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MODAL LOGICS WITH TWO KINDS OF NECESSITY AND POSSIBILITY

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0. Historical Remarks. **0.1** The first logic of modalities owes its origin to Aristotle. Already at this time he developed two different aspects of modal logics: a theory of modal statements in general—mainly considered in (PHe) ch. 9,12 and 13 and in (APr) I,ch.3 and 13—and a theory of modal syllogisms which is described primarily in (APr) I, ch. 8-22[1]. **0.2** Theophrast introduced a new aspect: in his theory the statement as a whole is determined by the modus. In scholastic philosophy this kind of modality was called "modales de dicto" (modality de dicto)[2]. Aristotle however interprets the structure of a modal statement—according to A. Becker and Bocheński—in such a way that the modus is applied either to the subject or to the predicate of a subject—predicate statement. If the statement in question is the universal statement $(x)(Fx \supset Gx)$ and if L is the modal operation 'necessary', then Aristotle's view of the modal statement is such that it can be represented either by the statement $(x)(Fx \supset L(Gx))$ or by the statement $(x)(L(Fx) \supset L(Gx))$. In scholastic philosophy those kinds of modality were called "modales de re" (modality de re)[3]. **0.3** Some interesting contributions to the logic of modalities have been made by the stoic and megaric schools: the first attempt to interpret the modal operators by time-operators is due to them[4]. **0.4** In scholastic philosophy both interpretations of the modal statements, as described in 0.2, were well-known (cf. Abaelard (Dia) p. 204ff.). A precise distinction of these kinds of modality based on a syntactical criterion—namely, the position of the modus (modal operator) in the sentence—can be found in the work "De Propositionibus Modalibus" of Thomas Aquinas (PMo) cf. Bocheński (AMo). In this work Aquinas considers also the relations between modalities and quantification and gives an interpretation of the modal operators with the help of quantifiers. In 1952 0. Becker did the same and he called this interpretation "statistical interpretation (Deutung) of the modal calculus" (UMo) p. 16ff. Further contributions to modal logics in scholastic philosophy have been made by Albert the Great (PAP), Petrus Hispanus (SuL) 7,26, Pseudo Scott (PrA) II, p. 143-159 and Paulus Venetus (LgM) I,21[5]. **0.5** Modern modal logics begins essentially with the contributions of C. I. Lewis (SSL) (SLg). Since the publication of the first works of Lewis (about 1918) a great number and variety of papers and books about modal logics have appeared[6].

1. Structure of the system SS1. **1.01** Introductory remarks: To expedite matters it seems desirable to have a modal logics which distinguishes two kinds of necessity and two kinds of possibility, for instance, interpretable as logical and empirical necessity (or natural necessity) or possibility respectively. There are some contemporary contributions which have been made to this problem already. The two most fundamental seem to be an essay of Popper (UDN) and one of Montague (LPE). There are three wellknown methods to construct deductive systems: the axiomatic method, the method of natural deduction and the matrix method. For the construction of the following deductive system (abbreviated as SS1) the matrix method was chosen. This method originates from Peirce (ALg) and Schröder (VAL). Since 1920 Łukasiewicz applied this method from the first to construct many valued systems of propositional calculus (MSA). Łukasiewicz and Tarski established the two-valued propositional calculus by means of this method (UAK). Bernays used the matrix method for the first time for investigations about independency—proofs (AUA). Tarski established the base for a meta-calculus in order to set down the laws for the construction of deductive systems with the help of the matrix-method (UAK) Def. 3,4; Theorem 2,3,4[7].

1.03 In the following the so called Polish notation (which is due to Łukasiewicz) is used. **1.04** In the propositional calculus as in the uninterpreted SS1 '*p*', '*q*', '*r*', '*s*' are propositional variables and '*N*', '*A*', '*K*', '*C*' and '*E*' are propositional constants standing for negation-sign, disjunction-sign, conjunction-sign, (material) implication-sign and (material) equivalence-sign respectively. **1.05** In SS1 there are the further one-place operators '*L*', '*LL*', '*M*', '*MM*', '*ML*' and '*LM*' which are interpreted in SS1M as (empirical) necessity, logical necessity, (empirical) possibility, logical possibility, possible necessity and necessary possibility respectively.

1.06 To the wellformed formulas (*wffs*) of the two-valued propositional calculus two truth-values true and false can be assigned. To the *wffs* of SS1 the six truth-values 1,2,3,4,5,6 are assigned, the values 1,2,3 for true, the values 4,5,6 for false.

1.07 Formula (sentence) of the system. **1.071** Every propositional variable is a formula (atomic formula)[8]. **1.072** If '*p*' is a formula, '*Np*' and '*Lp*' are formulas. According to **1.16-1.20** it follows that also '*LLp*', '*Mp*', '*MMp*', '*MLp*', '*LMp*' are formulas. **1.073** If '*p*' and '*q*' are formulas, '*Apq*' is a formula. According to **1.12-1.14** it follows that also '*Kpq*', '*Cpq*', '*Epq*' are formulas.

1.08 Definition of the system SS1. The system SS1 can be defined as the set of all formulas (sentences) which are satisfied by the matrix $Mat = \langle T,F,N,A,L \rangle$ where $T = \{1,2,3,\}$, $F = \{4,5,6\}$ and the operations N, A and L are defined by the following formulas:

$N(1)=6,\ N(2)=5,\ N(3)=4,\ N(4)=3,\ N(5)=2,\ N(6)=1$

$A(1,1)=A(1,2)=A(1,3)=A(1,4)=A(1,5)=A(1,6)=A(2,1)=A(3,1)=A(4,1)=$
$A(5,1)=A(6,1)=A(2,5)=A(3,4)=A(4,3)=A(5,2)=1$

$A(2,2)=A(2,3)=A(2,4)=A(2,6)=A(3,2)=A(4,2)=A(6,2)=2$

$A(3,3) = A(3,5) = A(3,6) = A(5,3) = A(6,3) = 3$
$A(4,4) = A(4,5) = A(4,6) = A(5,4) = A(6,4) = 4$
$A(5,5) = A(5,6) = A(6,5) = 5$
$A(6,6) = 6$
$L(1) = 1, L(2) = 3, L(3) = 6, L(4) = 6, L(5) = 6, L(6) = 6$

1.081 It follows from 1.08 that every sentence (formula) of SS1 is unambiguously determined by a certain matrix which is an instance of the matrix $Mat = \langle T,F,N,A,L \rangle$ (as defined in **1.08**). And on the other hand to every special matrix which is an instance of $Mat = \langle T,F,N,A,L \rangle$ a sentence of SS1 corresponds. The meaning of the expression 'a sentence is determined (or satisfied) by a matrix' need not to be outlined here; it will become sufficiently clear (for the understanding of the paper) in ch. **1.4** and **1.5**. For a detailed and formal definition of this expression see Tarski (UAK) def. 4.

