot\!\cdot$
$[B]: B\ \varepsilon\, \mathrm{vrb} . B\ \varepsilon\, \mathrm{ingr}(A) .\supset.$

$[\exists C] . C\, \varepsilon\, a . B\, \varepsilon\, \mathrm{ingr}(C) .\because.$

$[B,C,D] : B\, \varepsilon\, a . C\, \varepsilon\, a .$

$D\, \varepsilon\, \mathrm{vrb} . D\, \varepsilon\, \mathrm{ingr}(B) . D\, \varepsilon\, \mathrm{ingr}(C) . \supset . B\, \varepsilon\, \mathrm{Id}(C) .\because.$

$[B] : B\, \varepsilon\, a . \supset.$

$B\, \varepsilon\, \mathrm{expr} \cap \mathrm{ingr}(A)$

where '$B\, \varepsilon\, \mathrm{Id}(C)$' and '$B\, \varepsilon\, \mathrm{expr} \cap \mathrm{ingr}(A)$' correspond to '$B =_\infty C$' and '$B\, \varepsilon_\infty\, \mathbf{expr} . B\, \varepsilon_\infty\, \mathbf{ingr}\langle A \rangle$' respectively. Since this explanation involves quantification over general names, it cannot be presented as primitive recursive—our quantifiers must be restricted to individual (numerical) names. However, it is clear from the above that *complexes* of things, as defined in *T.E. VII*, are only expressions of those things. Thus, Leśniewski's uses of 'Cmpl' can be avoided in favor of mentioning some specific concatenation. Hence no terminological explanation corresponding to *T.E. VII* is given in this exposition.

D2.8 $[A] .\because. A\, \varepsilon_\infty\, \mathsf{qntf} . \equiv :$
1) $\mathbf{Ingr} \langle 1A \rangle\, \varepsilon_\infty\, \mathsf{vrb1}$.
2) $\mathbf{Uingr} \langle A \rangle\, \varepsilon_\infty\, \mathsf{vrb2}$.
3) $[\exists B \leqq A] . B\, \varepsilon_\infty\, \mathsf{int} \langle A \rangle :$
4) $[B \leqq A] : B\, \varepsilon_\infty\, \mathsf{int} \langle A \rangle . \supset . B\, \varepsilon_\infty\, \mathsf{trm} :$
5) $[BC \leqq A] : B\, \varepsilon_\infty\, \mathsf{int} \langle A \rangle . C\, \varepsilon_\infty\, \mathsf{int} \langle A \rangle . B\, \varepsilon_\infty\, \mathbf{cnf} \langle C \rangle . \supset . B =_\infty C$

A is a *quantifier*. Thus, quantifiers are unempty expressions bounded by lower corners containing only non-repetitious terms.

D2.9 $[A] :: A\, \varepsilon_\infty\, \mathsf{sbqntf} . \equiv :$
1) $[\exists B \leqq A] . B\, \varepsilon_\infty\, \mathsf{int} \langle A \rangle .\because.$
2) $[B \leqq A] .\because. B =_\infty \mathbf{Ingr} \langle 1A \rangle . \vee . B\, \varepsilon_\infty\, \mathsf{int} \langle A \rangle : \supset : [CD \leqq A] : C\, \varepsilon_\infty\, \mathsf{vrb3}$.
$C\, \varepsilon_\infty\, \mathbf{ingr} \langle A \rangle . C\, \varepsilon_\infty\, \mathbf{scd} \langle BA \rangle . D\, \varepsilon_\infty\, \mathsf{vrb4} . D\, \varepsilon_\infty\, \mathbf{ingr} \langle A \rangle$.
$D\, \varepsilon_\infty\, \mathbf{scd} \langle BA \rangle . \supset . \mathsf{N} \langle CA \rangle < \mathsf{N} \langle DA \rangle .\because.$
3) $[B \leqq A] .\because. B\, \varepsilon_\infty\, \mathsf{int} \langle A \rangle . \vee . B\, \varepsilon_\infty\, \mathbf{Uingr} \langle A \rangle : \supset : [CD \leqq A] : C\, \varepsilon_\infty\, \mathsf{vrb4}$.
$C\, \varepsilon_\infty\, \mathbf{ingr} \langle A \rangle . C\, \varepsilon_\infty\, \mathbf{prcd} \langle BA \rangle . D\, \varepsilon_\infty\, \mathsf{vrb3} . D\, \varepsilon_\infty\, \mathbf{ingr} \langle A \rangle$.
$D\, \varepsilon_\infty\, \mathbf{prcd} \langle BA \rangle . \supset . \mathsf{N} \langle CA \rangle < \mathsf{N} \langle DA \rangle$

A is a *subquantifier*. Clearly, all subquantifiers are unempty expressions structured in such a way that all upper corners occurring in them are uniquely paired. In this explanation, Leśniewski's clause:

$$[B] .\because. B\, \varepsilon\, \mathrm{1ingr}(A) . \vee . B\, \varepsilon\, \mathrm{int}(A) : \supset . (\mathrm{vrb3} \cap \mathrm{ingr}(A) \cap \mathrm{scd}(B)) \propto (\mathrm{vrb4} \cap \mathrm{ingr}(A) \cap \mathrm{scd}(B))$$

that is, upper left corners in *A* preceding *B* are strictly less in number than upper right corners in *A* preceding *B*, has been rendered as the second clause above, using the "number of times *C* occurs in *A* is strictly less than the number of times *D* occurs in *A*" as will be the case for similar clauses. Incidentally, it is necessary to avoid Leśniewski's use of "$a \propto b$" (as well as his use of "$a \infty b$") since it would introduce non-recursive concepts. However, since the arguments of Leśniewski's functor for less equinumerosity (and equinumerosity) are always ingredients of expressions, the effect of his use of "less equinumerous" (or "equinumerous") is always obtainable by comparing the specific count of the ingredients to be compared. Such an approach keeps this exposition faithful to Leśniewski's explanations while allowing it to remain recursive.

D2.10 $[A]: A\ \varepsilon_\infty\ \mathsf{gnrl}\ .\equiv.\ [\exists BC \leqq A].\ B\ \varepsilon_\infty\ \mathsf{qntf}\ .\ C\ \varepsilon_\infty\ \mathsf{sbqntf}\ .\ A =_\infty B * C$

A is a *generalization.* Generalizations are expressions made up entirely of some quantifier immediately followed by some subquantifier. Leśniewski uses 'Cmpl' in his *T.E.X* (given below), but this has been avoided in favor of concatenation with subsequent simplification of the explanation:

T.E.X $[A]::A\ \varepsilon\ \mathrm{gnrl}\ .{=}.\!\cdot\ [\exists B].\ B\ \varepsilon\ \mathrm{qntf}\ .\ B\ \varepsilon\ \mathrm{ingr}(A)\ .$
$1\mathrm{ingr}(A)\ \varepsilon\ \mathrm{ingr}(B):$
$[\exists B].\ B\ \varepsilon\ \mathrm{sbqntf}\ .\ B\ \varepsilon\ \mathrm{ingr}(A)\ .$
$\mathrm{Uingr}(A)\ \varepsilon\ \mathrm{ingr}(B):$
$[B,C]: B\ \varepsilon\ \mathrm{qntf}\ .\ B\ \varepsilon\ \mathrm{ingr}(A)\ .$
$C\ \varepsilon\ \mathrm{sbqntf}\ .\ C\ \varepsilon\ \mathrm{ingr}(A)\ .\ 1\mathrm{ingr}(A)\ \varepsilon\ \mathrm{ingr}(B)\ .$
$\mathrm{Uingr}(A)\ \varepsilon\ \mathrm{ingr}(C)\ .\supset.\ A\ \varepsilon\ \mathrm{Cmpl}(B \cup C)$

Thus, for Leśniewski as well, generalizations are expressions made up entirely of some quantifier immediately followed by some subquantifier.

D2.11 $[AB]: A\ \varepsilon_\infty\ \mathsf{Qntf}\langle B\rangle\ .\equiv.\ B\ \varepsilon_\infty\ \mathsf{gnrl}\ .\ A\ \varepsilon_\infty\ \mathsf{qntf}\ .\ A\ \varepsilon_\infty\ \mathbf{ingr}\langle B\rangle\ .$
$\mathbf{Ingr}\langle 1B\rangle\ \varepsilon_\infty\ \mathbf{ingr}\langle A\rangle$

A is *the quantifier* of B. The quantifier *of* a generalization is the quantifier with which the generalization begins, and is to be distinguished from a quantifier merely *in* the generalization. Thus, the numerical name given here is unique (as its capitalization indicates).

D2.12 $[AB]: A\ \varepsilon_\infty\ \mathsf{Sbqntf}\langle B\rangle\ .\equiv.\ B\ \varepsilon_\infty\ \mathsf{gnrl}\ .\ A\ \varepsilon_\infty\ \mathsf{sbqntf}\ .\ A\ \varepsilon_\infty\ \mathbf{ingr}\langle B\rangle\ .$
$\mathbf{Uingr}\langle B\rangle\ \varepsilon_\infty\ \mathbf{ingr}\langle A\rangle$

A is *the subquantifier* of B. The subquantifier of a generalization is the subquantifier with which the generalization ends. Thus, a subquantifier *of* an expression is unique and should be distinguished from a subquantifier merely *in* a generalization.

D2.13 $[AB].\!\cdot\ A\ \varepsilon_\infty\ \mathsf{Essnt}\langle B\rangle\ .\equiv.\ \mathsf{Sbqntf}\langle B\rangle =_\infty \mathsf{vrb3} * A * \mathsf{vrb4}\ .\ \mathsf{v}\ .\ A\ \varepsilon_\infty\ \mathsf{expr}\ .$
$A =_\infty B\ .\ {\sim}(A\ \varepsilon_\infty\ \mathsf{gnrl})$

A is *the nucleus* of B. The nucleus of an expression which is not a generalization is the expression itself, while the nucleus of a generalization is that expression which is generally said to fall within the "scope" of its quantifier. In either case the nucleus of an expression is unique. In this explanation, Leśniewski's clause:

$$A\ \varepsilon\ \mathrm{Cmpl}(\mathrm{int}(\mathrm{Sbqntf}(B)))$$

is rendered in terms of concatenation.

D2.14 $[ABC] .\because. A \,\varepsilon_\infty\, \mathrm{var}\langle BC\rangle .\equiv:$
1) $B \,\varepsilon_\infty\, \mathrm{int}\langle \mathrm{Qntf}\langle C\rangle\rangle .$
2) $A \,\varepsilon_\infty\, \mathbf{cnf}\langle B\rangle .$
3) $A \,\varepsilon_\infty\, \mathbf{ingr}\langle \mathrm{Essnt}\langle C\rangle\rangle :$
4) $[DE \leqq C]: D \,\varepsilon_\infty\, \mathbf{ingr}\langle C\rangle . E \,\varepsilon_\infty\, \mathrm{int}\langle \mathrm{Qntf}\langle D\rangle\rangle .$
$A \,\varepsilon_\infty\, \mathbf{cnf}\langle E\rangle . A \,\varepsilon_\infty\, \mathbf{ingr}\langle D\rangle .\supset. D =_\infty C$

A is a *variable bound* by *B* in *C*. Here is a simple example of Leśniewski's use of contextual definition, since bound variables are defined only in a given context, namely, as certain ingredients of a given expression. This explanation also makes it clear that there is great freedom of choice in selecting symbols as variables: a symbol that is a variable in one expression need not necessarily be a variable in another expression.

D2.15 $[ABC] . A \,\varepsilon_\infty\, \mathrm{cnvar}\langle BC\rangle .\equiv. [\exists D \leqq C] . A \,\varepsilon_\infty\, \mathrm{var}\langle DC\rangle .$
$[\exists D \leqq C] . B \,\varepsilon_\infty\, \mathrm{var}\langle DC\rangle . A \,\varepsilon_\infty\, \mathbf{cnf}\langle B\rangle$

A is a *variable equiform* with *B* in *C*. That is, *A* and *B* are variables bound by the same term of a given generalization. Thus, equiformity of variables is context dependent.

D2.16 $[A] :: A \,\varepsilon_\infty\, \mathrm{prntm} .\equiv.\because$
1) $[\exists B \leqq A] : B \,\varepsilon_\infty\, \mathrm{int}\langle A\rangle .\because$
2) $[B \leqq A] .\because. B \,\varepsilon_\infty\, \mathbf{ingr}\langle 1A\rangle . \mathrm{v} . B \,\varepsilon_\infty\, \mathrm{int}\langle A\rangle :\supset: [CD \leqq A] : C \,\varepsilon_\infty\, \mathbf{ingr}\langle A\rangle .$
$C \,\varepsilon_\infty\, \mathbf{scd}\langle BA\rangle . C \,\varepsilon_\infty\, \mathbf{cnf}\langle \mathbf{Ingr}\langle 1A\rangle\rangle . D \,\varepsilon_\infty\, \mathbf{ingr}\langle A\rangle . D \,\varepsilon_\infty\, \mathbf{scd}\langle BA\rangle .$
$D \,\varepsilon_\infty\, \mathbf{prntsym}\langle \mathbf{Ingr}\langle 1A\rangle\rangle .\supset. \mathrm{N}\langle CA\rangle\langle \mathrm{N}\langle DA\rangle .\because$
3) $[B \leqq A] .\because. B \,\varepsilon_\infty\, \mathrm{int}\langle A\rangle . \mathrm{v} . B \,\varepsilon_\infty\, \mathbf{Uingr}\langle A\rangle :\supset: [CD \leqq A] : C \,\varepsilon_\infty\, \mathbf{ingr}\langle A\rangle .$
$C \,\varepsilon_\infty\, \mathbf{prcd}\langle BA\rangle . C \,\varepsilon_\infty\, \mathbf{prntsym}\langle \mathbf{Ingr}\langle 1A\rangle\rangle . D \,\varepsilon_\infty\, \mathbf{ingr}\langle A\rangle .$
$D \,\varepsilon_\infty\, \mathbf{prcd}\langle BA\rangle . D \,\varepsilon_\infty\, \mathbf{prntl} . D \,\varepsilon_\infty\, \mathbf{cnf}\langle \mathbf{Ingr}\langle 1A\rangle\rangle .\supset.$
$\mathrm{N}\langle CA\rangle\langle \mathrm{N}\langle DA\rangle$

A is a *parentheme*. Parenthemes are parenthetical clauses (including their outermost parentheses) in which all words equiform to their outermost parentheses are uniquely paired.

