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AN INFIXED, PUNCTUATION-FREE NOTATION[1]

W. C. WILCOX and R. D. CARNES

I. The Current Notation In what follows, the capital script letters '$\mathcal{P}$', '$\mathcal{Q}$', '$\mathcal{R}$', '$\mathcal{S}$', and '$\mathcal{T}$' are syntactic variables ranging over the wffs (regardless of complexity) of *Principia*-like, Polish, and the current notations. These variables are not symbols within any of these notations, and the context makes it quite clear which of the three notations is involved. One other symbol occurs below that is not part of the current notation, *viz*., the double-arrow, '$\longleftrightarrow$', which is used to indicate extensional equivalence. The wffs at either end may be replaced by the wff at the other end. The symbols within the current notation are as follows:

(1) the sentence variables 'p', 'q', 'r', 's', and 't', with or without primes;

(2) the monadic connective '( )';

(3) the binary connectives '|' and '——'; and

(4) the therefore-indicator, '$\vdash$', used in writing formalizations of arguments.

The wffs of the current notation are specified by the following recursive definition:

(a) Every sentence variable is a wff.
(b) If $\mathcal{P}$ is a wff, then $(\mathcal{P})$ is a wff.
(c) If $\mathcal{P}$ and $\mathcal{Q}$ are wffs, then $\mathcal{P}|\mathcal{Q}$ is a wff. (In this wff, the wffs $\mathcal{P}$ and $\mathcal{Q}$ are said to be in series.)
(d) If $\mathcal{P}$ and $\mathcal{Q}$ are wffs, then $\frac{\mathcal{P}}{\mathcal{Q}}$ is a wff. (In this wff, the wffs $\mathcal{P}$ and $\mathcal{Q}$ are said to be in parallel.)

All and only formulas which qualify under (a)-(d) above are wffs in the current notation.

The truth conditions for the wffs of the current notation are as follows:

(T1) $(\mathcal{P})$ is **T** if and only if $\mathcal{P}$ is **F**; otherwise, $(\mathcal{P})$ is **F**.
(T2) $\mathcal{P}|\mathcal{Q}$ is **T** if and only if both $\mathcal{P}$ and $\mathcal{Q}$ are **T**; otherwise, $\mathcal{P}|\mathcal{Q}$ is **F**. We call molecular wffs of this form *series formulas*.

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(T3) $\frac{\mathcal{P}}{\mathcal{Q}}$ is **T** if and only if at least one of $\mathcal{P}$ and $\mathcal{Q}$ is **T**; otherwise, $\frac{\mathcal{P}}{\mathcal{Q}}$ is **F**. We call molecular wffs of this form *parallel formulas*.

As the title indicates, the current notation is infixed, but punctuation free.. The following table provides translation from the traditional notations to the current notation.

Principia	Polish	Current
$\sim\mathcal{P}$	$N\mathcal{P}$	$(\mathcal{P})$
$\mathcal{P}\&\mathcal{Q}$	$K\mathcal{P}\mathcal{Q}$	$\mathcal{P}\vert\mathcal{Q}$
$\mathcal{P}\vee\mathcal{Q}$	$A\mathcal{P}\mathcal{Q}$	$\frac{\mathcal{P}}{\mathcal{Q}}$
$\mathcal{P}\supset\mathcal{Q}$	$C\mathcal{P}\mathcal{Q}$	$\frac{(\mathcal{P})}{\mathcal{Q}}$
$\mathcal{P}\equiv\mathcal{Q}$	$E\mathcal{P}\mathcal{Q}$	$\frac{(\mathcal{P})}{\mathcal{Q}}\Big\vert\frac{(\mathcal{Q})}{\mathcal{P}}$

Examples of translation:

(1) '$p\&(q\vee r)$' is translated '$p\Big|\frac{q}{r}$'[2]

(2) '$(p\&q)\vee r$' is translated '$\frac{p|q}{r}$'

(3) '$CpCqNr$' is translated '$\frac{(p)}{\frac{(q)}{(r)}}$'

There are two transformation rules in the current notation.

(1) DeMorgan's Laws (**DM**)

(a) $(\mathcal{P}|\mathcal{Q}) \longleftrightarrow \frac{(\mathcal{P})}{(\mathcal{Q})}$

(b) $\left(\frac{\mathcal{P}}{\mathcal{Q}}\right) \longleftrightarrow (\mathcal{P})|(\mathcal{Q})$

(2) Double Negation (**DN**)

$((\mathcal{P})) \longleftrightarrow \mathcal{P}$

II. Validity and Theoremhood We set forth the following terminological definitions.

(1) The spaces flanking binary connectives are called *bins*.

(2) A wff determined to be **F** is cancelled with a diagonal line.