1.09 Basic matrix. Every atomic formula has the basic matrix:

1 2 3 4 5 6

1.10 Negation (N). Iff[9] p has the basic matrix 1 2 3 4 5 6 then Np has the matrix: 6 5 4 3 2 1

1.11 Disjunction (A). Iff p and q have the basic matrix then Apq has the matrix:

Apq	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	1	2
3	1	2	3	1	3	3
4	1	2	1	4	4	4
5	1	1	3	4	5	5
6	1	2	3	4	5	6

1.111 With the help of these two operations one can—as it is wellknown—build up all other compound sentences of the classical assertoric propositional calculus. Instead of the two operations of negation and disjunction as a base for the classical assertoric propositional calculus one could use also only one, for instance Sheffers function.

1.12 Conjunction (K). Kpq has the same matrix as $NANpNq$; i.e. iff p and q have the basic matrix, then Kpq has the matrix:

Kpq	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	6	6
3	3	3	3	6	5	6
4	4	4	6	4	5	6
5	5	6	5	5	5	6
6	6	6	6	6	6	6

1.13 Material Implication (C). Cpq has the same matrix as $ANpq$ or as $NKpNq$; i.e. iff p and q have the basic matrix, then Cpq has the matrix:

Cpq	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	4	5	5
3	1	2	1	4	4	4
4	1	2	3	1	3	3
5	1	2	2	2	1	2
6	1	1	1	1	1	1

1.14 Material Equivalence (E). Epq has the same matrix as $KCpqCqp$ or as $AKpqKNpNq$; i.e. iff p and q have the basic matrix then Epq has the matrix:

Epq	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	1	3	4	6	5
3	3	3	1	6	4	4
4	4	4	6	1	3	3
5	5	6	4	3	1	2
6	6	5	4	3	2	1

1.15 L, in SS1M interpreted as the operation of necessity. Iff p has the basic matrix, then Lp has the matrix: 1 3 6 6 6 6.

1.151 The operation L together with the operations N and A (or Sheffer's function instead of N and A) suffice to build up the system SS1; with additional interpretations and definitions also the systems SS1M and SS1I can be constructed. It is not necessary to give a certain interpretation of the operations L, LL, M, MM, ML, LM right from the beginning. The system SS1 determined by the matrix given in **1.08** can be interpreted in different ways and it would have been possible to leave open the question of interpretation all the way through. This has not been done for the following reasons:

1. The system SS1 is much better to understand for the reader in its modal interpretation SS1M.
2. The purpose of constructing SS1 was to find a modal system with two kinds of necessity and possibility; thus SS1 was constructed with its modal interpretation SS1M in mind.

Therefore in the following the system SS1 is described through its modal interpretation SS1M.

1.16 LL, in SS1M: logical necessity. LLp has the same matrix as $L(Lp)$; i.e. iff p has the basic matrix, then LLp has the matrix: 1 6 6 6 6 6.

1.17 M, in SS1M: possibility. Mp has the same matrix as $NLNp$; i.e. iff p has the basic matrix, then Mp has the matrix: 1 1 1 1 4 6.

1.18 MM, in SS1M: logical possibility. MMp has the same matrix as $M(Mp)$; i.e. iff p has the basic matrix, then MMp has the matrix: 1 1 1 1 1 6.

1.19 ML, in SS1M: possible necessity. MLp has the same matrix as $M(Lp)$; i.e. iff p has the basic matrix, then MLp has the matrix: 1 1 6 6 6 6.

1.20 LM, in SS1M: necessary possibility. LMp has the same matrix as $L(Mp)$; i.e. iff p has the basic matrix, then LMp has the matrix: 1 1 1 1 6 6.

1.21 Table of the one place operations of SS1.

p	Np	LLp	Lp	MLp	p	LMp	Mp	MMp
1	6	1	1	1	1	1	1	1
2	5	6	3	1	2	1	1	1
3	4	6	6	6	3	1	1	1
4	3	6	6	6	4	1	1	1
5	2	6	6	6	5	6	4	1
6	1	6	6	6	6	6	6	6

From this table it can be observed that the values 1 and 6 are not changed by any of the modal operations. One can see further that the possibilities for positive (proper) modalities in SS1M are exhausted by the six given above. Analogously there are six negative (proper) modalities in SS1M: $LLNp$, LNp, $MLNp$, $LMNp$, MNp and $MMNp$. Their matrices can be obtained by overturning the above matrices of the positive (proper) modalities respectively. From this it is clear that in SS1M there are exactly 12 proper and 2 improper modalities (p, Np). The number of modalities is therefore the same as in the system S4 by C. I. Lewis, although S4 and SS1M have essential differences.

1.22 Strict implication (LC). $LCpq$ has the same matrix as $L(Cpq)$; i.e. iff p and q have the basic matrix, then $LCpq$ has the matrix:

$LCpq$	1	2	3	4	5	6
1	1	3	6	6	6	6
2	1	1	6	6	6	6
3	1	3	1	6	6	6
4	1	3	6	1	6	6
5	1	3	3	3	1	3
6	1	1	1	1	1	1

1.23 Strong implication (LLC). $LLCpq$ has the same matrix as $L(LCpq)$; i.e. iff p and q have the basic matrix, then $LLCpq$ has the matrix:

$LLCpq$	1	2	3	4	5	6
1	1	6	6	6	6	6
2	1	1	6	6	6	6
3	1	6	1	6	6	6
4	1	6	6	1	6	6
5	1	6	6	6	1	6
6	1	1	1	1	1	1

1.24 Between the seven positive modalities the following implicational relations exist:

$LLC\ LLp\ Lp$
$LLC\ Lp\ MLp$
$LC\ MLp\ p$
$LC\ p\ LMp$
$LLC\ LMp\ Mp$
$LLC\ Mp\ MMp$

This can be seen easily from the given matrices in **1.22** and **1.23.** The implicational sequence of the positive modalities is also shown by the right part of the table (from left to right) in **1.21.**

1.25 Strict equivalence (LE). $LEpq$ has the same matrix as $L(Epq)$; i.e. iff p and q have the basic matrix then $LEpq$ has the matrix:

$LEpq$	1	2	3	4	5	6
1	1	3	6	6	6	6
2	3	1	6	6	6	6
3	6	6	1	6	6	6
4	6	6	6	1	6	6
5	6	6	6	6	1	3
6	6	6	6	6	3	1

1.26 Strong equivalence (*LLE*). *LLEpq* has the same matrix as *L*(*LEpq*); i.e. iff *p* and *q* have the basic matrix then *LLEpq* has the matrix:

LLEpq	1	2	3	4	5	6
1	1	6	6	6	6	6
2	6	1	6	6	6	6
3	6	6	1	6	6	6
4	6	6	6	1	6	6
5	6	6	6	6	1	6
6	6	6	6	6	6	1

1.27 Substitution. Two formulas (sentences) *p* and *q* can be substituted for one another iff *LLEpq* holds i.e. iff they are strongly equivalent. In SS1M: Two sentences *p* and *q* can be substituted for one another iff they are logically necessary equivalent. As it can be seen from the matrix in **1.26.** *LLEpq* holds only if the matrices of *p* and *q* are identical.