In order to avoid Leśniewski's use of 'Cmpl' in:

T.E.XVII $[A, a, B] :: A \,\varepsilon\, \mathrm{prntm}(B, a) .\equiv [C] : C \,\varepsilon a .\supset. C \,\varepsilon\, \mathrm{prntm}$
$B \,\varepsilon \mathrm{Cmpl}(1\mathrm{ingr}(B) \cup a) .$
$1\mathrm{ingr}(B) \,\varepsilon\, \mathrm{trm} . A \,\varepsilon a$

and the quantification over general names in:

T.E.XVIII $[A, B] : A \,\varepsilon\, \mathrm{prntm}(B) .\equiv. [\exists a] . A \,\varepsilon\, \mathrm{prntm}(B, a)$

we define the *n*th parentheme of an expression recursively as follows:

D2.17a $[AB] : A \,\varepsilon_\infty\, \mathrm{prntm}\langle 0B\rangle .\equiv. A \varepsilon_\infty\, \mathrm{prntm} . [C \leqq B] . B =_\infty \mathbf{Ingr}\langle 1B\rangle * A * C .$
$\mathbf{Ingr}\langle 1B\rangle \,\varepsilon_\infty\, \mathrm{trm}$
D2.17b $[ABn] : A \,\varepsilon_\infty\, \mathrm{prntm}\langle n+1B\rangle .\equiv. A \,\varepsilon_\infty\, \mathrm{prntm} . [\exists CD \leqq B] .$
$B =_\infty C * \mathrm{prntm}\langle nB\rangle * A * D$

A is the *nth parentheme* of *B*. Thus, the *n*th parentheme of an expression in unique.

D2.18 $[AB]:A\,\varepsilon_\infty\,\mathsf{prntm}\langle B\rangle\,.\!\equiv\!.\,[\exists n\leqq \mathsf{L}\langle B\rangle]\,.\,A\varepsilon_\infty\,\mathsf{prntm}\langle nB\rangle$

A is a *parentheme* of B. Notice that this explanation closely parallels Leśniewski's: in both cases parenthemes *of* expression are to be distinguished from parenthemes merely *in* expressions.

D2.18 $[AB]\therefore A\,\varepsilon_\infty\,\mathsf{Uprntm}\langle B\rangle\,.\!\equiv\!:\,A\,\varepsilon_\infty\,\mathsf{prntm}\langle B\rangle:[nmC\leqq B]:$
$A\,\varepsilon_\infty\,\mathsf{prntm}\langle nB\rangle\,.\,C\,\varepsilon_\infty\,\mathsf{prntm}\langle mB\rangle\,.\!\supset\!.\,m\leqq n$

A is the *last parentheme* of B.

D2.19 $[A]:A\,\varepsilon_\infty\,\mathsf{fnct}\,.\!\equiv\!.\,[\exists B\leqq A]\,.\,B\,\varepsilon_\infty\,\mathsf{prntm}\langle A\rangle$

A is a *function*.

D2.20a $[AB]\therefore A\,\varepsilon_\infty\,\mathsf{arg}\langle 0B\rangle\,.\!\equiv\!:\,B\,\varepsilon_\infty\,\mathsf{prntm}\,.\,[\exists C\leqq B]\,.\,B=_\infty \mathbf{Ingr}\langle 1B\rangle * A * C\,.$
$A\,\varepsilon_\infty\,\mathsf{trm}\,.\,\mathsf{v}\,.\,A\,\varepsilon_\infty\,\mathsf{gnrl}\,.\,\mathsf{v}\,.\,A\,\varepsilon_\infty\,\mathsf{fnct}$

D2.20b $[ABn]\therefore A\,\varepsilon_\infty\,\mathsf{arg}\langle n+1B\rangle\,.\!\equiv\!:B\,\varepsilon_\infty\,\mathsf{prntm}\,.\,[\exists CD\leqq B]\,.$
$B=_\infty C*\mathsf{arg}\langle nB\rangle * A * C: A\,\varepsilon_\infty\,\mathsf{trm}\,.\,\mathsf{v}\,.\,A\,\varepsilon_\infty\,\mathsf{gnrl}\,.\,\mathsf{v}\,.\,A\,\varepsilon_\infty\,\mathsf{fnct}$

A is the *nth argument* of B. Thus the nth argument of an expression is unique.

D2.21 $[AB]:A\,\varepsilon_\infty\,\mathsf{arg}\langle B\rangle\,.\!\equiv\!.\,[\exists n\leqq \mathsf{L}\langle B\rangle]\,.\,A\,\varepsilon_\infty\,\mathsf{arg}\langle nB\rangle$

A is an *argument* of B. Thus, arguments *of* parenthemes are to be distinguished from arguments merely in parenthemes. As in D2.17 and D2.18 the difficulties inherent in Leśniewski's use of 'Cmpl' are again avoided in favor of concatenation.

D2.21 $[AB]\therefore A\varepsilon_\infty\,\mathsf{Uarg}\langle B\rangle\,.\!\equiv\!:A\,\varepsilon_\infty\,\mathsf{arg}\langle B\rangle:[nmC\leqq B]:$
$A\,\varepsilon_\infty\,\mathsf{arg}\langle nB\rangle\,.\,C\,\varepsilon_\infty\,\mathsf{arg}\langle mB\rangle\,.\!\supset\!.\,m\leqq n$

A is the *last argument* of B.

D2.22 $[AB]:A\,\varepsilon_\infty\,\mathsf{Sgnfcnt}\langle B\rangle\,.\!\equiv\!.A\,\varepsilon_\infty\,\mathsf{expr}\,.\,[\exists C\leqq B]\,.\,C\,\varepsilon_\infty\,\mathsf{prntm}\langle B\rangle\,.$
$B=_\infty A*C$

A is *the functor* of B. Thus the functor of an expression is unique. Since the functor of an expression is all but the last parentheme of some given function, it may be many-linked, that is, its own function may have parameters. In this explanation, concatenation replaces the use of 'Cmpl' in Leśniewski's:

T.E.XXII $[A,B]:A\,\varepsilon\,\mathrm{Sgnfnct}(B)\,.\!=\!.\,A\,\varepsilon\,\mathrm{expr}\,.$
$A\,\varepsilon\,\mathrm{ingr}(B)\,.$
$\mathrm{Cmpl}(\mathrm{vrb}\cap\mathrm{ingr}(B)\cap$
$(\mathrm{ingr}(A)))\,\varepsilon\,\mathrm{prntm}(B)$

D2.23 $[AB]\therefore A\,\varepsilon_\infty\,\mathsf{simprntm}\langle B\rangle\,.\!\equiv\!:$
1) $A\,\varepsilon_\infty\,\mathsf{prntm}\,.$
2) $B\,\varepsilon_\infty\,\mathsf{prntm}\,.$
3) $\mathbf{Ingr}\langle 1A\rangle\,\varepsilon_\infty\,\mathbf{cnf}\langle\mathbf{Ingr}\langle 1B\rangle\rangle:$
4) $[n\leqq \mathsf{L}\langle A\rangle+\mathsf{L}\langle B\rangle]:\mathsf{Uarg}\langle A\rangle\,\varepsilon_\infty\,\mathsf{arg}\langle nA\rangle\,.\!\equiv\!.\,\mathsf{Uarg}\langle B\rangle\,\varepsilon_\infty\,\mathsf{arg}\langle nB\rangle$

A is a *similar parentheme* to B. Thus, similar parenthemes have equiform outermost parentheses and the same number of arguments. Here Leśniewski's clause:

$$\arg(A) \infty \arg(B)$$

that is, the number of arguments of A is the same as the number of B, is rendered as the last clause above, as will be the case with similar clauses.

D2.24 $[AB]::A\,\varepsilon_\infty\,\mathsf{genfnct}<B>.\equiv.\because$
1) $A\,\varepsilon_\infty\,\mathsf{fnct}:$
2) $[nm\leqq \mathsf{L}<B>].\,\mathsf{Uprntm}<A>\varepsilon_\infty\,\mathsf{prntm}<nA>.$
$\mathsf{Uprntm}<B>\varepsilon_\infty\,\mathsf{prntm}<mB>.\supset.n\leqq m:$
3) $[CDnmr\leqq A+B]:C\,\varepsilon_\infty\,\mathsf{prntm}<nA>.$
$D\,\varepsilon_\infty\,\mathsf{prntm}<mB>.\,\mathsf{prntm}<n+rA>\varepsilon_\infty\,\mathsf{Uprntm}<A>.$
$\mathsf{prntm}<m+rB>\varepsilon_\infty\,\mathsf{Uprntm}<B>.\supset.C\,\varepsilon_\infty\,\mathsf{simprntm}<D>$

A is a *generating function* with respect to B. Thus A is a generating function with respect to B if it is a function that has no more parenthemes than B has; and whose parenthemes are similar to the terminal parenthemes of B. Generating functions are very useful in determining the semantical categories of newly defined functors (cf. D2.39). For instance, '$!\{a\}$' is a generating function in respect to '$*\langle\varphi\rangle\langle b\rangle\{a\}$' and thus determines the semantical category of '$*\langle\varphi\rangle\langle b\rangle$'.

D2.25 $[ABCD].\because.A\,\varepsilon_\infty\,\mathsf{Anarg}<BCD>.\equiv.C\,\varepsilon_\infty\,\mathsf{simprntm}<D>.\,A\,\varepsilon_\infty\,\mathsf{arg}<C>.$
$B\,\varepsilon_\infty\,\mathsf{arg}<D>:[n\leqq \mathsf{L}<C>+\mathsf{L}<D>]:A\,\varepsilon_\infty\,\mathsf{arg}<nC>.\equiv.\,B\,\varepsilon_\infty\,\mathsf{arg}<nD>$

A is the *argument in C analogous* to B in D.

D2.26 $[ABCD]:A\,\varepsilon_\infty\,\mathsf{Ansgnfnct}<BCD>.\equiv.A\,\varepsilon_\infty\,\mathsf{Sgnfnct}<B>.$
$B\,\varepsilon_\infty\,\mathsf{Sgnfnct}<D>.[\exists EF\leqq B+C].E\,\varepsilon_\infty\,\mathsf{prntm}<C>.E\,\varepsilon_\infty\,\mathbf{scd}<AC>.$
$F\,\varepsilon_\infty\,\mathsf{prntm}<D>.F\,\varepsilon_\infty\,\mathbf{scd}<BD>.\,E\,\varepsilon_\infty\,\mathsf{simprntm}<F>$

A is the *functor in C analogous* to B in D.

D2.27 $[ABCD].\because.A\,\varepsilon_\infty\,\mathsf{An}<BCD>.\equiv:A\,\varepsilon_\infty\,\mathsf{Anarg}<BCD>.\vee.$
$A\,\varepsilon_\infty\,\mathsf{Ansgnfnct}<BCD>$

A in C is the *analogue* of B in D.

D2.28 $[AB]:A\,\varepsilon_\infty\,\mathsf{arg1}<B>.\equiv.[\exists C\leqq \mathsf{Axp}].C\,\varepsilon_\infty\,\mathbf{ingr}<\mathsf{Axp}>.$
$A\,\varepsilon_\infty\,\mathsf{Anarg}<\mathbf{Ingr}<10\,\mathsf{Axp}>BC>$

A is the *first argument* of B. Thus this explanation determines that the last word preceding the first argument of B is a parenthesis of the semantical category whose number is $2^4\times 3^1\times 5^1\times 7^1$.

D2.29 $[AB]:A\,\varepsilon_\infty\,\mathsf{arg2}<B>.\equiv.[\exists C\leqq \mathsf{Axp}].C\,\varepsilon_\infty\,\mathbf{ingr}<\mathsf{Axp}>.$
$A\,\varepsilon_\infty\,\mathsf{Anarg}<\mathbf{Ingr}<11\,\mathsf{Axp}>BC>$

A is the *second argument* of B.

D2.30 $[AB]: A \,\varepsilon_\infty\, \mathrm{Eqvl1} < B > .\equiv.\ \mathrm{Sgnfnct} < B > \varepsilon_\infty\, \mathbf{cnf} < \mathbf{Ingr} < 7\,\mathrm{Axp} >> .$
$[\exists C \leqq B].\, C\ \varepsilon_\infty\, \mathrm{prntm} < B > .\, A\, \varepsilon_\infty\, \mathrm{Arg1} < C >$

A is the *coimplicans* of B. Thus this explanation, together with the next, can be considered as a stipulation for determining the shape of the symbol to be used for complication.

D2.31 $[AB]: A\, \varepsilon_\infty\, \mathrm{Eqvl2} < B > .\equiv.\ \mathrm{Sgnfnct} < B > \varepsilon_\infty\, \mathbf{cnf} < \mathbf{Ingr} < 7\,\mathrm{Axp} >> .$
$[\exists C \leqq B].\, C\ \varepsilon_\infty \mathrm{prntm} < B > .\, A\, \varepsilon_\infty\, \mathrm{Arg2} < C >$

A is the *complicate* of B.

D2.32 $[ABC]: A\, \varepsilon_\infty\, \mathbf{thp} < BC > .\equiv.\ C\, \varepsilon_\infty\, \mathbf{lst}\, .\, \mathrm{Axp}\ \varepsilon_\infty\, \mathrm{gl} < 1C > .$
$[\exists nm \leqq \mathrm{L} < C >].\, A =_\infty \mathrm{gl} < nC > .\, B =_\infty \mathrm{gl} < mC > .\, n \leqq m .$

A is a *theses of protothetic* relative to B in a list C (assuming that every term in C is a thesis of protothetic). In order to retain limited quantification, a thesis of protothetic must be explained relative to a given list (of theses) instead of relative to a given thesis as is the case in:

T.E.XXXII $[A,B] .\therefore A\, \varepsilon\, \mathrm{thp}(B) .=: A\, \varepsilon\, \mathrm{thp} .$
$B\, \varepsilon\, \mathrm{thp} :$
$A\ \varepsilon\, \mathrm{prcd}(B) .\, \mathrm{v} .\, A\, \varepsilon\, \mathrm{Id}(B)$

Thus, all explanations which depend on D2.32 will also be relativized to a given list (of theses). Actually, Leśniewski takes 'thp' as primitive: its first use occurs in his *T.E.XXXII* when he defines '$A\,\varepsilon\,\mathrm{thp}(B)$'. However, the concept is only needed in an inductive caluse for the directive of protothetic and so the concept given here suffices for the exposition.