(3) A *basic wff* is either a sentence variable or its negation.

(4) Given a series formula $\mathcal{S}$, the only way for $\mathcal{S}$ to be **T** is for the wff in each bin to be **T**. If, on such an assignment, consistency requires the cancellation of an entire wff in a bin, then $\mathcal{S}$ is a *self-contradiction*. This is obvious since (a) the cancellation of an entire wff means that that wff is **F**

and, hence, that $\mathscr{S}$ is **F**; and (b) since any other assignment to the wffs in series, other than the original assignment, renders $\mathscr{S}$ **F**.

(5) Given a parallel formula $\mathscr{P}$, the only way for $\mathscr{P}$ to be **F** is for the wff in each bin to be **F**. If on such an assignment, consistency requires that the wff in one bin remain uncancelled, then $\mathscr{P}$ is a *tautology*. (By reasoning similar to that in (4).)

(6) A wff is *contingent* if it is neither a tautology nor a self-contradiction.

The assessment of the validity of arguments is quite simple in the current system. Given an argument, translate the premisses and conclusion into the current notation, put the premisses and the denial of the conclusion in series, and if the result of cancelling ((4) and (5) above) is a self-contradiction, the argument is valid; otherwise, invalid. Of course, one could form the corresponding conditional and test for tautologousness, but the authors have found the previous method more direct. Consider the following argument formula.

$$\left|\begin{array}{l} CpKsAqr \\ Cqp \\ \underline{Ns} \\ ANqt \end{array}\right.$$

After translation, the premisses and the denial of the conclusion are placed in series giving us

$$\frac{(p)}{s\left|\begin{array}{c} q \\ \hline r \end{array}\right.} \;\Bigg|\; \frac{(q)}{p} \;\Bigg|\; (s) \;\Bigg|\; \left(\frac{(q)}{t}\right).$$

Application of **DM** to the wff in the last bin yields

$$\frac{(p)}{s\left|\begin{array}{c} q \\ \hline r \end{array}\right.} \;\Bigg|\; \frac{(q)}{p} \;\Bigg|\; (s) \;\Bigg|\; ((q)) \;\Bigg|\; (t)\ .$$

Application of **DN** to '$((q))$' produces

$$\frac{(p)}{s\left|\begin{array}{c} q \\ \hline r \end{array}\right.} \;\Bigg|\; \frac{(q)}{p} \;\Bigg|\; (s) \;\Bigg|\; q \;\Bigg|\; (t)\ .$$

Now assume that the basic wffs in the last three bins are $\mathbf{T}^3$ (otherwise, the whole series formula is **F**). Since '(s)' is **T**, cancellation of 's' in the wff in bin one is called for. Similarly, for the '(q)' in the wff in bin three. So far we have

$$\frac{(p)}{\cancel{s}\left|\begin{array}{c} q \\ \hline r \end{array}\right.} \;\Bigg|\; \frac{\cancel{(q)}}{p} \;\Bigg|\; (s) \;\Bigg|\; q \;\Bigg|\; (t)$$

The cancellation of 's' in '$s \mid \frac{q}{r}$' requires cancellation of that whole wff (since it is a series formula). This gives us

$$\frac{(p)}{\cancel{s} \mid \frac{\cancel{q}}{r}} \;\Big|\; \frac{\cancel{(q)}}{p} \;\Big|\; (s) \;\Big|\; q \;\Big|\; (t)\,.$$

But in order for the wff in bin one to be **T** (as it must be, if the whole series formula is to be **T**), '(p)' must be **T**. But this calls for cancellation of the 'p' in the wff in bin two, yielding

$$\frac{(p)}{\cancel{s} \mid \frac{\cancel{q}}{r}} \;\Big|\; \frac{\cancel{(q)}}{\cancel{p}} \;\Big|\; (s) \;\Big|\; q \;\Big|\; t\,.$$

Since the wff in bin two is entirely cancelled, that wff is **F**, and the whole series formula is a self-contradiction (since it is not possible for all the wffs in the series to be **T** simultaneously, it is not possible for that series formula to have any truth-value but **F**).

Let's look at another example as it would occur in practice.

$$\begin{array}{|l} ApNq \\ CsNp \\ \underline{Ctq} \\ NKst \end{array} \qquad \frac{\cancel{p}^{3}}{\cancel{(q)}^{4}} \;\Big|\; \frac{\cancel{(s)}^{2}}{(p)} \;\Big|\; \frac{\cancel{(t)}^{2}}{q} \;\Big|\; \cancel{(}(s \mid t)\cancel{)}.^{1}$$

1 is the result of applying **DN** to the denial of the conclusion. 2 indicates the cancellation of '(s)' and '(t)' on the assumption that 's' and 't' are **T**. Since '$(p$') must be **T**, in order for the wff in bin two to be **T**, 3 indicates the subsequent cancellation of 'p' in bin one. Similarly, since 'q' must be **T**, in order for the wff in bin three to be **T**, 4 indicates the subsequent cancellation of '(q)' in bin one. The cancellation of the entire formula in bin one renders the series formula a self-contradiction and proves the argument valid.