1.3 Matrices of compound sentences. As an example the matrix of modus ponens *CKpCpqq* and of its strict form *LCKpCpqq* is taken:

L	*C*	*K*	*p*	*C*	*p*	*q*	*q*
1	1	1		1	1		1
1	1	2		2			2
1	1	3		3			3
1	1	4		4			4
1	1	5		5			5
1	1	6		6			6
1	1	2		1	2		1
1	1	2		1			2
1	1	3		3			3
1	1	4		4			4
1	1	6		5			5
1	1	6		5			6
1	1	3		1	3		1
3	2	3		2			2
1	1	3		1			3
1	1	6		4			4
1	1	6		4			5
1	1	6		4			6
1	1	4		1	4		1
3	2	4		2			2
1	1	6		3			3
1	1	4		1			4
1	1	6		3			5
1	1	6		3			6
1	1	5		1	5		1
1	1	6		2			2
1	1	6		2			3
1	1	6		2			4
1	1	5		1			5
1	1	6		2			6
1	1	6		1	6		1
1	1	6		1			2
1	1	6		1			3
1	1	6		1			4
1	1	6		1			5
1	1	6		1			6

1.4 Truth in SS1 and in SS1M. **1.41** A sentence (formula) is logically true (or: valid) in SS1 and SS1M iff its matrix contains exclusively values between 1 and 3. **1.411** A sentence (formula) is logically false in SS1 and SS1M iff its negation is logically true in SS1 and SS1M, i.e. iff its matrix contains exclusively values between 4 and 6. **1.42** A sentence is strongly logically true (or: strongly valid) in SS1 and SS1M iff its matrix contains exclusively the value 1, in other words: iff the highest value of its matrix is 1. **1.421** A sentence is strongly logically false in SS1 and SS1M iff its negation is strongly logically true in SS1 and SS1M, i.e. iff its matrix contains exclusively the value 6, in other words: iff the lowest value of its matrix is 6. **1.43** A sentence is strictly logically true (or: strictly valid) in SS1 and SS1M iff the highest value of its matrix is 2. **1.431** A sentence is strictly logically false in SS1 and SS1M iff its negation is strictly logically true, i.e. iff the lowest value of its matrix is 5. **1.44** A sentence is materially logically true (or materially valid) in SS1 and SS1M iff the highest value of its matrix is 3. **1.441** A sentence is materially logically false in SS1 and SS1M iff its negation is materially logically true, i.e. iff the lowest value of its matrix is 4. **1.45** A sentence is contingent (or contingently true) in SS1 and SS1M iff its matrix contains at least one value between 1 and 3 and at the same time at least one value between 4 and 6.

1.46 The following table shows the distribution of the truth-values for logically true, logically false and contingent sentences in SS1 and SS1M:

Highest value of the matrix (=characteristical value)

1	strongly logically true
2	strictly logically true
3	materially logically true

The matrix contains at least both of the values:

1 and 4	contingent
2 and 4	contingent
3 and 4	contingent
1 and 5	contingent
2 and 5	contingent
3 and 5	contingent
1 and 6	contingent
2 and 6	contingent
3 and 6	contingent

Lowest value of the matrix

4	materially logically false
5	strictly logically false
6	strongly logically false

1.5 Characteristical value (*cv*) of a sentence (formula). The characteristical value of validity (or short: characteristical value) of a sentence in SS1 and SS1M is the highest value between 1 and 6 which occurs in its matrix. **1.51** The definitions **1.42, 1.43** and **1.44** can also be formulated by replacing 'the highest value of its matrix' by 'its characteristical value'. Thus a sentence is logically true (valid) in SS1 and SS1M iff its *cv* is either 1,2 or 3; it is strongly valid iff its *cv* is 1, strictly valid iff its *cv* is 2 and materially valid iff its *cv* is 3. For instance if one looks at the form of modus ponens in 1.3—which is not the metalinguistic rule but the sentence in the object language of propositional calculus—one observes that *C KpCpq q* is strictly valid in SS1 and SS1M, because its *cv* is 2 and *LC KpCpq q* is materially valid in SS1 and SS1M, because its *cv* is 3. **1.52** If a sentence in SS1 or SS1M has a *cv* which is higher than 3 (i.e. 4,5 or 6) then this sentence is either contingent or logically false in SS1 and SS1M (df. **1.46**).

1.53 In many systems of modal logic the following rule holds: If *p* is valid in the system, then *Lp* (necessarily *p*) is also valid in the system. This rule does not hold in SS1 or SS1M or in any of the other systems which are interpretations or extensions of SS1 or SS1M. Instead of this rule a number of much more detailed statements hold in SS1 and SS1M. Let *p* be any sentence of SS1 or SS1M. Then the following statements hold:

If the *cv* of *p* is 1, then *LLp* is valid.
If the *cv* of *p* is 2, then *Lp* is valid.
If the *cv* of *p* is 3, then *p* is valid.
If the *cv* of *p* is 4, then *LMp* (and *Mp*) are valid.
If the *cv* of *p* is 5, then *MMp* is valid.

1.6 The concept of logical consequence in SS1 and SS1M. In SS1 and SS1M three concepts of logical consequence are distinguished which correspond to the three kinds of implication: the strong concept of consequence in SS1 and SS1M is defined by the matrix of strong implication (cf. **1.23**); the strict concept of consequence is defined by the matrix of strict implication (cf. **1.22**); the material concept of consequence is defined by the matrix of material implication (cf. **1.13**). In all these cases one may just say that a conclusion (consequence) *q* follows (strongly, strictly, materially) from a premiss *p*, iff *p* implies (strongly, strictly, materially) *q*. **1.61** The concept of logical consequence in SS1 and SS1M can also be defined in the following way:

1.611 Strong concept of consequence. The conclusion *q* follows (strongly) from the premiss *p* iff at least one of the following conditions (a) or (b) are satisfied:

(a) *p* and *q* have identical matrices; i.e. all the values of the matrices of *p* and *q* which are coordinated to one another because they are on the same line (the matrix of modus ponens in **1.3** for example has 36 lines, the coordinated values are in the rows under *K* and *q*) are identical.

(b) For all coordinated pairs of values of the matrices of *p* and *q* (a coordinated pair consists of two values which are on the same line): the value of the matrix of *p* is 6 or the value of the matrix of *q* is 1 (or both cases hold). From this it is clear that the strong concept of consequence in SS1 and SS1M is also satisfied if the premiss is a contradiction (in which case every value of the matrix is 6) and the conclusion is any sentence or else if the conclusion is strongly logically true (in which case *cv* is 1) and the premiss is any sentence. In other words: from a contradiction every sentence follows and a sentence which is strongly valid follows from every sentence (or also: a sentence which is strongly valid follows from the nullclass of sentences).

1.612 Strict concept of consequence. The conclusion q follows (strictly) from the premiss p iff at least one of the following conditions (a), (b) or (c) are satisfied:

(a) as in **1.611** (a)

(b) as in **1.611** (b)

(c) For all coordinated pairs of values of the matrices of p and q: the value of the matrix of p is 5 or the value of the matrix of q is 2 (or both cases hold).