D2.33 $[ABB'] .\therefore A\, \varepsilon_\infty\, \mathrm{frp} < BB' > .\equiv: A\ \varepsilon_\infty\, \mathbf{thp} < BB' > .\, \mathrm{v} .$
$[\exists CD \leqq B'].\, C\, \varepsilon_\infty\, \mathbf{thp} < BB' > .\, D\, \varepsilon_\infty\, \mathbf{ingr} < C > .\, A\, \varepsilon_\infty\, \mathrm{Arg1} < D > .\, \mathrm{v} .$
$[\exists CD \leqq B'].\, C\, \varepsilon_\infty\, \mathbf{thp} < BB' > . D\ \varepsilon_\infty\, \mathbf{ingr} < C > . A\ \varepsilon_\infty\, \mathrm{Arg2} < D > .\, \mathrm{v} .$
$[\exists CD \leqq B'].C\ \varepsilon_\infty\, \mathbf{thp} < BB' > . D\ \varepsilon_\infty\, \mathrm{sbqntf} .\, D\, \varepsilon_\infty\, \mathbf{ingr} < C > .$
$D =_\infty \mathrm{vrb3} * A * \mathrm{vrb4}$

A is a *propositional phrase* relative to B in B'. The use of 'Cmpl' in *T.E.XXXIII* is avoided in the same manner as in *T.E.XIII*.

D2.34a $[ABCC'] A\, \varepsilon_\infty \mathrm{homosemp} < 0BCC' > .\equiv. A\ \varepsilon_\infty\, \mathrm{frp} < CC' > .$
$B\, \varepsilon_\infty\, \mathrm{frp} < CC' > .\, \mathrm{v} .\, [\exists DE \leqq C'].D\ \varepsilon_\infty\, \mathbf{thp} < CC' > .\, E\, \varepsilon_\infty\, \mathbf{ingr} < D > .$
$A\, \varepsilon_\infty\, \mathrm{cnvar} < BE > .\, \mathrm{v} .\, [\exists DEFG \leqq C'].D\ \varepsilon_\infty\, \mathbf{thp} < CC' > .\, E\, \varepsilon_\infty\, \mathbf{ingr} < D > .$
$F\, \varepsilon_\infty\, \mathbf{thp} < CC' > . G\ \varepsilon_\infty\, \mathbf{ingr} < F > .\, A\, \varepsilon_\infty\, \mathrm{An} < BFG >$

D2.34b $[ABCC'n]: A\, \varepsilon_\infty\, \mathrm{homosemp} < n+1BCC' > .\equiv.\ [\exists D \leqq C'].$
$A\, \varepsilon_\infty\, \mathrm{homosemp} < 0DCC' > .\, D\, \varepsilon_\infty \mathrm{homosemp} < nBCC' >$

A is the *nth homosome* of B relative to C in C'. That is, the semantical category of A is determined to be the same as that of B (relative to C in C') within n-number of determinations. The inductive clause of this explanation has been added to Leśniewski's *T.E.XXXIV*, which is the first case in the recursive definition, in order to avoid his quantification over general names in:

$T.E.XXXV$ $[A,B,C]\therefore A\ \varepsilon\, \mathrm{homosemp}(B,C) .\equiv\therefore A \varepsilon\ 1\mathrm{homosemp}(A,C)$. $B\ \varepsilon\ 1\mathrm{homosemp}(B,C) \therefore$

$[a]:: [D]: D\ \varepsilon\ a. \supset.$

$E\ \varepsilon a .\because. B\ \varepsilon a .\because. \supset. A \varepsilon\ a$

Now, in the place of the above, there is the following.

D2.35 $[ABCC']: A\ \varepsilon_\infty \mathrm{homosemp} < BCC' > .\equiv. [\ n \leqq\ ! < C >].$
$A\ \varepsilon_\infty \mathrm{homosemp} < nBCC' >$

A is of the *same semantical category* as B relative to C in C'.

D2.36 $[ABB'CDE] .\because. A\varepsilon_\infty \mathrm{constp} < BB'CDE > .\equiv:$
1) $D\ \varepsilon_\infty \mathrm{homosemp} < EBB' >:$
2) $[FG \leqq B']: G\ \varepsilon_\infty \mathbf{thp} < B,B' > . F\varepsilon_\infty \mathbf{ingr} < G > .\supset. \sim(D\ \varepsilon_\infty \mathrm{convar} < DF >):$
3) $A\varepsilon_\infty \mathbf{cnf} < D >.$
4) $[\exists FGH \leqq B']. F\varepsilon_\infty \mathbf{ingr} < C > . G\ \varepsilon_\infty \mathbf{thp} < BB' > . H\ \varepsilon_\infty \mathbf{ingr} < G >$
$A\varepsilon_\infty \mathrm{An} < EFH >$

A, in C and analogue of E, is *suited to be a constant equisignificant* to D relative to B in B'. That is, any argument (or functor) A in some expression C is suited to be a constant equisignificant to D, provided A is equiform to D; D is not a variable; and A is the analogue of some E, where E is of the same semantical category as D relative to B in B'. This explanation is given in order to insure that symbols employed as constants have some fixed semantical category—see the following explanation.

D2.37 $[ABB'C]: A\varepsilon_\infty \mathrm{constp} < BB'C > .\equiv. [\exists DE \leqq B']. A\varepsilon_\infty \mathrm{constp} < BB'CDE >$

A in C is *suited to be a constant* relative to B in B'.

D2.38 $[ABCC'DEF]: A\ \varepsilon_\infty \mathrm{quasihomosemp} < BCC'DEF > .\equiv.$
1) $E\varepsilon_\infty \mathrm{homosemp} < FCC' >.$
2) $[\exists GHI \leqq C']. G\ \varepsilon_\infty \mathbf{ingr} < D > . H\varepsilon_\infty \mathbf{thp} < BB' > . I\ \varepsilon_\infty \mathbf{ingr} < H >$
$A\ \varepsilon_\infty \mathrm{An} < EGI >.$
3) $[\exists GHI \leqq C']. G\ \varepsilon_\infty \mathbf{ingr} < D > . H\varepsilon_\infty \mathbf{thp} < BB' > . I\ \varepsilon_\infty \mathbf{ingr} < H >$
$B\varepsilon_\infty \mathrm{An} < FGI >$

A is a *quasihomoseme* of B in respect to D,E, and F relative to C in C'. That is, A and B are eligible to belong to the same semantical category as their respective analogues E and F, relative to C in C'. This explanation is useful in explaining protothetical definitions where it is necessary to speak about the semantical categories of a pair of words.

D2.39 $[ABB'CDE]: A\ \varepsilon_\infty \mathrm{fnctp} < BB'CDE > .\equiv.$
1) $D\varepsilon_\infty \mathrm{homosemp} < EBB' >.$
2) $A\ \varepsilon_\infty \mathrm{genfnct} < D >.$
3) $[\exists FGH \leqq B']. F\varepsilon_\infty \mathbf{ingr} < C > . G\varepsilon_\infty \mathbf{thp} < BB' > . H\varepsilon_\infty \mathbf{ingr} < G >.$
$A\ \varepsilon_\infty \mathrm{An} < EFH >$

A in C is *suited to be a function belonging to the semantical category* of D and E relative to B in B'. This explanation and the remaining ones of this section are all useful in explaining protothetical definitions. In each case what is needed is a way of determining the semantical categories of the components of expressions.

D2.40 $[ABCC'DEF]: A\,\varepsilon_\infty\,\mathrm{varp}<BCC'DEF>.\equiv.$
1) $E\,\varepsilon_\infty\,\mathrm{homosemp}<BCC'>.$
2) $[\exists GHI \leqq C'].G\,\varepsilon_\infty\,\mathbf{ingr}<D>.H\,\varepsilon_\infty\,\mathbf{thp}<BC'>.I\,\varepsilon_\infty\,\mathbf{ingr}<H>.$
$F\,\varepsilon_\infty\,\mathrm{An}<HGI>.$
3) $F\,\varepsilon_\infty\,\mathbf{ingr}<\mathrm{Eqvl1}<\mathrm{Essnt}<D>>>.$
4) $A\,\varepsilon_\infty\,\mathrm{cnvar}<FD>$

A in D is *suited to be a variable belonging to the same semantical category as* $B, E,$ *and* F relative to C in C'.

D2.41 $[ABB'CDE]:: A\,\varepsilon_\infty\,\mathrm{prntmp}<BB'CDE>.\equiv.\because$
1) $D\,\varepsilon_\infty\,\mathrm{homosemp}<BBB'>.$
2) $E\,\varepsilon_\infty\,\mathrm{prntm}<D>.$
3) $A\,\varepsilon_\infty\,\mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<C>>>:$
4) $[n \leqq \mathrm{L}<A>+\mathrm{L}<E>]: \mathrm{Uarg}<A> =_\infty \mathrm{arg}<nA>.\equiv.\mathrm{Uarg}<E> =_\infty \mathrm{arg}<nE>.\because$
5) $[nFG \leqq A+E].\because.F\,\varepsilon_\infty\,\mathrm{arg}<nA>.\equiv.G\,\varepsilon_\infty\,\mathrm{arg}<nE>:\supset.[\exists HI \leqq B'].$
$F\,\varepsilon_\infty\,\mathrm{varp}<GBB'CHI>$

A in C is *suited to be similar to the parentheme* E *of the semantical category of* D relative to B in B'.

D2.42 $[ABB'CDE]: A\,\varepsilon_\infty\,\mathrm{1prntm}<BB'CDE>.\equiv.A\,\varepsilon_\infty\,\mathrm{prntm}<BB'CDE>.$
$\mathbf{Uingr}<D>\varepsilon_\infty\,\mathbf{ingr}<E>$

A in C is *suited to be similar to the last parentheme* E of the semantical category of D relative to B in B'.

D2.43 $[ABB'CDEFG]: A\,\varepsilon_\infty\,\mathrm{2prntm}<BB'CDEFG>.\equiv.A\,\varepsilon_\infty\,\mathrm{prntm}<BB'CDE>.$
$F\varepsilon_\infty\,\mathrm{prntm}<D>.\mathbf{Uprcd}<FD>\varepsilon_\infty\,\mathbf{ingr}<E>.G\,\varepsilon_\infty\,\mathrm{simprntm}<F>$

A in C is *suited to be similar to the parentheme* E of D which immediately precedes an F similar to G relative to B in B'.

This ends the preliminary explanations, and it is now possible to explain the rule of protothetic, which is taken up in the next section.

3. *The rule of protothetic* In this section, the terminological explanations for the rule of protothetic are given and are numbered according to [5]. There are, thus, only five terminological explanations to list—one for each part of the rule. However, since there are generally many defining conditions in each explanation (eighteen for protothetical definition), each of the five explanations has been sub-divided into its important components.

D2.44 $[ABB']:: A\,\varepsilon_\infty\,\mathrm{defp}<BB'>.\equiv::$
1) $\sim(\mathbf{Ingr}<1\,\mathrm{Essnt}<A>>\varepsilon_\infty\,\mathrm{cnvar}<\mathbf{Ingr}<1A>A>).$
2) $\sim(\mathbf{Ingr}<1\,\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>\varepsilon_\infty\mathrm{cnvar}<\mathbf{Ingr}<1\,\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>A>).$
3) $\sim(\mathbf{Ingr}<1\,\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>\varepsilon_\infty\mathrm{constp}<BB'A>)::$
4) $[C \leqq A].\because.C\,\varepsilon_\infty\,\mathrm{trm}.C\,\varepsilon_\infty\,\mathbf{ingr}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>.\supset:[\exists D \leqq A].D\,\varepsilon_\infty\,\mathrm{qntf}.$
$D\,\varepsilon_\infty\,\mathbf{ingr}<A>.C\,\varepsilon_\infty\,\mathrm{int}<D>.\mathrm{v}.[\exists DE \leqq A].D\varepsilon_\infty\,\mathbf{ingr}<A>.$
$C\,\varepsilon_\infty\,\mathrm{var}<E,D>.\mathrm{v}.C\,\varepsilon_\infty\,\mathrm{constp}<BB'A>::$

5) $[CD \leqq A]\colon D\,\varepsilon_\infty\,\mathrm{qntf}\,.\,D\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,C\,\varepsilon_\infty\,\mathrm{int}<D>.\supset.\,[\exists EF \leqq A]\,.$
$E\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,F\,\varepsilon_\infty\,\mathrm{var}<CE>.\therefore$

6) $[CDE \leqq A]\colon C\,\varepsilon_\infty\,\mathrm{int}<\mathrm{Qntf}<A>>.\,E\,\varepsilon_\infty\,\mathrm{prntm}<\mathrm{Essnt}<A>>.$
$D\,\varepsilon_\infty\,\mathrm{arg}<E>.\supset.\,[\exists F \leqq A].\,F\,\varepsilon_\infty\,\mathbf{ingr}<D>.\,F\,\varepsilon_\infty\,\mathrm{var}<CA>::$

7) $[CDE \leqq A].\therefore C\,\varepsilon_\infty\,\mathbf{ingr}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>.\,E\,\varepsilon_\infty\,\mathbf{ingr}<A>.$
$D\,\varepsilon_\infty\,\mathrm{cnvar}<CE>.\,D\,\varepsilon_\infty\,\mathbf{ingr}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>.\supset\colon D =_\infty C\,.\,\mathrm{v}\,.$
$[\exists FG \leqq B'].\,D\,\varepsilon_\infty\,\mathrm{quasihomosemp}<CBB'AFG>::$