To check whether or not a given wff is a theorem of the propositional calculus, we simply take its denial and check to see if it is a self-contradiction. If it is, then we know the original wff was a theorem. For example, consider the corresponding conditional of the preceding argument formula.

$$CKKApNqCsNpCtqNKst$$

Translating it into the current notation, the denial of the above wff is

$$\left(\frac{\left(\frac{p}{(q)} \;\Big|\; \frac{(s)}{(p)} \;\Big|\; \frac{(t)}{q}\right)}{(s \mid t)}\right)$$

Applying **DM** and **DN** we get

$$\frac{p}{(q)} \,\Big|\, \frac{(s)}{(p)} \,\Big|\, \frac{(t)}{q} \,\Big|\, s \,\Big|\, t$$

which, as we saw before, is a self-contradiction.

III. Normal Forms Although the theory of normal forms, in the propositional calculus, is simple, anyone who has ever tried to perform the required computations knows how difficult they can be. All of the existing notations seem to get in the way of the computations, rather than aiding them. The current notation, rather than obscuring the moves to be made, seems to indicate them. The following transformations are to be allowed, as in any notation.

(1) DeMorgan's Laws (**DM**), as in the previous section, with (1a) generalised as

$$(p|q|\ldots|r) \longleftrightarrow \begin{array}{c}(p)\\ \hline (q)\\ \hline \vdots \\ \hline (r)\end{array}$$

and similarly for (1b).

(2) Double Negation (**DN**), as in the previous section.

(3) Commutation (**Com**). Since '|' and '——' are both associative and commutative, formulas either in parallel or in series can be placed in any desired order.

(4) Distribution (**Dist**)

(4a) $p \,\Big|\, \frac{q}{r} \longleftrightarrow \frac{p|q}{p|r}$

(4b) $\frac{p}{q|r} \longleftrightarrow \frac{p}{q} \,\Big|\, \frac{p}{r}$

(5) Tautology (**Taut**)

(5a) $p \longleftrightarrow p|p$

(5b) $p \longleftrightarrow \frac{p}{p}$

(6) Seriology (**Ser**)

$$p \longleftrightarrow p \,\Big|\, \frac{q}{(q)}$$

(7) Paradiction (**Par**)

$$p \longleftrightarrow \frac{p}{q|(q)}$$

The names of the last two standard moves in normal form computation derive, respectively, from putting a tautology in series (conjoining it) with a formula, and from putting a contradiction in parallel (disjoining it) with a formula. Seriology is particularly useful for deriving conjunctive Boolean expansions, paradiction for deriving disjunctive Boolean expansions.

There are in fact two different things which go under the name 'normal forms'. There are Boolean normal forms, or expansions, used for determining whether a formula is contingent, contradictory, or tautologous, and ordinary normal forms, used for finding simpler equivalent formulas. As for this latter task, one may proceed in either of two ways: either by attacking the formula directly, or by first forming the Boolean expansion and then simplifying that. The former is often more direct, but the latter has the advantage of a *somewhat* routine procedure.

We shall first give an example of the derivation of a disjunctive Boolean expansion. The expansion will be a disjunction of conjunctions, each disjunct of which contains, in alphabetical order, every variable in the formula or its negation. If there are 2^n disjuncts after eliminating duplicates, where n is the number of variables, then the original formula is a tautology.

As an example, we shall take the formula '*CpCqp*'. Translating this into the current notation, we get

$$\begin{array}{c} (p) \\ \hline (q) \\ \hline p \end{array}$$

To get each of the sentence variables into each of the parallel bins, we use Seriology.

$$\begin{array}{c|c} (p) & \begin{array}{c} q \\ \hline (q) \end{array} \\ \hline (q) & \begin{array}{c} p \\ \hline (p) \end{array} \\ \hline p & \begin{array}{c} q \\ \hline (q) \end{array} \end{array}$$

Distribution through each of the parallel bins then gives us

$$\begin{array}{c|c} (p) & q \\ \hline (p) & (q) \\ \hline (q) & p \\ \hline (q) & (p) \\ \hline p & q \\ \hline p & (q) \end{array} \qquad \text{and by commutation} \qquad \begin{array}{c|c} (p) & q \\ \hline (p) & (q) \\ \hline p & (q) \\ \hline (p) & (q) \\ \hline p & q \\ \hline p & (q) \end{array}$$

Eliminating duplicates by the law of tautology we get

$$\begin{array}{c|c} (p) & q \\ \hline (p) & (q) \\ \hline p & (q) \\ \hline p & q \end{array}$$

and this is in Boolean disjunctive normal form. Since its Boolean expansion contains four (2^2) parallel elements, the original formula is a tautology.