By the strict concept of consequence already a large number of the so called "paradoxes of implication" are excluded. As an example one can consider the statement *A Cpq CpNq* which is a theorem of the classical propositional calculus; its *cv* is 2, i.e. this statement is strictly valid in SS1 and SS1M. If however one replaces the material implications of this statement by strict ones then the resulting statement, *A LCpq LCpNq* is no longer valid in SS1 or SS1M; the values of its matrix are between 6 and 1, thus this statement is contingent in SS1 and SS1M (cf. **1.46**). There are a number of reasons which make it very probable that by the strong concept of consequence (**1.611**) almost all of the serious paradoxes of implication are excluded.

1.613 Material concept of consequence. The conclusion q follows materially from the premiss p iff at least one of the following conditions (a), (b), (c) or (d) are satisfied:

(a) as in **1.611** (a)

(b) as in **1.611** (b)

(c) as in **1.612** (c)

(d) For all coordinated pairs of values of the matrices of p and q: the value of the matrix of p is 4 or the value of the matrix of q is 3 (or both cases hold).

1.62 There is no need for any rule of derivation in SS1 or SS1M. The reason is this: From **1.081** it is clear that every sentence of SS1 (and also of SS1M) is determined by a certain matrix. All what one has to do in order to decide whether a sentence is a theorem or is not a theorem of SS1 or SS1M is to check the *cv* of its matrix (cf. **1.46** and **1.51**). (In complicated cases, if the compound sentences contain many different propositional variables, this can be done by a computor). Thus the answer to the question whether a sentence is or is not a theorem of SS1 or SS1M does not require to know whether this sentence follows from certain premisses or not, this answer can be given quite independently of such a knowledge, i.e. on grounds of the matrix (and the *cv*) of the sentence in question. On the other hand the concept of consequence as defined in **1.6-1.613** is not superfluous.

With its help one can decide which sentences are premisses of other sentences or which sentences are conclusions of other sentences in SS1 and SS1M. Also one can compare the consequence class of a sentence in any kind of propositional calculus with that of a sentence in SS1 or SS1M.

1.63 Consequence class (Cl).[10] To each of the three distinguished concepts of consequence there are three corresponding concepts of consequence-class in SS1 and SS1M. **1.631** Classes (or sets) of sentences are viewed in SS1 and SS1M as conjunctions of sentences. Thus it follows from **1.081** that every class (set) of sentences of SS1 or SS1M is determined by a certain matrix. Applied to classes of premisses (Pr) and classes of consequences (Cl) in SS1 and SS1M this means that any Pr and any Cl is determined by a certain matrix.

1.632 Strong consequence class. The strong consequence class Cl_1 of a set of premises Pr is the set of all sentences which are satisifed by the matrix $Mat = <T, F, C_T>$ where $T = \{1,2,3\}$, $F = \{4,5,6\}$ and C_T is defined by the following set (a) of formulas in which (. . . , . . .) is a coordinated value pair (a coordinated value pair consists of two different values of matrices which are on the same line; cf. 1.3 the rows under K and q), one value belonging to the matrix of Pr, the other belonging to the matrix of Cl_1:

(a) $C_T(1,1) = C_T(2,2) = C_T(3,3 = C_T(4,4) = C_T(5,5) = C_T(6,6) = C_T(2,1) =$
$= C_T(3,1) = C_T(4,1) = C_T(5,1) = C_T(6,1) = C_T(6,2) = C_T(6,3) = C_T(6,4) =$
$C_T(6,5) = 1$

1.633 Strict consequence class. The strict consequence class Cl_2 of a set of premisses Pr is the set of all sentences which are satisfied by the matrix $Mat = <T,F,C_T>$ where $T = \{1,2,3\}$, $F = \{4,5,6\}$ and C_T is defined by the following sets (a) and (b) of formulas in which (. . . , . . .) is a coordinated value pair, one value belonging to the matrix of Pr, the other to the matrix of Cl_2:

(a) as in **1.632** (a)
(b) $C_T(1,2) = C_T(3,2) = C_T(4,2) = C_T(5,2) = C_T(5,3) = C_T(5,4) = C_T(5,6) = 2$

1.634 Material consequence class. The material consequence class Cl_3 of a set of premisses Pr is the set of all sentences which are satisfied by the matrix $Mat = <T,F,C_T>$ where $T = \{1,2,3\}$, $F = \{4,5,6\}$ and C_T is defined by the following sets (a), (b) and (c) of formulas in which (. . . , . . .) is a coordinated value pair, one value belonging to the matrix of Pr, the other to the matrix of Cl_3:

(a) as in **1.632** (a)
(b) as in **1.633** (b)
(c) $C_T(1,3) = C_T(2,3) = C_T(4,3) = C_T(4,5) = C_T(4,6) = 3$

1.7 Consistency of SS1 and SS1M. **1.71** A sentence (of SS1 or SS1M) is a theorem (of SS1 or SS1M) iff it is logically true (or: valid) in SS1 or SS1M (cf. 1.41). In other words: A sentence is a theorem iff its cv is either 1 or 2 of 3. **1.72** A system is consistent iff it contains no sentence such that both the sentence and its negation are provable as theorems within it[11].

1.73 By the help of **1.71** and **1.72** it is easily seen that SS1 and SS1M are consistent:
Case 1: The matrices of the sentences in question consist of values 1 exclusively or 2 exclusively or 3 exclusively. Then these sentences are theorems of SS1 and SS1M (according **1.71**). The matrices of their negations however consist then of the values 6 exclusively, 5 exclusively or 4 exclusively as it is clear from the definition of the operation of negation (N) in **1.08.** Thus according to **1.71** these negations cannot be theorems of SS1 or SS1M because their cv is higher than 3.
Case 2: The matrices of the sentences in question consist of the values 1 and 2, 1 and 3, 2 and 3 or 1, 2 and 3 which are mixed up. Then these sentences are theorems of SS1 and SS1M (**1.71**). The matrices of their negations however consist then of the values 6 and 5, 6 and 4, 5 and 4 or 6, 5 and 4 respectively mixed up (**1.08**). Thus according to **1.71** these negations cannot be theorems of SS1 or SS1M because their cv is higher than 3.
Case 1 and 2 cover all different distributions of the values in the matrices of theorems of SS1 and SS1M. Thus the consistency proof is completed.

1.74 SS1 and SS1M are consistent (**1.71-1.73**).

1.8 Decision procedure for SS1 and SS1M. **1.81** A method which suffices to answer, either by "yes" or by "no", the question whether or not any sentence of SS1 or SS1M is a theorem (**1.71**) of SS1 or SS1M is called a decision precedure (or: decision method, or: algorithm) for SS1 or SS1M[12].