8) $[C \leqq A]\colon C\,\varepsilon_\infty\,\mathrm{gnrl}\,.\,C\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,C \neq_\infty A.\supset.\,[\exists EDFG \leqq B'].$
$D\,\varepsilon_\infty\,\mathrm{homosemp}<BBB'>.\,E\,\varepsilon_\infty\,\mathbf{thp}<BB'>.\,F\,\varepsilon_\infty\,\mathbf{ingr}<E>.$
$G\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,D\,\varepsilon_\infty\,\mathrm{Anarg}<CFG>::$

9) $[CD \leqq A].\therefore C\,\varepsilon_\infty\,\mathrm{gnrl}\,.\,C\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,D\,\varepsilon_\infty\,\mathrm{Essnt}<C>.\supset\colon D\,\varepsilon_\infty\,\mathbf{vrb}\,.\,\mathrm{v}\,.$
$[\exists E \leqq B'].\,E\,\varepsilon_\infty\,\mathrm{frp}<BB'>.\,D\,\varepsilon_\infty\,\mathrm{genfnct}<E>::$

10) $[C \leqq A].\therefore C\,\varepsilon_\infty\,\mathrm{fnct}\,.\,C\,\varepsilon_\infty\,\mathbf{ingr}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>.\supset\colon [\exists D \leqq A].$
$D\,\varepsilon_\infty\,\mathrm{gnrl}\,.\,D\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,C\,\varepsilon_\infty\,\mathrm{Essnt}<D>.\,\mathrm{v}\,.\,[\exists DE \leqq B'].$
$C\,\varepsilon_\infty\,\mathrm{fnctp}<BB'ADE>::$

11) $[C \leqq A]\colon C\,\varepsilon_\infty\,\mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>.\supset.\,[\exists D \leqq C].\,D\,\varepsilon_\infty\,\mathrm{arg}<C>.\therefore$

12) $[CD \leqq A]\colon C\,\varepsilon_\infty\,\mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>.\,D\,\varepsilon_\infty\,\mathrm{arg}<C>.\supset.$
$[\exists E \leqq A].\,D\,\varepsilon_\infty\,\mathrm{var}<EA>.\therefore$

13) $[CD \leqq A]\colon C\,\varepsilon_\infty\,\mathrm{trm}\,.\,C\,\varepsilon_\infty\,\mathbf{ingr}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>.\,D\,\varepsilon_\infty\,\mathrm{trm}\,.$
$D\,\varepsilon_\infty\,\mathbf{ingr}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>.\,C\,\varepsilon_\infty\,\mathbf{cnf}<D>.\supset.\,C =_\infty D.\therefore$

14) $[CD \leqq A]\colon C\,\varepsilon_\infty\,\mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>.$
$D\,\varepsilon_\infty\,\mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>.\,C\,\varepsilon_\infty\,\mathrm{simprntm}<D>.\supset.\,C =_\infty D.\therefore$

15) $[CDE \leqq B']\colon C\,\varepsilon_\infty\,\mathrm{1prntmp}<BB'ADE>.\,\mathbf{Uingr}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>.$
$\varepsilon_\infty\,\mathbf{ingr}<C>.\supset.\,C\,\varepsilon_\infty\,\mathrm{simprntm}<E>\colon$

16) $[CDEFG \leqq B']\colon C\,\varepsilon_\infty\,\mathrm{2prntmp}<BB'ADEFG>.\,G\,\varepsilon_\infty\,\mathbf{ingr}<A>.$
$\mathbf{Uprcd}<GA>\varepsilon_\infty\,\mathbf{ingr}<C>.\supset.C\,\varepsilon_\infty\,\mathrm{simprntm}<E>\colon$

17) $[CDE \leqq B'].\,C\,\varepsilon_\infty\,\mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>\colon$
$\mathbf{Uingr}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>\,\varepsilon_\infty\,\mathbf{ingr}<C>.\,D\,\varepsilon_\infty\,\mathbf{thp}<BB'>.$
$E\,\varepsilon_\infty\,\mathbf{ingr}<D>.\,C\,\varepsilon_\infty\,\mathrm{simprntn}<E>.\supset.\,[\exists FG \leqq B'].$
$C\,\varepsilon_\infty\,\mathrm{1prntm}<BB'AFG>\colon$

18) $[CDEF \leqq B']\colon C\,\varepsilon_\infty\,\mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>.\,D\,\varepsilon_\infty\,\mathrm{prntm}\,.$
$D\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,\mathbf{Uprcd}<DA>\,\varepsilon_\infty\,\mathbf{ingr}<C>.\,E\,\varepsilon_\infty\,\mathbf{thp}<BB'>.$
$F\,\varepsilon_\infty\,\mathbf{ingr}<E>.\,C\,\varepsilon_\infty\,\mathrm{simprntm}<F>.\supset.\,[\exists GHI \leqq B'].$
$C\,\varepsilon_\infty\,\mathrm{2prntmp}<BB'AGHID>$

A is a *protothetical definition* relative to B in B'. In discussing a definition A, the coimplicans of the nucleus of A shall be called the definiens of A, and the coimplicate of the nucleus of A shall be called the definiendum of A. Thus, the first three clauses of D3.44 indicate that definitions are generalizations of coimplications, where the first word of the definiendum is neither a variable nor a previously defined or primitive constant—it is in fact the constant which is being defined. The fourth clause indicates that any terms occurring in the definiens are either variables in quantifiers, variables bound by quantifiers, or constants which already have a fixed semantical category. Clauses 5 and 6 indicate that any of the quantifiers which occur in a definition (either as the quantifier of the definition or merely as quantifiers in the definition) are not vacuous.

Clauses 7 through 10 make certain stipulations about the definiens of a definition. Equiform variables in the definiens must belong to the same semantical category. Any generalizations must be propositional and their nuclei must either be single words or generating functions for some determined semantical category. Any function in the definiens is either the nucleus of a generalization or belongs to some determined semantical category.

Finally, the remaining clauses of D3.44 deal with the definiendum. In clauses 11 through 14 it is stipulated that parenthemes in the definiendum are unempty and contain only variables, while there is no duplication of terms (and hence variables) nor of similar parenthemes. While the last four clauses taken together stipulate that each parentheme occurring in the definiendum can be assigned unambiguously a suitable semantical category.

D3.45 $[AB]::A\,\varepsilon_\infty\,\mathrm{cnsqrprtqntf}\langle B\rangle\,.\!\equiv\!.\!\cdot\!.$

1) $\mathrm{Essnt}\langle\mathrm{Eqvl1}\langle\mathrm{Essnt}\langle A\rangle\rangle\rangle\,\varepsilon_\infty\,\mathbf{cnf}\langle\mathrm{Essnt}\langle\mathrm{Eqvl1}\langle\mathrm{Essnt}\langle B\rangle\rangle\rangle\rangle\,.$
2) $\mathrm{Essnt}\langle\mathrm{Eqvl2}\langle\mathrm{Essnt}\langle A\rangle\rangle\rangle\,\varepsilon_\infty\,\mathbf{cnf}\langle\mathrm{Essnt}\langle\mathrm{Eqvl2}\langle\mathrm{Essnt}\langle B\rangle\rangle\rangle\rangle:$
3) $[C\leqq A]:C\,\varepsilon_\infty\,\mathrm{int}\langle\mathrm{Qntf}\langle A\rangle\rangle\,.\supset.\,[\exists D\leqq B]\,.\,D\,\varepsilon_\infty\,\mathbf{cnf}\langle C\rangle\,.$
$D\,\varepsilon_\infty\,\mathbf{ingr}\langle\mathrm{Qntf}\langle B\rangle\rangle\,.\!\cdot\!.$
4) $[CDEFGH\leqq A+B]::F\,\varepsilon_\infty\,\mathrm{prntm}\langle\mathrm{Essnt}\langle A\rangle\rangle\,.\,G\,\varepsilon_\infty\,\mathrm{prntm}\langle\mathrm{Essnt}\langle B\rangle\rangle\,.$
$C\,\varepsilon_\infty\,\mathrm{Anarg}\langle DFG\rangle\,.E\,\varepsilon_\infty\,\mathrm{var}\langle HB\rangle\,.E\,\varepsilon_\infty\,\mathbf{ingr}\langle D\rangle\,.\supset:\,[\exists I\leqq A]:$
$I\,\varepsilon_\infty\,\mathbf{cnf}\langle E\rangle:I\,\varepsilon_\infty\,\mathrm{int}\langle\mathrm{Qntf}\langle A\rangle\rangle.\,\mathrm{v}\,.\,I\,\varepsilon_\infty\,\mathrm{int}\langle\mathrm{Qntf}\langle C\rangle\rangle.\!\cdot\!.$
5) $[CDEFG\leqq A+B]:F\,\varepsilon_\infty\,\mathrm{prntm}\langle\mathrm{Essnt}\langle A\rangle\rangle\,.\,G\,\varepsilon_\infty\,\mathrm{prntm}\langle\mathrm{Essnt}\langle B\rangle\rangle\,.$
$C\varepsilon_\infty\,\mathrm{Anarg}\langle DFG\rangle\,.\,E\,\varepsilon_\infty\,\mathrm{int}\langle\mathrm{Qntf}\langle D\rangle\rangle.\supset.\,[\exists H\leqq C]\,.\,H\,\varepsilon_\infty\,\mathbf{cnf}\langle E\rangle\,.$
$H\,\varepsilon_\infty\,\mathbf{ingr}\langle\mathrm{Qntf}\langle C\rangle\rangle\,.\!\cdot\!.$
6) $[CDEFG\leqq A+B].\!\cdot\!.\,F\,\varepsilon_\infty\mathrm{prntm}\langle\mathrm{Essnt}\langle A\rangle\rangle\,.\,G\,\varepsilon_\infty\,\mathrm{prntm}\langle\mathrm{Essnt}\langle B\rangle\rangle\,.$
$C\,\varepsilon_\infty\mathrm{Anarg}\langle DFG\rangle\,.\,E\,\varepsilon_\infty\,\mathrm{int}\langle\mathrm{Qntf}\langle C\rangle\rangle\,.\supset:\,[\exists H\leqq D]:\,H\,\varepsilon_\infty\,\mathbf{cnf}\langle E\rangle\,.$
$H\,\varepsilon_\infty\,\mathbf{ingr}\langle D\rangle\,.\,[\exists I\leqq B]\,.\,H\,\varepsilon_\infty\,\mathrm{var}\langle IB\rangle\,.\,\mathrm{v}\,.\,H\,\varepsilon_\infty\,\mathrm{int}\langle\mathrm{Qntf}\langle D\rangle\rangle\,.\!\cdot\!.$
7) $[CDEFGH\leqq A+B]:F\,\varepsilon_\infty\,\mathrm{prntm}\langle\mathrm{Essnt}\langle A\rangle\rangle\,.\,G\,\varepsilon_\infty\,\mathrm{prntm}\langle\mathrm{Essnt}\langle B\rangle\rangle\,.$
$C\,\varepsilon_\infty\,\mathrm{Anarg}\langle DFG\rangle\,.\,H\,\varepsilon_\infty\,\mathrm{int}\langle\mathrm{Qntf}\langle A\rangle\rangle\,.\,E\,\varepsilon_\infty\,\mathbf{cnf}\langle H\rangle\,.$
$E\,\varepsilon_\infty\,\mathbf{ingr}\langle\mathrm{Qntf}\langle C\rangle\rangle\,.\supset.\,[\exists I\leqq D]\,.\,I\,\varepsilon_\infty\,\mathbf{cnf}\langle E\rangle\,.\,I\,\varepsilon_\infty\,\mathbf{ingr}\langle\mathrm{Qntf}\langle D\rangle\rangle$

A is a *consequence by distribution* of the quantifier of *B*. The first two clauses of this explanation indicate that the result of distributing quantifiers through a biconditional only effects the quantifier of the biconditional and the quantifiers of its coimplicans and coimplicate. The next three clauses indicate that any variable in the quantifier of *A* previously occurs in the quantifier of *B*, every variable in *A* is bound by some quantifier, and any variable bound by a quantifier in *B* remains so bound in *A*. Clause 6 indicates that variables bound by interior quantifiers in *A* either were so bound in *B* or have become so bound by distribution of the quantifier of *B*. Finally, the last clause indicates that the result of distributing the quantifier of *B* shall not bind any variables already bound by interior quantifiers of *B*.

D3.46 $[ABC]:A\,\varepsilon_\infty\,\mathrm{cnsqeqvl}\langle BC\rangle\,.\!\equiv\!.\,C\,\varepsilon_\infty\,\mathbf{cnf}\langle\mathrm{Eqvl1}\langle B\rangle\rangle\,.$
$A\,\varepsilon_\infty\,\mathbf{cnf}\langle\mathrm{Eqvl2}\langle B\rangle\rangle$

A is a *consequence by detachment* from *B* and *C*. It is worth noting that detachment under quantifiers is not officially allowed in protothetic, though it can be justified as a derived rule of the system.