A conjunctive Boolean expansion is a conjunction of disjunctions, each conjunct of which contains, in alphabetical order, every variable in the formula or its negation. If there are 2^n conjuncts after eliminating duplicates, then the original formula is a contradiction. For our example, we shall take '*CApNpKqNq*'.

$$\dfrac{\left(\dfrac{p}{(p)}\right)}{q\,|\,(q)}$$

We proceed as follows:

$$\dfrac{(p)\,|\,p}{q\,|\,(q)} \qquad \text{(by DM and DN)}$$

$$\dfrac{(p)}{q\,|\,(q)}\ \Bigg|\ \dfrac{p}{q\,|\,(q)} \qquad \text{(by Dist)} \qquad \dfrac{(p)}{q}\ \Bigg|\ \dfrac{(p)}{(q)}\ \Bigg|\ \dfrac{p}{q}\ \Bigg|\ \dfrac{p}{(q)} \qquad \text{(by Dist)}$$

Since this is in conjunctive Boolean normal form and contains four (2^2) conjuncts, '*CApNpKqNq*' is a contradiction.

We can find the simplest equivalent of '*CpCqp*' by taking its Boolean expansion

$$\begin{array}{c|c} p & q \\ \hline p & (q) \\ \hline (p) & (q) \\ \hline (p) & q \end{array}$$

and proceeding as follows:

$$\dfrac{p\ \Bigg|\ \dfrac{(q)}{q}}{(p)\ \Bigg|\ \dfrac{(q)}{q}} \qquad \text{(by Dist and Com)} \qquad \dfrac{p}{(p)}\ \Bigg|\ \dfrac{(q)}{q} \qquad \text{(by Dist)}$$

$$\dfrac{p}{(p)} \qquad \text{(by Ser)}$$

We shall now exhibit both methods of finding a simpler formula with the following contingent formula, taken from Copi's *Symbolic Logic* (p. 326).

$$(p\&q)\vee(q\&r)\vee(p\&r)\vee r$$

Its Boolean expansion is

$$\begin{array}{c|c|c} p & q & r \\ \hline p & q & (r) \\ \hline p & (q) & r \\ \hline (p) & (q) & r \\ \hline (p) & q & r \end{array}$$

and we proceed as follows:

$$\begin{array}{c|c|c} r & p & q \\ \hline r & p & (q) \\ \hline r & (p) & q \\ \hline r & (p) & (q) \\ \hline (r) & p & q \end{array}$$ (by **Com**)

$$\frac{r \mid \frac{p}{(p)} \mid \frac{q}{(q)}}{(r) \mid p \mid q}$$ (by **Dist**)

$$\frac{r}{(r) \mid p \mid q}$$ (by **Ser**)

$$\frac{r}{(r)} \mid \frac{r}{p} \mid \frac{r}{q}$$ (by **Dist**)

$$\frac{r}{p} \mid \frac{r}{q}$$ (by **Ser**)

$$\frac{r}{p \mid q}$$ (by **Dist**)

The derivation starting from the formula itself and not from its expansion will now be given. The formula in current notation is

$$\begin{array}{c} p \mid q \\ \hline q \mid r \\ \hline p \mid r \\ \hline r \end{array}$$

and the derivation goes

$$\begin{array}{c} p \mid q \\ \hline p \mid r \\ \hline r \mid q \\ \hline r \mid r \end{array}$$ (by **Com** and **Taut**)

$$\frac{p \mid \frac{q}{r}}{r \mid \frac{q}{r}}$$ (by **Dist**)

$$\frac{p}{r} \mid \frac{q}{r}$$ (by **Dist**)

$$\frac{p \mid q}{r}$$ (by **Dist**)

NOTES

1. The current notation was discovered by Wilcox while reflecting upon the use of Boolean algebra for the analysis of electrical circuits. This genesis of the current notation is reflected in the terms "series formulas" (conjunctions) and "parallel formulas" (disjunctions). Wilcox also provided the treatment of normal forms found in section III. Carnes is responsible for the specification of the system and the cancellation technique found in the first two sections.

2. As one can see from this and example (2), no punctuation is required, as in the *Principia* wffs, to resolve the ambiguity of 'p&qvr'.

3. This is the analogue of Quine's fell-swoop with one advantage: this technique applies in every case, not just in the range of cases to which Quine's technique is restricted.

State University College
Oswego, New York