1.82 A decision procedure for any sentence of SS1 and SS1M is afforded by the process of calculating the matrix of the sentence in question for its cv (characteristical value): The sentence is a theorem iff its cv is either 1 or 2 or 3. The sentence is not a theorem iff its cv is higher than 3 (i.e. either 4 or 5 or 6); in this case the sentence may be either provable contingent (cf. **1.46**) or provable false (cf. **1.46**). The sentence is provable false iff the lowest value of its matrix is either 4 or 5 or 6. The sentence is provable contingent iff its matrix contains at least one value between 1 and 3 and at least one value between 4 and 6. **1.83** Such a decision procedure has been carried out by an electronic computer for a great number of sentences and their variations (cf. **1.9**) of SS1 and SS1M. These sentences and their variations are given in chapters 2 and 3. The results of the decision procedure for a certain sentence (or variation of it) is given by stating its cv. Iff the sentence (or variation of it) is provable contingent then this result is given by writing a 'k' instead of a certain value-number (cf. **2.24**).

1.84 Completeness-proofs for SS1 and SS1M (also one with respect to classical propositional calculus) are given in **3.06**.

1.9 Variations of formulas (sentences). A formula p becomes a variation of p (in SS1M: a modal variation of p) iff any of the operations L, LL, ML, M, MM, LM is applied to either its atomic formulas or to its two-place operations A, K, C, E. In producing the variations (in order to decide whether they are theorems or not) in the following chapters the operation L, LL, ML, M, MM, LM have been applied always in the order just stated.

1.91 Not-separated variations of the atomic formulas of a formula (VA). A formula p becomes a VA (of p) iff one of the operations L, LL, ML, M, MM, LM (abbreviated: L-LM) is applied to all of its atomic formulas at the same time.

Example: VA of $CpCqp$ are: $CLpCLqLp$, $CLLpCLLqLLp$ . . . etc. It is clear that there are exactly six VA of $CpCqp$ as for any other formula.

1.92 Separated variations of the atomic formulas of a formula (VAG). A formula p becomes a VAG (of p) iff one of the operations L-LM is applied exactly and at the same time to all of its atomic formulas which have the same index.

Example: VAG of $CKp_2Cp_1q_1q_2$ are: $CKLp_2Cp_1q_1Lq_2$, $CKLLp_2Cp_1q_1LLq_2$ . . . etc. The application of the operations L-LM begins always with these atomic formulas which have the highest index (in this special case the index 2) and continues to the lower index in such a way that for every application of an operation (say L) to the p_1 and q_1 all the six VAG of p_2 and q_2 are carried through. The following table shows this (in the example discussed the number of VAG is 49):

C	K	p_2	C	p_1	q_1	q_2
		Lp_2				Lq_2
		.				
		.				
		LMp_2				LMq_2
				Lp_1	Lq_1	
		Lp_2				Lq_2
		.				.
		.				.
		LMp_2				LMp_2
				LLp_1	LLq_1	
				.	.	
				.	.	
				.	.	
				LMp_1	LMq_1	

1.93 Variations of the two-place operations (A, K, C, E) of a formula (VO). A formula p becomes VO (of p) iff one of the operations L-LM is applied to all of its two-place operations of the same kind at the same time (i.e. to all operations A at the same time, to all operations K at the same time . . . etc.). The order of application is such that the first two-place operation-sign on the left of a formula (i.e. this operation-sign which is the main-connective)—together with the operation-signs of the same kind—is the last in the order of application.

Example: VO of $CKpCpqq$ are: $CLKpCpqq$, $CLLKpCpqq$ . . . etc. $LCKpLCpqq$, $LCLKpLCpqq$— $LCLMKpLCpqq$, $LLCKpLLCpqq$ . . . etc.

1.94 *VO+VA* variations. A formula becomes a *VO+VA* (of p) iff to a certain *VO* of p a *VA* variation is applied. To every *VO* six *VA* variations belong.

Example: *VO+VA* of $CpApq$ are:

$$CpLApq,\ CLpLALpLq,\ CLLpLALLpLLq \ldots \text{etc.}$$
$$CpLLApq,\ CLpLLALpLq,\ CLLpLLALLpLLq \ldots \text{etc.}$$

$$LCpApq,\ LCLpALpLq \ldots \text{etc.}$$
$$LCpLApq,\ LCLpLALpLq \ldots \text{etc.}$$

(in this example the number of *VO+VA* is 336; i.e. 343 minus 7 *VA*).

1.95 Separated variations of the two-place operations (A,K,C,E) of a formula (*VOG*). A formula p becomes a *VOG* (of p) iff one of the operations *L-LM* is applied exactly and at the same time to all of its two-place operations A,K,C,E which have the same index. The application of the operations *L-LM* begins always with these operations A,K,C,E which have the highest index and continues to the lower index ending up with the index 1 which is always given to the main-connective (i.e. the first two-place operation-sign on the left of a formula). The order of application is shown by the following table

Example: *VOG* of $C_1C_2pC_4qrC_2K_3pqr$

C_1	C_2p	C_4qr	C_2	K_3pqr
		LC_4		
		LLC_4		
		.		
		.		
				LK_3
		LC_4		
		.		
		.		
				LLK_3
				.
				.
	LC_2		LC_2	
		LC_4		
		.		
		.		
				LK_3
				.
				.
	LLC_2		LLC_2	
		LC_4		
		.		
		.		

LK_3
.
.
. .
. .
LC_1
LC_4
.
.
LK_3
.
.
LC_2 LC_2
. .
. .
LLC_1
.
.

Note: In order to decide the *cv* of the *VOG* variations of the main-connective—in the above example C_1—which has always the index 1 (if there are indices at all) it is not necessary to calculate through these variations (namely: LC_1, $LLC_1 \ldots LMC_1$). The *cv* of it can be recognized easily with with the help of the statements of **1.53**.

1.96 *VOG+VA* variations. A formula becomes a *VOG+VA* (of p) iff to a certain *VOG* of p a *VA* variation is applied. To every *VOG* six *VA* variations belong.

Example: *VOG+VA* of C_1pC_2qp are: C_1pLC_2qp, C_1LpLC_2LqLp, $C_1LLpLC_2LLqLLp \ldots$ etc.
C_1pLLC_2qp, C_1LpLLC_2LqLp, $C_1LLpLLC_2LLqLLp \ldots$ etc.