D3.47 $[ABB'CDE]\therefore A\,\varepsilon_\infty\,\mathsf{cnsqsbstp}<BB'CDE>.\equiv::$

1) $[F\leqq A].\because F\,\varepsilon_\infty\,\mathbf{ingr}<\mathsf{Essnt}<A>>.\,F\,\varepsilon_\infty\,\mathbf{vrb}.\supset:[\exists G\leqq C].$
$F\,\varepsilon_\infty\,\mathsf{assoc}<GED\,\mathsf{Essnt}<A>\,\mathsf{Essnt}<C>>.\mathsf{v}.[\exists nG\leqq A].\,F\,\varepsilon_\infty\,\mathbf{ingr}<G>.$
$G\,\varepsilon_\infty\,\mathsf{Occ}<nE\,\mathsf{Essnt}<A>>::$
2) $[F\leqq A].\because F\,\varepsilon_\infty\,\mathsf{int}<\mathsf{Sbqntf}<C>>.\supset.[\exists G\leqq A].$
$F\,\varepsilon_\infty\,\mathsf{assoc}<GDE\,\mathsf{Essnt}<C>\,\mathsf{Essnt}<A>>.\mathsf{v}.[\exists n\leqq \mathsf{L}<C>].$
$F\,\varepsilon_\infty\,\mathsf{Occ}<nD\,\mathsf{Essnt}<C>>::$
3) $D\,\varepsilon_\infty\,\mathsf{int}<\mathsf{Qntf}<C>>:$
4) $E\,\varepsilon_\infty\,\mathsf{trm}.\mathsf{v}.E\,\varepsilon_\infty\,\mathsf{gnrl}.\mathsf{v}.E\,\varepsilon_\infty\,\mathsf{fnct}:$
5) $[FG\leqq A+C]:F\,\varepsilon_\infty\,\mathsf{int}<\mathsf{Sbqntf}<C>>.$
$G\,\varepsilon_\infty\,\mathsf{assoc}<FED\,\mathsf{Essnt}<A>\,\mathsf{Essnt}<C>>.\supset.G\,\varepsilon_\infty\,\mathbf{cnf}<F>:$
6) $[FGHI\leqq C]::F\,\varepsilon_\infty\,\mathbf{ingr}<\mathsf{Essnt}<C>>.\,G\,\varepsilon_\infty\,\mathsf{int}<\mathsf{Qntf}<F>>.$
$H\,\varepsilon_\infty\,\mathsf{var}<IC>.\,H\,\varepsilon_\infty\,\mathbf{ingr}<F>.\supset.\because[JKLM\leqq A]:$
$J\,\varepsilon_\infty\,\mathsf{assoc}<GED\,\mathsf{Essnt}<A>\,\mathsf{Essnt}<C>>.$
$K\,\varepsilon_\infty\,\mathsf{assoc}<HED\,\mathsf{Essnt}<A>\,\mathsf{Essnt}<C>>.\mathsf{v}.[\exists n\leqq A].$
$K\,\varepsilon_\infty\,\mathsf{Occ}<nE\,\mathsf{Essnt}<A>>.\,M\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,L\,\varepsilon_\infty\,\mathsf{var}<JM>:\supset.$
$\sim(L\,\varepsilon_\infty\,\mathbf{ingr}<K>)::$
7) $[FG\leqq A+C].\because F\,\varepsilon_\infty\,\mathsf{int}<\mathsf{Qntf}<A>>.\,G\,\varepsilon_\infty\,\mathbf{cnf}<F>.\,G\,\varepsilon_\infty\,\mathbf{ingr}<C>.\supset:$
$[\exists H\leqq C].\,H\,\varepsilon_\infty\,\mathsf{qntf}.\,H\,\varepsilon_\infty\,\mathbf{ingr}<C>.\,G\,\varepsilon_\infty\,\mathsf{int}<H>.\mathsf{v}.[HI\leqq C].$
$H\,\varepsilon_\infty\,\mathbf{ingr}<C>.\,G\,\varepsilon_\infty\,\mathsf{var}<IH>::$
8) $B\,\varepsilon_\infty\,\mathbf{lst}::$
9) $[F\leqq A].\because F\,\varepsilon_\infty\,\mathsf{trm}.\,F\,\varepsilon_\infty\,\mathbf{ingr}<A>.\supset:[\exists G\leqq A].\,G\,\varepsilon_\infty\,\mathsf{qntf}.\,G\,\varepsilon_\infty\,\mathbf{ingr}<A>.$
$F\,\varepsilon_\infty\,\mathsf{int}<G>.\mathsf{v}.[\exists GH\leqq A].\,G\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,F\,\varepsilon_\infty\,\mathsf{var}<HG>.\mathsf{v}.$
$F\,\varepsilon_\infty\,\mathsf{constp}<BB'A>::$
10) $[FG\leqq A]:G\,\varepsilon_\infty\,\mathsf{qntf}.\,G\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,F\,\varepsilon_\infty\,\mathsf{int}<G>.\supset.[\exists HI\leqq A].$
$H\,\varepsilon_\infty\,\mathbf{ingr}<A>.I\,\varepsilon_\infty\,\mathsf{var}<FH>::$
11) $[FGH\leqq A].\because G\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,H\,\varepsilon_\infty\,\mathsf{cnvar}<FG>.\supset:H=_\infty F.\mathsf{v}.$
$[\exists IJ\leqq B'].\,H\,\varepsilon_\infty\,\mathsf{quasihomosemp}<FBB'AIJ>::$
12) $[F\leqq A]:F\,\varepsilon_\infty\,\mathsf{gnrl}.\,F\,\varepsilon_\infty\,\mathbf{ingr}<A>.\sim(F=_\infty A).\supset.[\exists GHIJ\leqq B'].$
$G\,\varepsilon_\infty\,\mathsf{homosemp}<BBB'>.\,H\,\varepsilon_\infty\mathbf{thp}<BB'>.I\,\varepsilon_\infty\,\mathbf{ingr}<H>.$
$J\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,G\,\varepsilon_\infty\,\mathsf{Anarg}<FIJ>::$
13) $[FG\leqq A].\because F\,\varepsilon_\infty\,\mathsf{gnrl}.\,F\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,G\,\varepsilon_\infty\,\mathsf{Essnt}<F>.\supset:G\,\varepsilon_\infty\,\mathsf{vrb}.\mathsf{v}.$
$[\exists H\leqq B'].\,H\,\varepsilon_\infty\,\mathsf{frp}<BB'>.\,G\,\varepsilon_\infty\,\mathsf{genfnct}<H>::$
14) $[F\leqq A].\because F\,\varepsilon_\infty\,\mathsf{fnct}.\,F\,\varepsilon_\infty\,\mathbf{ingr}<A>.\supset:F=_\infty A.\mathsf{v}.[\exists G\leqq A].\,G\,\varepsilon_\infty\,\mathsf{gnrl}.$
$G\,\varepsilon_\infty\,\mathbf{ingr}<A>.\,F\,\varepsilon_\infty\,\mathsf{Essnt}<G>.\mathsf{v}.[\exists GH\leqq B'].\,F\,\varepsilon_\infty\,\mathsf{fnctp}<BB'AGH>$

A is a *consequence by substitution* in C of E for D relative to B in B. The first, second, and fifth clauses of this explanation indicate that the nucleus of the result of making a substitution in C is, symbol by symbol, equiform to the nuclues of C, except where the substitution occurred. While clauses 3 and 4 indicate that substitution is made only for variables bound by the quantifier of C and only terms, generalizations or functions may be substituted for such variables.

Clauses 6 through 10 make certain stipulations about quantification.

No variable substituted into C is to be bound by some (previously present) interior quantifier of C. Variables in the quantifier of A, if they are equiform to words in C, must be equiform to variables in C. Every term in A is either a quantifier, bound by a quantifier, or is a suitable constant. Finally, no quantifier in A is vacuous.

The last four clauses of D3.47 guarantee that the result of substitution is a proposition (compare clauses 7 through 10 of D3.44). Thus, all equiform variables must belong to the same semantical category. Generalizations in A are propositional. The nucleus of any generalization in A is either a word or a generating function for a propositional phrase. And finally, any function in A is either identical with A, the nucleus of a generalization in A, or belongs to some determined semantical category.

This explanation differs from Leśniewski's in that the consequence by substitution in C is relativized to given expressions in itself and C (namely, E and D respectively), rather than to a general name. Thus, the first and second clauses given here do the work of Leśniewski's:

$$\mathrm{Essnt}(A)\,\varepsilon\,\mathrm{Cmpl}(a)$$

and

$$a \infty \mathrm{int}(\mathrm{Sbqntf}(C))$$

respectively, while his third and fourth clauses:

$$[D,E] \therefore D\,\varepsilon\,\mathrm{int}(\mathrm{Sbqntf}(C)) \,.\, E\,\varepsilon\, a \,.\, (a \cap \mathrm{prcd}(E)) \infty (\mathrm{int}(\mathrm{Sbqntf}(C)) \cap \mathrm{prcd}(D)) \,.\supset:$$
$$[\exists F] \,.\, D\,\varepsilon\,\mathrm{var}(F,C) \,.\vee .\, D\,\varepsilon\,\mathrm{cnf}(E)$$

and

$$[D,E] \therefore D\,\varepsilon\,\mathrm{int}(\mathrm{Sbqntf}(C)) .\, E\,\varepsilon\, a \,.\, (a \cap \mathrm{prcd}(E)) \infty (\mathrm{int}(\mathrm{Sbqntf}(C)) \cap \mathrm{prcd}(D)) \,.\supset:$$
$$E\,\varepsilon\,\mathrm{trm} \,.\vee .\, E\,\varepsilon\,\mathrm{gnrl} \,.\vee .\, E\,\varepsilon\,\mathrm{fnct} \,.\vee .\, E\,\varepsilon\,\mathrm{cnf}(D)$$

are replaced by the simpler third and fourth clauses above. Finally, in Leśniewski's fifth, sixth, and seventh clauses, the use of equinumerosity is avoided by employing 'assoc' as could be done for his third clause above bv giving:

$$[FG \leqq A + C] \therefore F\,\varepsilon_\infty\,\mathrm{int} < \mathrm{Sbqntf} < C >> .$$
$$G\,\varepsilon_\infty\,\mathrm{assoc} < FED\ \mathrm{Essnt} < A > \mathrm{Essnt} < C >> .\supset:$$
$$[\exists H \leqq C] .\, F\,\varepsilon_\infty\,\mathrm{var} < HC > .\vee .\, F\,\varepsilon_\infty\,\mathrm{cnf} < G >$$

where D and E are the parameters in question.

The only other difference to be noted in this explanation and Leśniewski's is the eighth clause of D3.47, which replaces:

$$B\,\varepsilon\,\mathrm{expr}$$

but which implies the above and is needed because of the addition of "B'" in this explanation. Hence, instead of Leśniewski's:

T.E.XLVIII $[A,B,C] : A\,\varepsilon\,\mathrm{cnsqsbstp}(B,C)$
$[\exists a] .\, A\,\varepsilon\,\mathrm{cnsqsbstp}(B,C,a)$

this exposition uses:

D3.48 $[ABB'C]:A\ \varepsilon_\infty \mathrm{cnsqsbstp}<BB'C>.\equiv.[\exists DE \leqq A+C].$
$A\ \varepsilon_\infty \mathrm{cnsqsbstp}<BB'CDE>$

A is a *consequence by substitution* in *C* relative to *B* in *B'*. It should be noted that Leśniewski's original explanation allows simultaneous substitution for one or more variables while the above does not.

D3.49 $[ABB']::A\ \varepsilon_\infty \mathrm{extnsnlp}<BB'>.\equiv::$

1) $[\exists CD \leqq A].C\ \varepsilon_\infty\ \mathrm{int}<\mathrm{Qntf}<A>>.D\ \varepsilon_\infty\ \mathrm{int}<\mathrm{Qntf}<A>>.$
$C\varepsilon_\infty \mathbf{prcd}<DA>.\therefore$
2) $[CD \leqq A]:D\ \varepsilon_\infty \mathrm{qntf}.D\ \varepsilon_\infty \mathbf{ingr}<A>.C\ \varepsilon_\infty\ \mathrm{int}<D>.\supset.[\exists EF \leqq A].$
$E\ \varepsilon_\infty \mathbf{ingr}<A>.F\ \varepsilon_\infty \mathrm{var}<CE>.\sim(F\ \varepsilon_\infty \mathbf{cnf}<\mathbf{Ingr}<1\ \mathrm{Essnt}<A>>>).\therefore$
3) $[\exists C \leqq A].\therefore C\ \varepsilon_\infty \mathrm{prntm}<\mathrm{Eqvl1}<\mathrm{Essnt}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>>>:[DE \leqq A]:$
$D\ \varepsilon_\infty \mathrm{prntm}<\mathrm{Sbqntf}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>>.$
$E\ \varepsilon_\infty \mathrm{prntm}<\mathrm{Sbqntf}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>>.\supset.D =_\infty E:[FG \leqq C]:$
$F\ \varepsilon_\infty\ \mathrm{int}<C>.G\ \varepsilon_\infty\ \mathrm{int}<C>.\supset.F =_\infty G.$
$\mathbf{Ingr}<1\ \mathrm{Eqvl1}<D>>\ \varepsilon_\infty \mathrm{cnvar}<FA>.\therefore$
4) $[\exists C \leqq A].\therefore C\ \varepsilon_\infty \mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>>>:$
$[DE \leqq C]:D\ \varepsilon_\infty\ \mathrm{int}<C>.E\ \varepsilon_\infty\ \mathrm{int}<C>.\supset.D =_\infty E.$
$\mathbf{Ingr}<1\ \mathrm{Eqvl2}<\mathrm{Essnt}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>>>\ \varepsilon_\infty \mathrm{cnvar}<DA>.\therefore$
5) $[C \leqq A].\therefore C\ \varepsilon_\infty \mathrm{fnct}.C\ \varepsilon_\infty \mathbf{ingr}<A>.\supset:[\exists D \leqq A].D\ \varepsilon_\infty \mathrm{gnrl}.$
$D\ \varepsilon_\infty \mathbf{ingr}<A>.C\ \varepsilon_\infty \mathrm{Essnt}<D>.\mathrm{v}.[\exists DE \leqq B'].$
$C\ \varepsilon_\infty \mathrm{fnctp}<BB'ADE>::$
6) $[CDEF \leqq A]:D\ \varepsilon_\infty \mathrm{prntm}<\mathrm{Eqvl1}<\mathrm{Essnt}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>>>.$
$E\ \varepsilon_\infty \mathrm{prntm}<\mathrm{Eqvl2}<\mathrm{Essnt}<\mathrm{Eqvl1}<\mathrm{Essnt}<A>>>>>.$
$F\ \varepsilon_\infty \mathrm{Anarg}<CDE>.\supset.F\ \varepsilon_\infty \mathrm{cnvar}<C\ \mathrm{Eqvl1}<\mathrm{Essnt}<A>>>::$
7) $[CDE \leqq A]:D\ \varepsilon_\infty \mathbf{ingr}<A>.E\ \varepsilon_\infty \mathrm{cnvar}<CD>.\supset.[\exists FG \leqq B'].$
$E\ \varepsilon_\infty \mathrm{quasihomosemp}<CBB'AFG>.\therefore$
8) $[CD \leqq A]:D\ \varepsilon_\infty \mathrm{cnvar}<C\ \mathrm{Eqvl1}<\mathrm{Essnt}<A>>>.\supset.[\exists EF \leqq A].$
$E\ \varepsilon_\infty \mathbf{ingr}<A>.F\ \varepsilon_\infty \mathbf{ingr}<A>.D\ \varepsilon_\infty \mathrm{Anarg}<CEF>.\therefore$
9) $[CDE \leqq A].\therefore C\ \varepsilon_\infty \mathrm{prntm}<\mathrm{Essnt}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>>.D\ \varepsilon_\infty \mathrm{arg}<C>.$
$E\ \varepsilon_\infty \mathrm{Sqnfnct}<D>.\supset:[FG \leqq A].F\ \varepsilon_\infty\ \mathrm{int}<\mathrm{Qntf}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>>.$
$G\ \varepsilon_\infty\ \mathrm{int}<\mathrm{Qntf}<\mathrm{Eqvl2}<\mathrm{Essnt}<A>>>>.\supset.F =_\infty G.$
$E\ \varepsilon_\infty \mathrm{var}<F\ \mathrm{Eqvl2}<\mathrm{Essnt}<A>>>$

A is a *protothetical thesis of extensionality* relative to *B* in *B'*. The first two clauses of this explanation indicate that there are at least two variables in the quantifier of *A*, that no quantifiers in *A* are vacuous, and that the first word in the nucleus of *A* is not a variable.