2. Basic laws of SS1 and SS1M

2.1 Identity and Negation

			characteristical value (*cv*)
2.11	$E\ p\ \ p$[13]	principle of identity (in prop. calc.)	1
(7)	all *VA*		1
2.12	$E\ NNp\ \ p$	double negation	1
(7)	all *VA*		1
2.13	$E\ NKpq\ \ ANpNq$	De Morgan's law	1
(7)	all *VA*		1
2.131	$E\ Kpq\ \ NANpNq$	De Morgan's law	1
(7)	all *VA*		1
2.132	$E\ NApq\ \ KNpNq$	De Morgan's law	1
(7)	all *VA*		1
2.133	$E\ Apq\ \ NKNpNq$	De Morgan's law	1
(7)	all *VA*		1

			cv
2.14	*N KpNp*	principle of non-contradiction	1
(7)	all *VA*		1
2.141	*C KqNq p*		1
2.15	*Ap Np*	tertium non datur	1
(7)	all *VA*		1
2.151	*C p AqNq*		1
2.16	*E Lp NMNp*		1
2.161	*E NLp MNp*		1
2.162	*E Mp NLNp*		1
2.163	*E NMp LNp*		1
2.17	*E LLp NMMNp*		1
2.171	*E NLLp MMNp*		1
2.172	*E MMp NLLNp*		1
2.173	*E NMMp LLNp*		1
2.18	*E MLp NLMNp*		1
2.181	*E NMLp LMNp*		1
2.182	*E LMp NMLNp*		1
2.183	*E NLMp MLNp*		1
2.19	*C LLp Lp*		1
2.191	*C Lp MLp*		1
2.192	*C MLp p*		2
2.193	*C LLp p*		1
2.194	*C Lp p*		2
2.195	*C p LMp*		2
2.196	*C LMp Mp*		1
2.197	*C Mp MMp*		1
2.198	*C p Mp*		2
2.199	*C p MMp*		1
2.2	Conjunction		
2.21	*E p Kpp*		1
(7)	all *VA*		1
2.211	*E p KpApq*		1
2.212	*E p KpAqNq*		1
2.22	*C Kpq p*		2
2.221	*C Kpq q*		2
2.23	*E Kpq Kqp*	commutation	1
(196)	all *VA* and *VO+VA*		1
2.24	*C KKpqr KpKqr*	association	2
(343)	all *VA*		1
(314)	the *VO LLK, MLK*[14]		1
	the *VO LK, MK, MMK, LMK*; the *VO LLC . . . K*		*k*[15]
	all *VA* of *VO LC-LMC*		1
2.241	*E KKpqr KpKqr*	association	2
2.25	*C KpAqr AKpqKpr*	distribution	1
2.251	*C AKpqKpr KpAqr*	distribution	2

			cv
2.252	*E KpAqr AKpqKpr*	distribution	2
2.26	*E LLKpq KLLpLLq*	distribution of modalities	1
2.261	*C LLKpq LKpq*	distribution of modalities	1
2.262	*E LKpq KLpLq*		1
2.263	*C LKpq MLKpq*		1
2.264	*E MLKpq KMLpMLq*		1
2.265	*C MLKpq Kpq*		2
2.266	*C LKpq Kpq*		2
2.267	*C LLKpq Kpq*		1
2.27	*C Kpq LMKpq*		2
2.271	*C LMKpq KLMpLMq*		1
2.272	*C KLMpLMq KMpMq*		1
2.273	*C LMKpq MKpq*		1
2.274	*C MKpq KMpMq*		1
2.275	*C KMpMq KMMpMMq*		1
2.276	*C MKpq MMKpq*		1
2.277	*C MMKpq KMMpMMq*		1
2.278	*C Kpq KLMpLMq*		2
2.2781	*C Kpq MKpq*		2
2.2782	*C Kpq KMpMq*		2
2.279	*C Kpq MMKpq*		1
2.2791	*C Kpq KMMpMMq*		1
2.28	*C LKpq Lp*		1
	All variations which arise from replacing both of the operations *L* in **2.28** by *LL, ML, M, MM, LM*		1
2.3	Disjunction		
2.31	*E p App*		1
2.311	*C App p*		1
(196)	all *VA*		1
(161)	the *VO LA, MLA*		2
	all *VA* of *LA* and *MLA*		1
	all *VO+VA* of *LLA*		1
	the *VO MA, MMA, LMA*		*k*
	its *VA Mp*[16]		*k*
	its *VA Lp*		3
	all other *VA*		1
2.312	*E p ApKpq*		1
2.313	*E p ApKqNq*		1
2.32	*C p Apq*	addition	2
(196)	all *VA*		1
(160)	the *VO LA, LLA, MLA*		*k*
	its *VA Lp*		*k*
	its *VA Mp*		3
	the *VO MA, LMA*		2
	the *VO MMA*		1
	the *VA* of *LMA, MMA*		1

			cv
2.321	$C\ q\ Apq$		2
2.33	$E\ Apq\ Aqp$	commutation	1
(196)	all VA and $VO+VA$		1
2.34	$E\ ApAqr\ AApqr$	association	2
2.35	$C\ KApqApr\ ApKqr$	distribution	1
2.351	$C\ ApKqr\ KApqApr$		2
2.352	$E\ KApqApr\ ApKqr$		2
2.36	$C\ ALLpLLq\ LLApq$	distribution of modalities	1
2.361	$C\ LLApq\ LApq$	distribution of modalities	1
2.362	$C\ ALLpLLq\ ALpLq$		1
2.363	$C\ ALpLq\ LApq$		1
2.364	$C\ LApq\ MLApq$		1
2.365	$C\ ALpLq\ AMLpMLq$		1
2.366	$C\ AMLpMLq\ MLApq$		1
2.367	$C\ MLApq\ Apq$		2
2.368	$C\ ALLpLLq\ Apq$		1
2.3681	$C\ LLApq\ Apq$		1
2.3682	$C\ ALpLq\ Apq$		2
2.3683	$C\ LApq\ Apq$		2
2.3684	$C\ AMLpMLq\ Apq$		2
2.37	$C\ Apq\ LMApq$		2
2.371	$E\ LMApq\ ALMpLMq$		1
2.372	$C\ LMApq\ MApq$		1
2.373	$E\ MApq\ AMpMq$		1
2.374	$C\ MApq\ MMApq$		1
2.375	$E\ MMApq\ AMMpMMq$		1
2.376	$C\ Apq\ MApq$		2
2.377	$C\ Apq\ MMApq$		2
2.38	$C\ Lp\ LApq$		1
	All variations which arise from replacing both of the operations L in **2.38** by LL, ML, M, MM, LM		1

2.4 Implication and Equivalence

2.41	$E\ Epq\ Eqp$		1
(49)	all VA and $VO+VA$		1
2.411	$E\ Epq\ ENpNq$		1
(49)	all VA and $VO+VA$		1
2.412	$C\ Epq\ Cpq$		1
(7)	all VA		1
2.4121	$C\ Epq\ Cqp$		1
(7)	all VA		1
2.413	$E\ Cpq\ ANpq$		1
2.414	$E\ Cpq\ NKpNq$		1
2.4141	$E\ NCpq\ KpNq$		1
2.415	$E\ Cpq\ CNqNp$	transposition	1
(49)	all VA and $VO+VA$		1

				cv
2.416	*E*	*CpNq CqNp*		1
(49)		all *VA* and *VO+VA*		1
2.4161	*E*	*CNpq CNqp*		1
(49)		all *VA* and *VO+VA*		1
2.417	*E*	*p CNpp*		1
(7)		all *VA*		1
2.4171	*E*	*Np CpNp*		1
(7)		all *VA*		1
2.42	*C*	*q Cpq*		2
2.421	*C*	*Np Cpq*		2
2.422	*C*	*Kpq Epq*		2
2.423	*C*	*KNpNq Epq*		2
2.424	*C*	*Cpq EpKpq*		1
2.4241	*C*	*EpKpq Cpq*		2
2.4242	*E*	*Cpq EpKpq*		2
2.425	*A*	*Cpq Cqp*		2
2.426	*A*	*Cpq CpNq*		2

The corresponding sentences with strict or strong implication i.e. *ALCpqLCqp*, *ALCpqLCpNq* and *ALLCpqLLCqp*, *ALLCpqLLCpNq* are not valid in SS1 (and SS1M).