The remaining clauses indicate that *A* is a generalization of some coimplication, while both the coimplicans and coimplicate of the nucleus of *A* are themselves generalizations of coimplications. Let us call the coimplicans of the nucleus of *A* the basis of *A*, and the coimplicate of the nucleus of *A* the extension of *A*. Then, clauses 3 and 4 indicate that the coimplicans (coimplicate) of the nucleus of the extension of *A* has a single argument, which is an equiform variable with the first word in the coimplicans (coimplicate) of the nucleus of the basis of *A*. Indeed, these are the variables bound by the quantifier of *A*.

Clause 5 through 8 indicate that, first of all, any function in A is either the nucleus of some generalization or belongs to some determined semantical category. And lastly, analogous arguments of the coimplicans and coimplicate of the nucleus of the basis are equiform variables, and any equiform variables in the basis are analogous arguments of the basis.

Finally, the last clause indicates that the quantifier of the extension of A has only one variable and it binds the functor of the coimplicans (coimplicate) of the nucleus of the extension of A.

This explanation differs significantly from Leśniewski's only in its third, fourth, and last clauses. Where for example Leśniewski gives:

$$[\exists C].\, C \,\varepsilon\, \mathrm{prntm}(\mathrm{Eqvl1}(\mathrm{Essnt}(\mathrm{Eqvl2}(\mathrm{Essnt}(A))))) \,.$$
$$1\, \mathrm{ingr}(\mathrm{Eqvl1}(\mathrm{Cmpl}(\mathrm{int}(\mathrm{Sbqntf}(\mathrm{Eqvl1})\mathrm{Essnt}(A)))))) \,\varepsilon\, \mathrm{cnvar}(\mathrm{Cmpl}(\mathrm{int}(C)), A)$$

the third clause of D3.49 is given, and the use of 'Cmpl' is avoided in the the fourth and last clauses in a similar manner. With *T.E.XLIX* stated, Leśniewski in [5] writes:

> Unter der Voraussetzung, dass eine These A die letzte der Thesen ist, die schon zu dem System gehören, darf man zu ihm als neue These einen Ausdruck B nur in dem Fall hinzufügen, wenn wenigstens eine der fünf folgenden Bedingugnen erfüllt ist:
> 1) $B \,\varepsilon\, \mathrm{defp}(A)$
> 2) $[\exists C].\, C \,\varepsilon\, \mathrm{thp}(A) \,.\, B \varepsilon\, \mathrm{cnsqrprtntf}(C)$
> 3) $[\exists C,D].\, C \,\varepsilon\, \mathrm{thp}(A) \,.\, D \,\varepsilon\, \mathrm{thp}(A) \,.\, B \varepsilon\, \mathrm{cnsqeqvl}(C,D)$
> 4) $[\exists C].\, C \,\varepsilon\, \mathrm{thp}(A) \,.\, B \varepsilon\, \mathrm{cnsqsbstp}(A,C)$
> 5) $B \,\varepsilon\, \mathrm{extnsnlp}(A)$

Following this exposition, however, protothetic is legitimately developed under the following single rule.

> Supposing that a thesis A is the last thesis which already belongs to a list of theses A' of the system, then an expression B may be added as a new thesis only in case at least one of the following given conditions is fulfilled:
> 1) $B \,\varepsilon_\infty\, \mathrm{defp} \langle AA' \rangle$
> 2) $[\exists C \leqq A'].\, C \,\varepsilon_\infty\, \mathbf{thp} \langle AA' \rangle \,.\, B \,\varepsilon_\infty\, \mathrm{cnsqrprtntf} \langle C \rangle$
> 3) $[\exists CD \leqq A'].\, C \,\varepsilon_\infty\, \mathbf{thp} \langle AA' \rangle \,.\, D \,\varepsilon_\infty\, \mathbf{thp} \langle AA' \rangle \,.\, B \varepsilon_\infty\, \mathrm{cnsqeqvl} \langle CD \rangle$
> 4) $[\exists C \leqq A'].\, C \varepsilon_\infty\, \mathbf{thp} \langle AA' \rangle \,.\, B \varepsilon_\infty\, \mathrm{cnsqsbstp} \langle AC \rangle$
> 5) $B \varepsilon_\infty\, \mathrm{extnsnlp} \langle AA' \rangle$

Thus, if one wishes to give an exposition of protothetic, he asserts an adequate axiom and then chooses which theses he will next assert—his choice continually guided by the above effective rule. Clearly then, Leśniewski ultimately understands a systematic as an individual expression capable of being extended according to the choice of an author guided by directives which are adequate for any stage of the development of the systematic.

In this work it is enough to realize that the guiding directives are primitive recursive at any stage in the development of their systematic even though their significance is dependent upon the extent of the development. And just as Leśniewski would call theses only those things which are individual expressions of a given exposition, in this work something is a thesis if and only if there is a proof of it, where a proof of an individual expression is a list of expressions, each of which is either the axiom of protothetic or is a thesis of protothetic relative to a given expression in the list. Thus, the concept of thesis is no longer primitive recursive, and this is of importance (particularly) for the last section of this paper.

4. *The rule of onotology* The terminological explanations given in this section follow Leśniewski's [6] and are numbered identically to them. Ontology is based on a given development of protothetic. Any expression in such a development is called an effective thesis of protothetic (efthp). The axiom of onotlogy (assigned the number Axo under the assignment of numbers of this exposition) is the first thesis proper to ontology and the terminological explanations for ontology follow below.[3]

D4.32 $[ABC]::A\ \varepsilon_\infty$ **tho** $<BC>.\equiv.\because.A\ \varepsilon_\infty$ **thp** $<BC>.\vee.[\exists DE\leqq C]:$
$C=_\infty D*E.E\varepsilon_\infty$ **lst**. Axo ε_∞ gl $<1E>.[\exists nm\leqq$ L $<E>].$
$A=_\infty$ gl $<nE>.B=_\infty$ gl $<mE>.n\leqq m$

A is a *thesis of ontology* relative to B in a list C (assuming every expression in C is a thesis of ontology or an effective thesis of protothetic). Thus, this explanation, with minor exceptions, parallels Leśniewski's

$T.E.XXXII^\circ$ $[A,B].\because.A\ \varepsilon\ \mathrm{tho}(B).=:A\ \varepsilon\ \mathrm{efthp}.\vee.A\ \varepsilon\ \mathrm{tho}:$
$B\ \varepsilon\ \mathrm{tho}:$
$A\ \varepsilon\ \mathrm{prcd}(B).\vee.A\ \varepsilon\,\mathrm{Id}(B)$

In what follows many of the terminological explanations for ontology differ from those for protothetic only by relying on D4.32 instead of D2.32. When that is the case they shall be given in an abbreviated form. For instance:

D4.33 $[ABB']:A\ \varepsilon_\infty$ fro $<BB'>.\equiv.$ [D2.33 : **thp/tho**]

is, in unabbreviated form:

$[ABB'].\because.A\ \varepsilon_\infty$ fro $<BB'>.\equiv:A\ \varepsilon_\infty$ **tho** $<BB'>.\vee.[\exists CD\leqq B'].$
$C\ \varepsilon_\infty$ **tho** $<BB'>.D\ \varepsilon_\infty$ **ingr** $<C>.A\varepsilon_\infty$ Arg1 $<D>.\vee.[\exists CD\leqq B'].$
$C\ \varepsilon_\infty$ **tho** $<BB'>.D\ \varepsilon_\infty$ **ingr** $<C>.A\ \varepsilon_\infty$ Agr2 $<D>.\vee.[\exists CD\leqq B'].$
$C\ \varepsilon_\infty$ **tho** $<BB'>.D\ \varepsilon_\infty$ sbqntf $.D\ \varepsilon_\infty$ **ingr** $<C>.D=_\infty$ vrb3 $*A*$ vrb4

which differs from D2.33 only by having '**thp**' replaced by '**tho**' throughout. Hence, one may refer to sections 2 and 3 for the relevant discussions of many of the following explanations.

D4.34a $[ABCC']:A\ \varepsilon_\infty$ homosemo $<0BCC'>.\equiv.$ [D2.34a : frp/fro, **thp/tho**]
D4.34b $[ABCC'n]:A\ \varepsilon_\infty$ homosemp $<n+1BCC'>.\equiv.$ [D2.34b : homosemp/homosemo]
D4.35 $[ABCC']:A\ \varepsilon_\infty$ homosemo $<BCC'>.\equiv.$ [D2.35 homosemp/homosemo]
D4.36 $[ABB'CDE]:A\ \varepsilon_\infty$ consto $<BB'CDE>.\equiv.$ [D2.36 :
homosemp/homosemo, **thp/tho**]

D4.37 $[ABB'C]: A\,\varepsilon_\infty\,\mathsf{consto}\langle BB'C\rangle .\equiv. [\mathrm{D2.37}: \mathsf{constp/consto}]$

D4.38 $[ABCC'DEF]: A\,\varepsilon_\infty\,\mathsf{quasihomosemo}\langle BCC'DEF\rangle .\equiv.$
$[\mathrm{D2.38}: \mathsf{homosemp/homosemo}, \mathbf{thp/tho}]$

D4.39 $[ABB'CDE]: A\,\varepsilon_\infty\,\mathsf{fncto}\langle BB'CDE\rangle .\equiv. [\mathrm{D2.39}:$
$\mathsf{homosemp/homosemo}, \mathbf{thp/tho}]$

D4.40 $[ABCC'DEF]: A\,\varepsilon_\infty\,\mathsf{varo}\langle BCC'DEF\rangle .\equiv. [\mathrm{D2.40}:$
$\mathsf{homosemp/homosemo}, \mathbf{thp/tho}]$

D4.41 $[ABB'CDE]: A\,\varepsilon_\infty\,\mathsf{propprntmo}\langle BB'CDE\rangle .\equiv. [\mathrm{D2.41}:$
$\mathsf{homosemp/homosemo}, \mathsf{varp/varo}]$

D4.42 $[ABB'CDE]: A\,\varepsilon_\infty\,\mathsf{1propprntmo}\langle BB'CDE\rangle .\equiv. [\mathrm{D2.42}: \mathsf{prntmp/propprntmo}]$

D4.43 $[ABB'CDEFG]: A\,\varepsilon_\infty\,\mathsf{2propprntmo}\langle BB'CDEFG\rangle .\equiv.$
$[\mathrm{D2.43}: \mathsf{prntmp/propprntmo}]$

D4.44 $[ABB']: A\,\varepsilon_\infty\,\mathsf{1defo}\langle BB'\,] .\equiv. [\mathrm{D3.44}:$
$\mathsf{constp/consto}, \mathsf{quasihomosemp/quasihomosemo}, \mathbf{thp/tho}, \mathsf{frp/fro}, \mathsf{fnctp/fncto},$
$\mathsf{1prntmp/1propprntmo}, \mathsf{2prntmp/2propprntmo}]$

Notice that '1defo' is merely 'defp' adjusted to ontology. Since the explanations for distribution of quantifiers and detachment require no adjustment to ontology they have no proper counterparts in the ontological explanations and the enumeration skips to:

D4.47 $[ABB'CDE]: A\,\varepsilon_\infty\,\mathsf{cnsqsbsto}\langle BB'CDE\rangle .\equiv. [\mathrm{D3.47}:$
$\mathsf{constp/consto}, \mathsf{quasihomosemp/quasihomosemo}, \mathbf{thp/tho}, \mathsf{frp/fro}, \mathsf{fnctp/fncto}]$

D4.48 $[ABB'C]: A\,\varepsilon_\infty\,\mathsf{cnsqsbsto}\langle BB'C\rangle .\equiv. [\mathrm{D3.48}: \mathsf{cnsqsbstp/cnsqsbsto}]$

Thus, 'cnsqsbsto' is merely 'cnsqsbstp' adjusted to ontology.

D4.49 $[ABB']: A\,\varepsilon_\infty\,\mathsf{1extnsnlo}\langle BB'\rangle .\equiv. [\mathrm{D3.49}:$
$\mathsf{fnctp/fncto}, \mathsf{quasihomosemp/quasihomosemo}]$

Here, '1extnsnlo' is merely 'extnsnlp' adjusted to ontology. At this point all of the protothetical explanations have been adjusted so that they are applicable to ontology. This section concludes by giving those explanations which are proper to ontology. In particular the explanations for ontological definitions and extensionality will be given.

D4.50 $[AB] .\because. A\,\varepsilon_\infty\,\mathsf{cnjnct}\langle B\rangle .\equiv: \mathsf{Sgnfnct}\langle B\rangle\,\varepsilon_\infty\,\mathbf{cnf}\langle \mathbf{Ingr}\langle 21\,\mathsf{Axo}\rangle\rangle :$
$[\exists C \leqq B]: C\,\varepsilon_\infty\,\mathsf{prntm}\langle B\rangle : A\,\varepsilon_\infty\,\mathsf{Arg1}\langle C\rangle . \vee . A\,\varepsilon_\infty\,\mathsf{Arg2}\langle C\rangle$

A is a *conjunct* of *B*. This explanation fixes the shape of the symbol to be used for conjunction of propositions.