2.427	*C*	*Cpq CKrpKrq*	factor-theorem	2
(196)		all *VA*		1
(174)		the *VO LK, LLK, MLK*		*k*
		its *VA Lp*		*k*
		its *VA Mp*		3
		the *VO MK, MMK, LMK*		*k*
		its *VA Lp*		3
		its *VA Mp*		*k*
		all other *VA*		1
		the *VO+VA LC . . . MLK*	*LC . . . LMK*[17]	1
		all other *VO LC*		*k*
		its *VA*		1
		the *VO LLC . . . K*		*k*
		its *VA*		1
		all other *VO+VA LLC*		1
		the *VO+VA MLC . . . K, MLC . . . MLK, MLC . . . LMK*		1
		all other *VO MLC*[18]		*k*
		its *VA*		1
2.428	*C*	*Cpq CArpArq*	factor theorem	2
(196)		Variations: the same distribution of *cv* as in **2.427** with		
(174)		'*A*' for '*K*'		
2.43	*E*	*LCpq LANpq*		1
2.431	*E*	*NLCpq MKpNq*		1
2.432	*E*	*LCpq LNKpNq*		1
2.433	*E*	*Lp LCNpp*		1

			cv
2.4331	*E* *LNp* *LCpNp*		1
(6)(6)	All variations which arise from replacing both of the operations *L* in **2.433** and **2.4331** by *LL*, *ML*, *M*, *MM*, *LM*		1
2.4332	*E* *Lp* *LEpApN*$_{t'}$		1
2.4333	*E* *LNp* *LEpKpNp*		1
(6)(6)	All variations which arise from replacing both of the operations *L* in **2.4332** and **2.4333** by *LL*, *ML*, *M*, *MM*, *LM*		1
2.4334	*E* *Lq* *KLCpqLCNpq*		1
2.4335	*E* *LNp* *KLCpqLCpNq*		1
(3)(3)	All variations which arise from replacing all three operations *L* in **2.4334** and **2.4335** by *LL* and *ML*		1
2.434	*C* *Lq* *LCpq*		1
2.4341	*C* *LNp* *LCpq*		1
2.435	*C* *LKpq* *LEpq*		1
2.4351	*C* *LKNpNq* *LEpq*		1
2.436	*A* *MCpq* *LCpNq*		1
2.4361	*A* *LCpq* *MCpNq*		1
2.4362	*E* *NLCpq* *MKpNq*		1
2.4363	*C* *NMCpq* *LCpNq*		1
2.4364	*C* *NLCpq* *MCpNq*		1
2.437	*C* *LCMpLq* *LCpq*	law of I. Thomas	1
2.4371	*C* *LCMpLq* *Cpq*		1
2.4372	*C* *CMpLq* *Cpq*		2
2.45	*C* *LLCpq* *LLCLpLq*	distribution of	1
2.4501	*E* *LLCLpLq* *LCLpLq*	modalities	1
2.451	*C* *LCLpLq* *LLCLLpLLq*		1
2.4511	*E* *LLCLLpLLq* *LCLLpLLq*		1
2.4512	*E* *LCLLpLLq* *CLLpLLq*		1
2.452	*C* *LCLpLq* *CLpLq*		1
2.453	*C* *LCLpLq* *LLCMLpMLq*		1
2.4531	*E* *LLCMLpMLq* *LCMLpMLq*		1
2.4532	*E* *LCMLpMLq* *CMLpMLq*		1
2.454	*C* *LLCpq* *LCpq*		1
2.4541	*C* *LCpq* *CLpLq*		1
2.4542	*C* *LCpq* *Cpq*		2
2.4543	*C* *LCpq* *LLCMLpMLq*		1
2.455	*C* *LLCpq* *LLCMpMq*		1
2.4551	*E* *LLCMpMq* *LCMpMq*		1
2.456	*C* *LCMpMq* *LLCMMpMMq*		1
2.4561	*E* *LLCMMpMMq* *LCMMpMMq*		1
2.4562	*E* *LCMMpMMq* *CMMpMMq*		1
2.456	*C* *LCMpMq* *CMpMq*		1
2.458	*C* *LCMpMq* *LLCLMpLMq*		1
2.4581	*E* *LLCLMpLMq* *LCLMpLMq*		1
2.4582	*E* *LCLMpLMq* *CLMpLMq*		1

				cv
2.459	$C\ LCpq\ CMpMq$			1
2.4591	$C\ LCpq\ LLCLMpLMq$			1

Note: The following propositional-logical forms of Becker's rules are not valid in SS1 (and SS1M):

$C\ LCpq\ LCLpLq$ $C\ LCpq\ LCMpMq$

$C\ Cpq\ CLpLq$ $C\ Cpq\ CMpMq$

			cv
2.46	$C\ KCpqCqr\ Cpr$	hyp. syllogism	2
(196)	all *VA*		1
(184)	the *VO LK, MLK*		2
	its *VA*		1
	the *VO+VA LLK*		1
	the *VO MK, MMK, LMK*		*k*
	its *VA Lp, Mp*		*k*
	its *VA LLp, MLp, MMp, LMp*		1
	the *VO+VA LC . . . K, LC . . . LK, LC . . . LLK, LC . . . MLK*		1
	the *VO LC . . . MK, LC . . . MMK, LC . . . LMK*		*k*
	its *VA*		1
	the *VO+VA LLC* and *MLC . . . K-LMK*		1
2.461	$C\ Cpq\ CCqrCpr$		2
(28)	all *VA*		1
(27)	the *VO LC*		*k*
	its *VA*		1
	the *VO+VA LLC* and *MLC*		1
2.462	$C_1\ K_3C_2pqC_2pr\ C_2pK_3qr$	distribution	1
(196)	all *VA*		1
(155)	all *VOG* K_3 (i.e. LK_3, LLK_3 . . .)		*k*
	its *VA Lp*		*k*
	its *VA Mp*		3
	its other *VA* (i.e. *LLp, MLp, MMp, LMp*)		1
	the *VOG+VA* LC_2 . . . *K*; LLC_2 . . . *K*; MLC_2 . . . *K*		1
	the *VOG* LC_2 . . . *LK, LLK, MLK*; LLC_2 . . . *LK, LLK, MLK*; MLC_2 . . . *LK, LLK, MLK*		*k*
	its *VA Lp* *Mp*		*k*
	its other *VA*		1
	the *VOG* LC_2 . . . *MK, MMK, LMK*		3
	its *VA*		1
	the *VOG* LLC_2 . . . *MK, LMK*		*k*
	its *VA*		1
	the *VOG+VA* LLC_2 . . . *MMK*		1
	the *VOG+VA* MLC_2 . . . *MK, MMK, LMK*		1
(588)	the *VOG+VA* LC_1, LLC_1, MLC_1 have the same distribution		
(438)	of *cv* with exception of the *cv* 3 which turn into *k*		
2.4621	$E_1\ K_3C_2pqC_2pr\ C_2pK_3qr$	distribution	2
(196)	all *VA*		1
(126)	the *VOG* LK_3–LMK_3		*k*
	its *VA Lp*		*k*