D4.51 $[AB]: A\,\varepsilon_\infty\,\mathsf{Sbjct}\langle B\rangle .\equiv. \mathsf{Sgnfnct}\langle B\rangle\,\varepsilon_\infty\,\mathbf{cnf}\langle \mathbf{Ingr}\langle 8, \mathsf{Axo}\rangle\rangle .$
$[\exists CD \leqq B + \mathsf{Axo}] . C\,\varepsilon_\infty\,\mathsf{prntm}\langle B\rangle . D\,\varepsilon_\infty\,\mathbf{ingr}\langle \mathsf{Axo}\rangle .$
$A\,\varepsilon_\infty\,\mathsf{Anarg}\langle \mathbf{Ingr}\langle 10\,\mathsf{Axo}\rangle CD\rangle$

A is the *subject* of *B*. This explanation fixes the shape of the primitive symbol of ontology which is used to form a proposition from two name arguments—as well as indicating that the first argument of its parentheme is to be called the subject of the proposition.

D4.52 $[AB]: A \,\varepsilon_\infty\, \mathrm{Prdct} < B > . \equiv . \mathrm{Sgnfnct} < B > \varepsilon_\infty \mathbf{cnf} < \mathbf{Ingr} < 8\,\mathrm{Axo} >> .$
$[\exists CD \leqq B + \mathrm{Axo}] . C \,\varepsilon_\infty\, \mathrm{prntm} < B > . D \,\varepsilon_\infty\, \mathbf{ingr} < \mathrm{Axo} > .$
$A \,\varepsilon_\infty\, \mathrm{Anarg} < \mathbf{ingr} < 11\,\mathrm{Axo} > CD >$

A is *the predicate* of B.

D4.53 $[ABB'CDE] :: A \,\varepsilon_\infty\, \mathrm{nomprntmo} < BB'CDE > . \equiv . \therefore$
1) $D \,\varepsilon_\infty\, \mathrm{homosemo} < \mathbf{Ingr} < 10\,\mathrm{Axo} > BB' > .$
2) $E \,\varepsilon_\infty\, \mathrm{prntm} < D > .$
3) $A \,\varepsilon_\infty\, \mathrm{prntm} < \mathrm{Prdct} < \mathrm{Eqvl2} < \mathrm{Essnt} < C >>>> :$
4) $[n \leqq \mathrm{L} < A > + \mathrm{L} < E >] : \mathrm{Uarg} < A > \varepsilon_\infty \mathrm{arg} < nA > . \equiv .$
$\mathrm{Uarg} < E > \varepsilon_\infty \mathrm{arg} < nE > . \therefore$
5) $[nFG \leqq A + E] . \therefore F \,\varepsilon_\infty\, \mathrm{arg} < nA > . \equiv . G \,\varepsilon_\infty\, \mathrm{arg} < nE > : \supset .$
$[\exists HI \leqq B'] . F \,\varepsilon_\infty\, \mathrm{varo} < GBB'CHI >$

A in C is *suited to be similar to the nominative parentheme* E of the semantical category of D relative to B in B'.

D4.54 $[ABB'CDE] : A \,\varepsilon_\infty\, \mathrm{1nomprntmo} < BB'CDE > . \equiv . A \,\varepsilon_\infty\, \mathrm{nomprntmo} < BB'CDE > .$
$\mathbf{Uingr} < D > \varepsilon_\infty \mathbf{ingr} < E >$

A in C is *suited to be similar to the last nominative parentheme* E of the semantical category of D relative to B in B'.

D4.55 $[ABB'CDEFG] : A \,\varepsilon_\infty\, \mathrm{2nomprntmo} < BB'CDEFG > . \equiv .$
$A \,\varepsilon_\infty\, \mathrm{nomprntmo} < BB'CDE > . F \,\varepsilon_\infty\, \mathrm{prntm} < D > .$
$\mathbf{Uprcd} < FD > \varepsilon_\infty \mathbf{ingr} < E > . G \,\varepsilon_\infty\, \mathrm{simprntm} < F >$

A in C is *suited to be similar to the nominative parentheme* E of D immediately preceding an F similar to G relative to B in B'. This ends the preliminary explanations proper to ontology and it is now possible to explain the two proper parts of the rule of ontology.

D4.56 $[ABB'] :: A \,\varepsilon_\infty\, \mathrm{2defo} < BB' > . \equiv . \therefore$
1) $\sim(\mathbf{Ingr} < 1\,\mathrm{Essnt} < A >> \varepsilon_\infty \mathrm{cnvar} < \mathbf{Ingr} < 1\,\mathrm{Essnt} < A >> A >) .$
2) $\sim(\mathbf{Ingr} < 1\,\mathrm{Eqvl1} < \mathrm{Essnt} < A >>> \varepsilon_\infty \mathrm{cnvar} < \mathbf{Ingr} <$
$1\,\mathrm{Eqvl1} < \mathrm{Essnt} < A >>> A >) .$
3) $\sim(\mathbf{Ingr} < 1\,\mathrm{Eqvl2} < \mathrm{Essnt} < A >>> \varepsilon_\infty \mathrm{cnvar} < \mathbf{Ingr} <$
$1\,\mathrm{Eqvl2} < \mathrm{Essnt} < A >>> A >) .$
4) $\sim(\mathbf{Ingr} < 1\,\mathrm{Prdct} < \mathrm{Eqvl2} < \mathrm{Essnt} < A >>>> \varepsilon_\infty \mathrm{cnvar} <$
$\mathbf{Ingr} < 1\,\mathrm{Prdct} < \mathrm{Eqvl2} < \mathrm{Essnt} < A >>>> A >) .$
5) $\sim(\mathbf{Ingr} < 1\,\mathrm{Prdct} < \mathrm{Eqvl2} < \mathrm{Essnt} < A >>>> \varepsilon_\infty \mathrm{consto} < BB'A > . \therefore$
6) $[C \leqq A] . \therefore C \,\varepsilon_\infty\, \mathrm{trm} . C \,\varepsilon_\infty\, \mathbf{ingr} < \mathrm{Eqvl1} < \mathrm{Essnt} < A >>> . \supset : [\exists D \leqq A] .$
$D \,\varepsilon_\infty\, \mathrm{qntf} . D \,\varepsilon_\infty\, \mathbf{ingr} < A > . C \,\varepsilon_\infty\, \mathrm{int} < D > . \mathrm{v} . [\exists DE \leqq A] . D \,\varepsilon_\infty\, \mathbf{ingr} < A > .$
$C \,\varepsilon_\infty\, \mathrm{var} < ED > . \mathrm{v} . C \,\varepsilon_\infty\, \mathrm{consto} < BB'A > . \therefore$
7) $[CD \leqq A] : D \,\varepsilon_\infty\, \mathrm{qntf} . D \,\varepsilon_\infty\, \mathbf{ingr} < A > . C \,\varepsilon_\infty\, \mathrm{int} < D > . \supset . [\exists EF \leqq A] .$
$E \,\varepsilon_\infty\, \mathbf{ingr} < A > . F \,\varepsilon_\infty\, \mathrm{var} < CE > :$
8) $[CDE \leqq A] : C \,\varepsilon_\infty\, \mathrm{int} < \mathrm{Qntf} < A >> . E \,\varepsilon_\infty\, \mathrm{prntm} < \mathrm{Essnt} < A >> .$
$D \,\varepsilon_\infty\, \mathrm{arg} < E > . \supset . [\exists F \leqq A] . F \,\varepsilon_\infty\, \mathbf{ingr} < D > . F \,\varepsilon_\infty\, \mathrm{var} < CA > . \therefore$

9) $[CDE \leqq A] .\because. C \varepsilon_{\infty} \mathbf{ingr} < \mathsf{Eqvl1} < \mathsf{Essnt} < A >>> . E \varepsilon_{\infty} \mathbf{ingr} < A > .$
$D \varepsilon_{\infty} \mathsf{cnvar} < CE > . D \varepsilon_{\infty} \mathbf{ingr} < \mathsf{Eqvl1} < \mathsf{Essnt} < A >>> . \supset : D =_{\infty} C . \mathsf{v} .$
$[\exists FG \leqq B'] . D \varepsilon_{\infty} \mathsf{quasihomosemo} < CBB'AFG > .\because.$

10) $[C \leqq A] : C \varepsilon_{\infty} \mathsf{gnrl} . C \varepsilon_{\infty} \mathbf{ingr} < A > . \sim (C =_{\infty} A) . \supset . [\exists DEFG \leqq B'] .$
$D \varepsilon_{\infty} \mathsf{homosemo} < BBB' > . E \varepsilon_{\infty} \mathbf{tho} < BB' > . F \varepsilon_{\infty} \mathbf{ingr} < E > .$
$F \varepsilon_{\infty} \mathbf{ingr} < E > . G \varepsilon_{\infty} \mathbf{ingr} < A > . D \varepsilon_{\infty} \mathsf{Anarg} < CFG > .\because.$

11) $[CD \leqq A] .\because. C \varepsilon_{\infty} \mathsf{gnrl} . C \varepsilon_{\infty} \mathbf{ingr} < A > . D \varepsilon_{\infty} \mathsf{Essnt} < A > . \supset : D \varepsilon_{\infty} \mathbf{vrb} . \mathsf{v} .$
$[\exists E \leqq B'] . E \varepsilon_{\infty} \mathsf{fro} < BB' > . D \varepsilon_{\infty} \mathsf{genfnct} < E > .\because.$

12) $[C \leqq A] . C \varepsilon_{\infty} \mathsf{fnct} . C \varepsilon_{\infty} \mathbf{ingr} < \mathsf{Eqvl1} < \mathsf{Essnt} < A >>> . \supset : [\exists D \leqq A] .$
$D \varepsilon_{\infty} \mathsf{gnrl} . D \varepsilon_{\infty} \mathbf{ingr} < A > . C \varepsilon_{\infty} \mathsf{Essnt} < D > . \mathsf{v} . [\exists DE \leqq B'] .$
$C \varepsilon_{\infty} \mathsf{fncto} < BB'ADE > .\because.$

13) $[\exists C \leqq A] . C \varepsilon_{\infty} \mathsf{Eqvl1} < \mathsf{Essnt} < A >> . \mathsf{v} . C \varepsilon_{\infty} \mathsf{cnjnct} < \mathsf{Eqvl1} < \mathsf{Essnt} < A >>> :$
$\mathsf{Sbjct} < C > \varepsilon_{\infty} \mathsf{cnvar} < \mathsf{Sbjct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>> A > :$

14) $[C \leqq A] : C \varepsilon_{\infty} \mathsf{prntm} < \mathsf{Prdct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>> . \supset .$
$[\exists D \leqq C] . D \varepsilon_{\infty} \mathsf{arg} < C > :$

15) $[CD \leqq A] : C \varepsilon_{\infty} \mathsf{prntm} < \mathsf{Prdct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>> . D \varepsilon_{\infty} \mathsf{arg} < C > . \supset .$
$[\exists E \leqq A] . D \varepsilon_{\infty} \mathsf{var} < EA > :$

16) $[CD \leqq A] : C \varepsilon_{\infty} \mathsf{trm} . C \varepsilon_{\infty} \mathbf{ingr} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>> .$
$\sim (C \varepsilon_{\infty} \mathbf{Ingr} < 1\ \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>>) . D \varepsilon_{\infty} \mathsf{trm} .$
$D \varepsilon_{\infty} \mathbf{ingr} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>> . \sim (D \varepsilon_{\infty} \mathbf{Ingr} < 1\ \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>) .$
$C \varepsilon_{\infty} \mathbf{cnf} < D > . \supset . C =_{\infty} D :$

17) $[CD \leqq A] : C \varepsilon_{\infty} \mathsf{prntm} < \mathsf{Prdct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>> .$
$D \varepsilon_{\infty} \mathsf{prntm} < \mathsf{Prdct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>> . C \varepsilon_{\infty} \mathsf{simprntm} < D > . \supset . C =_{\infty} D :$

18) $[CDE \leqq B'] . C \varepsilon_{\infty} \mathsf{1nomprntmo} < BB'ADE > .$
$\mathbf{Uingr} < \mathsf{Prdct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>> \varepsilon_{\infty} \mathbf{ingr} < C > . \supset .$
$C \varepsilon_{\infty} \mathsf{simprntm} < E > :$

19) $[CDEFG \leqq B'] : C \varepsilon_{\infty} \mathsf{2nomprntmo} < BB'ADEFG > . G \varepsilon_{\infty} \mathbf{ingr} < A > .$
$\mathbf{Uprcd} < GA > \varepsilon_{\infty} \mathbf{ingr} < C > . \supset . C \varepsilon_{\infty} \mathsf{simprntm} < E > :$

20) $[CDE \leqq B'] : C \varepsilon_{\infty} \mathsf{prntm} < \mathsf{Prdct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>> .$
$\mathbf{Uingr} < \mathsf{Prdct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>> \varepsilon_{\infty} \mathbf{ingr} < C > .$
$D \varepsilon_{\infty} \mathbf{tho} < BB' > . E \varepsilon_{\infty} \mathbf{ingr} < D > . C \varepsilon_{\infty} \mathsf{simprntm} < E > . \supset .$
$[\exists FG \leqq B'] . C \varepsilon_{\infty} \mathsf{1nomprntm} < BB'AFG > :$

21) $[CDEF \leqq B'] : C \varepsilon_{\infty} \mathsf{prntm} < \mathsf{Prdct} < \mathsf{Eqvl2} < \mathsf{Essnt} < A >>>> . D \varepsilon_{\infty} \mathsf{prntm} .$
$D \varepsilon_{\infty} \mathbf{ingr} < A > . \mathbf{Uprcd} < DA > \varepsilon_{\infty} \mathbf{ingr} < C > . E \varepsilon_{\infty} \mathbf{tho} < BB' > .$
$F \varepsilon_{\infty} \mathbf{ingr} < E > . C \varepsilon_{\infty} \mathsf{simprntm} < F > . \supset . [\exists GHI \leqq B'] .$
$C \varepsilon_{\infty} \mathsf{2promprntmo} < BB'AGHID >$

A is a *nominative definition* relative to B in B'. In discussing a nominative definition A, the coimplicans of the nucleus of A shall be called the definiens of A and the predicate of the coimplicate of the nucleus of A shall be called the definiendum of A. Thus, the first five clauses in the above explanation indicate that nominative definitions are generalizations of coimplications where the first word of the definiendum is neither a variable nor a previously defined or primitive symbol—it is in fact the constant which is being defined. The second clause together with the thirteenth indicates that the definiens either has a subject term itself or is a conjunction which has a subject term, where this term is an equiform variable with the subject of the definiendum. Incidentally, this last requirement could be relaxed whenever one is certain that the definiens actually adopted implies the officially required definiens.

Clauses 6 through 12 make certain stipulations about the binding of variables in nominative definitions and the structure of the definiens. These clauses are in fact exactly analogous to clauses 4 through 10 for protothetical definitions. Thus, as before, any terms occurring in the definiens are either variables in quantifiers, variables bound by quantifiers, or constants which already have a fixed semantical category. Any of the quantifiers which occur in a definition (either as the quantifier of the definition or merely as quantifiers in the definition) are not vacuous. Equiform variables in the definiens must belong to the same semantical category. Any generalizations must be propositional and their nuclei must either be single words or generating functions for some determined semantical category. Finally, any function in the definiens is either the nucleus of a generalization or belongs to some determined semantical category.

Finally, the remaining clauses of D4.56 deal with the definiendum and differ insignificantly from the concluding clauses of the explanation for protothetical definition. As before, it is stipulated that parenthemes in the definiendum are unempty and contain only variables, while there is no duplication of terms (and hence variables) nor of similar parenthemes. While the last four clauses taken together stipulate that each parentheme occurring in the definiendum can be assigned unambiguously a suitable semantical category.

D4.57 $[ABB']:: A\,\varepsilon_\infty\,\mathrm{2extnsnlo} < BB' > .\equiv.\because$

1) $[\exists CD \leqq A]\,.\,C\,\varepsilon_\infty\,\mathrm{int} < \mathrm{Qntf} < A >> .\,D\,\varepsilon_\infty\,\mathrm{int} < \mathrm{Qntf} < A >> .\,C\,\varepsilon_\infty\,\mathbf{prcd} < DA >:$
2) $[CD \leqq A]: D\,\varepsilon_\infty\,\mathrm{qntf}\,.\,D\,\varepsilon_\infty\,\mathbf{ingr} < A > .\,C\varepsilon_\infty\,\mathrm{int} < D > .\supset.\,[\exists EF \leqq A].$
$E\,\varepsilon_\infty\,\mathbf{ingr} < A > .\,F\,\varepsilon_\infty\,\mathrm{var} < CE > .\sim(F\,\varepsilon_\infty\,\mathbf{cnf} < \mathbf{Ingr} < 1\,\mathrm{Essnt} < A >>>)\,\because$
3) $\sim(\mathbf{Ingr} < 1\,\mathrm{Eqvl}\,1 < \mathrm{Essnt} < \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>>> \varepsilon_\infty\,\mathrm{cnvar} < \mathbf{Ingr} <$
$1\,\mathrm{Eqvl}1 < \mathrm{Essnt} < \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>>>, A >)\,\because$
4) $[\exists C \leqq A]\,C\,\varepsilon_\infty\,\mathrm{prntm} < \mathrm{Eqvl}1 < \mathrm{Essnt} < \mathrm{Eqvl}2 < \mathrm{Essnt} < A >>>>> .$
$[DE \leqq C]: D\,\varepsilon_\infty\,\mathrm{int} < C > .\,E\,\varepsilon_\infty\,\mathrm{int} < C > .\supset.\,D =_\infty E\,.$
$\mathbf{Ingr} < 1\,\mathrm{Prdct} < \mathrm{Eqvl}1 < \mathrm{Essnt} < \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>>>> \varepsilon_\infty\,\mathrm{cnvar} < DA >\,\because$
5) $[\exists C \leqq A]\,\because\,C\,\varepsilon_\infty\,\mathrm{prntm} < \mathrm{Eqvl}2 < \mathrm{Essnt} < \mathrm{Eqvl}2 < \mathrm{Essnt} < A >>>>>:$
$[DE \leqq C]: D\,\varepsilon_\infty\,\mathrm{int} < C > .\,E\,\varepsilon_\infty\,\mathrm{int} < C > .\supset.\,D =_\infty E\,.$
$\mathbf{Ingr} < 1\,\mathrm{Prdct} < \mathrm{Eqvl}2 < \mathrm{Essnt} < \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>>>> \varepsilon_\infty\mathrm{cnvar} < DA >\,\because$
6) $[C \leqq A]\,\because\,C\,\varepsilon_\infty\,\mathrm{fnct}\,.\,C\,\varepsilon_\infty\,\mathbf{ingr} < A > .\supset: [\exists D \leqq A]\,.\,D\,\varepsilon_\infty\,\mathrm{gnrl}\,.$
$D\,\varepsilon_\infty\,\mathbf{ingr} < A > .\,C\,\varepsilon_\infty\,\mathrm{Essnt} < D > .\,\mathrm{v}\,: [\exists DE \leqq B']\,.\,C\,\varepsilon_\infty\,\mathrm{fncto} < BB'ADE >\,\because$
7) $\mathrm{Sbjct} < \mathrm{Eqvl}1 < \mathrm{Essnt} < \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>>> \varepsilon_\infty\,\mathrm{cnvar} < \mathrm{Sbjct} < \mathrm{Eqvl}2 <$
$\mathrm{Essnt} < \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>>> \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>:$
8) $[CDEF \leqq A]: D\,\varepsilon_\infty\,\mathrm{prntm} < \mathrm{Prdct} < \mathrm{Eqvl}1 < \mathrm{Essnt} < \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>>>> .$
$E\,\varepsilon_\infty\,\mathrm{prntm} < \mathrm{Prdct} < \mathrm{Eqvl}2 > \mathrm{Essnt} < \mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>>>> .$
$F\,\varepsilon_\infty\,\mathrm{Anarg} < CDE > .\supset.\,F\,\varepsilon_\infty\mathrm{cnvar} < C\,\mathrm{Eqvl}1 < \mathrm{Essnt} < A >>>:$
9) $[CDE \leqq A]: D\,\varepsilon_\infty\,\mathbf{ingr} < A > .\,E\,\varepsilon_\infty\mathrm{cnvar} < CD > .\supset.\,[\exists FG \leqq B'].$
$E\,\varepsilon_\infty\,\mathrm{quasihomosemo} < CBB'AFG >:$

10) $[CD \leqq A] : D\ \varepsilon_\infty\ \text{cnvar} < C\ \text{Eqvl1} < \text{Essnt} < A >>> . \supset . [\exists EF \leqq A] .$
$E\ \varepsilon_\infty\ \textbf{ingr} < A > . F\ \varepsilon_\infty\ \textbf{ingr} < A > . D\ \varepsilon_\infty\ \text{Anarg} < CEF > .^{\cdot}$
11) $[CDE \leqq A] .^{\cdot} . C\ \varepsilon_\infty\ \text{prntm} < \text{Essnt} < \text{Eqvl2} < \text{Essnt} < A >>>> . D\ \varepsilon_\infty\ \text{arg} < C > .$
$E\ \varepsilon_\infty\ \text{Sgnfnct} < D > . \supset : [FG \leqq A] : F\ \varepsilon_\infty\ \text{int} < \text{Qntf} < \text{Eqvl2} < \text{Essnt} < A >>>> .$
$G\ \varepsilon_\infty\ \text{int} < \text{Qntf} < \text{Eqvl2} < \text{Essnt} < A >>>> . \supset . F =_\infty G .$
$E\ \varepsilon_\infty\ \text{var} < F\ \text{Eqvl2} < \text{Essnt} < A >>>$

A is an *ontological thesis of extensionality* relative to B in B'. The first two clauses of this explanation indicate that there are at least two variables in the quantifier of A, that no quantifiers in A are vacuous, and that the first word in the nucleus of A is not a variable.

The remaining clauses indicate that A is a generalization of some coimplication, while both the coimplicans and coimplicate of the nucleus of A are themselves generalizations of coimplications. Let us call the coimplicans of the nucleus of A the basis of A and the coimplicate of the nucleus of A the extension of A. Then clauses 3, 4 and 5 indicate that the parentheme of the coimplicans (coimplicate) of the nucleus of the basis of A is nominative. Clause 7 indicates that the subjects of these parenthemes are equiform variables bound in the basis of A. In respect to their predicates: the coimplicans (coimplicate) of the nucleus of the extension of A has a parentheme containing a single argument which is a variable equiform with the first word in the predicate of the coimplicans (coimplicate) of the nucleus of the basis. Indeed, these are the variables bound by the quantifier of A.

The remaining clauses of this explanation closely parallel those for protothetical extensionality. Thus, as before, any function in A is either the nucleus of some generalization or belongs to some determined semantical category. Further, equiform variables in A belong to the same semantical category. Analogous arguments of the predicates of the coimplicans and coimplicate of the nucleus of the basis are equiform variables, and any equiform variables in the basis are analogous arguments of the basis. And finally, the last clause in D4.57 indicates that the quantifier of the extension of A has only one variable and it binds the functor of the coimplicans (coimplicate) of the nucleus of the extension.

This explanation differs significantly from Leśniewski's only in its fourth, fifth and last clauses and avoids the use of 'Cmpl' in these clauses in a manner similar to that of *T.E.XLIX*. With *T.E.LVII°* stated, Leśniewski in [6] writes:

Unter der Voraussetzung, dass eine These A die letzte der Thesen ist, die schon zu dem System gehören, darf man zu ihm als neue These einen Ausdruck B nur in dem Fall hinzufügen, wenn wenigstens eine der sieben folgenden Bedingungen erfüllt ist:

1) $B\ \varepsilon\ \text{1defo}(A)$
2) $B\ \varepsilon\ \text{2defo}(A)$
3) $[\exists C] . C\ \varepsilon\ \text{tho}(A) . B\ \varepsilon\ \text{cnsqrprtqntf}(C)$
4) $[\exists C,D] . C\ \varepsilon\ \text{tho}(A) . D\ \varepsilon\ \textbf{tho}(A) . B\ \varepsilon\ \text{cnsqeqvl}(C,D)$
5) $[\exists C] . C\ \varepsilon\ \text{tho}(A) . B\ \varepsilon\ \text{cnsqsbsto}(A,C)$

6) $B \varepsilon$ 1extnsnlo(A)
7) $B \varepsilon$ 2extnsnlo(A)

Following this exposition, however, ontology is legitimately developed under the following single rule.

Supposing that a thesis A is the last thesis which already belongs to a list of theses A' of the system, then an expression B may be added as a new thesis only in case at least one of the following seven conditions is fulfilled:

1) $B \varepsilon_\infty$ 1defo $\langle AA' \rangle$
2) $B \varepsilon_\infty$ 2defo $\langle AA' \rangle$
3) $[\exists C \leqq A]$. $C \varepsilon_\infty$ **tho** $\langle AA' \rangle$. $B \varepsilon_\infty$ cnsqrprtqntf $\langle C \rangle$
4) $[\exists CD \leqq A]$. $C \varepsilon_\infty$ **tho** $\langle AA' \rangle$. $D \varepsilon_\infty$ **tho** $\langle AA' \rangle$ $B \varepsilon_\infty$ cnsqrprtqntf $\langle C \rangle$
5) $[\exists C \leqq A]$. $C \varepsilon_\infty$ **tho** $\langle AA' \rangle$. $B \varepsilon_\infty$ cnsqsbsto $\langle AA'C \rangle$
6) $B \varepsilon_\infty$ 1extnsnlo $\langle AA' \rangle$
7) $B \varepsilon_\infty$ 2extnsnlo $\langle AA' \rangle$

Thus, if one wishes to give an exposition of ontology, he asserts an adequate axiom and then chooses which theses he will next assert—his choice continually guided by the above effective (primitive recursive) rule.

5. *Conclusion* Just as the rule for ontology incorporates an adjusted rule of protothetic, so too any extension of ontology will incorporate an adjusted rule of ontology—see for instance [6] where a particular extension of ontology, mereology, is discussed. All that is generally needed is an adjustment in the concept of thesis. Thus, the thirty-second terminological explanation is changed so that the concept of thesis for the extended system includes effective theses of ontology as well as the new axiom, for instance, the axiom for mereology or the axiom of infinity, etc. After that is done, the rule for the extended system of ontology is generated merely by replacing the previous concept of thesis, **tho**, by the new concept, **thm** or **thinf**, throughout the remaining terminological explanations.

One may assume that such a program is accomplished for ontology extended by the axiom of infinity, and that, therefore, there is available a rule for this extension of ontology analogous to that given above.

In [7], Leśniewski's original terminological explanations are presented axiomatically. Here, the terminological explanations are reduced to recursive concepts and are actually represented in ontology extended by the axiom of infinity. Naturally, Gödel's well known results of [4] follow for this system—the interested reader can consult [2] for a fuller statement of the incompleteness proof for ontology.

NOTES

1. Of course this is not a proper definition in the system of ontology, but only a definitional thesis as was indicated. That is, the thesis in question is not justified by the ontological directive for definitions although it is derivable in the system and is analogous to a proper definition. But with this point clear, because the availability of such theses is a direct result of there being an internal ontological

model for the numerical epsilon, such theses can be referred to as (numerical) definitions. See [1], especially section 2, for a fuller discussion of the internal model and numerical definitions.

2. The axiom of ontology used in this exposition introduces into the system of logic being formed by its assertion, the following semantical categories as primitive to ontology and not available in protothetic: names and proposition forming functors for two name arguments. These categories are introduced by the single primitive constant of ontology (ε). Any other categories are introduced into the system by defining a constant for the category in accordance with the definitional directives of the system. However, the identification of "basic semantical categories of ontology" with names and propositions is justified since all categories which can be introduced into ontology are definable in meta-logic in terms of these two.

3. As will become clear when the terminological explanations of this section are completed, ontology is not based on protothetic merely by appending an axiom to that system. Rather, ontology has its own single rule—parts of which are identical to protothetical directives except that they are understood as adjusted to the semantical categories available in ontology. Similarly, any extension of ontology, for example, mereology, is most accurately described as a system incorporating (some given development of) ontology within it.

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