		cv
	the *VA Mp* of the *VOG* LK_3, LLK_3, MLK_3	*k*
	the *VA Mp* of the *VOG* MK_3, MMK_3, LMK_3	3
	the other *VA LLp, MLp, MMp, LMp*	1
	the *VOG* $LC_2 \ldots K_3$	3
	its *VA*	1
	the *VOG* $LLC_2 \ldots K_3$	*k*
	its *VA*	1
	the *VOG+VA* MLC_2	1
	the *VOG* $LC_2 \ldots LK_3$, $LLK_3 \ldots LMK_3$; $LLC_2 \ldots LK_3$, $LLK_3 \ldots LMK_3$; $MLC_2 \ldots LK_3$, $LLK_3 \ldots LMK_3$	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
(196)	the *VOG+VA* LE_1 have the same distribution of *cv*	
(122)	as in **2.4621** with the following exceptions:	
	the *cv* 2 turn into 3 and the *cv* 3 into *k*	
(196)	the *VOG+VA* LLE_1: distribution as in **2.4621**, except:	
(121)	the *cv* 2 and 3 turn into *k*	
(196)	the *VOG+VA* MLE_1: distribution as in **2.4621**, except:	
(122)	the *cv* 2 turn into 1, the *cv* 3 into *k*.	
2.463 *C*	*KCprCqr CApqr* distribution	1
(1372)	all *VA*	1
(1071)	the *VO LK, MLK*	2
	its *VA*	1
	the *VO+VA LLK*	1
	the *VO MK, MMK, LMK*	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VO LA . . . K, LK, LLK, MLK; MLA . . . K, LK, LLK, MLK; LLA . . . LK; LLA . . . MLK*	2
	its *VA*	1
	the *VO+VA LLA . . . K, LLA . . . LLK*	1
	the *VO LA . . . MK, MMK, LMK; LLA, MLA* with the same; *MA . . . K, LK, LLK, MLK; MMA, LMA* with the same;	*k*
	its *VA Lp*	3
	its *VA Mp*	*k*
	its other *VA*	1
	the *VO MA . . . MK, MMK, LMK; MMA, LMA* with the same;	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VO+VA LC . . . K, LK, LLK, MLK; LC . . . LLA . . . K, LK, LLK, MLK*	1
	the *VO LC . . . MK, MMK, LMK; LC . . . LLA . . . MK, MMK, LMK*	*k*
	its *VA*	1
	the *VO LC . . . LA, MLA, MA, MMA, LMA* (with all *K*-var.)	*k*
	the *VA* of *VO LC . . . LA, MLA*	1
	the *VA Lp, Mp* of *VO LC . . . MA, MMA, LMA*	*k*

		cv
	the other *VA* of *VO LC . . . MA, MMA, LMA*	1
	the *VO+VA LLC . . . A, LLA* (with all *K*-var. i.e. *K-LMK*)	1
	the *VO LLC . . . LA, MLA, MA, MMA, LMA . . . K-LMK*	*k*
	the *VA* of *VO LLC . . . LA, MLA*	1
	the *VA Lp, Mp* of *VO LLC . . . MA, MMA, LMA*	*k*
	the other *VA* of *VO LLC . . . MA, MMA, LMA*	1
	the *VO+VA MLC . . . A, LA, LLA, MLA . . . K-LMK*	1
	the *VO MLC . . . MA, MMA, LMA . . . K-LMK*	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
2.464	$C_1\ C_2pC_4qr\ \ C_2K_3pqr$ importation	2
(784)	all *VA*	1
(639)	the *VOG* LC_4, LLC_4, MLC_4; LK_3, MLK_3 (with all C_4-var.)	2
	its *VA*	1
	the *VOG* $LLK_3 \dots C_4$, $LLK_3 \dots LLC_4$	1
	the *VOG* MK_3, MMK_3, $LMK_3 \dots C_4$-MLC_4	*k*
	its *VA Lp*	3
	its *VA Mp*	*k*
	its other *VA*	1
	the *VOG* $LC_2 \dots K_3$, LK_3, $MLK_3 \dots C_4$-MLC_4	3
	its *VA*	1
	the *VOG+VA* $LC_2 \dots LLK_3 \dots LC_4$, LLC_4	1
	the *VOG* $LC_2 \dots LLK_3 \dots MLC_4$	3
	its *VA*	1
	the *VOG* $LC_2 \dots MK_3$, MMK_3, $LMK_3 \dots C_4$-MLC_4	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VOG* $LLC_2 \dots K_3$, LK_3, ($LLK_3 \dots MLC_4$), $MLK_3 \dots C_4$- MLC_4	*k*
	its *VA*	1
	the *VOG+VA* $LLC_2 \dots LLK_3 \dots C_4$, LC_4, LLC_4	1
	the *VOG* $LLC_2 \dots MK_3$, MMK_3, $LMK_3 \dots C_4$-MLC_4	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VOG+VA* $MLC_2 \dots K_3$, LK_3, LLK_3, $MLK_3 \dots C_4$-MLC_4	1
	the *VOG* $MLC_2 \dots MK_3$, MMK_3, $LMK_3 \dots C_4$-MLC_4	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
(2352)	the variations LC_1, LLC_1, MLC_1: distribution	
(1829)	of *cv* as in **2.464** with slight differences (cf. the end of **2.465**)	
2.465	$C_1\ C_2K_3pqr\ \ C_2qC_4pr$ exportation	2
(784)	all *VA*	1
(567)	the *VOG* LC_4, LLC_4, MLC_4	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VOG* LK_3, LLK_3, $MLK_3 \dots C_4$-MLC_4	*k*
	its *VA Lp*	*k*

		cv
	its VA Mp	3
	its other VA	1
	the VOG $MK_3 \ldots C_4$; $MMK_3 \ldots C_4$; $LMK_3 \ldots C_4$	2
	its VA	1
	the VOG $MK_3 \ldots LC_4$, LLC_4, MLC_4; MMK_3, LMK_3 with the same	k
	its VA Lp	k
	its VA Mp	3
	its other VA	1
	the VOG $LC_2 \ldots C_4$; $LC_2 \ldots MK_3$, MMK_3, $LMK_3 \ldots C_4$	3
	its VA	1
	the VOG $LC_2 \ldots LC_4$, LLC_4, MLC_4; $LC_2 \ldots LK_3$, LLK_3, MLK_3	k
	its VA Lp, Mp	k
	its other VA	1
	the VOG $LC_2 \ldots MK_3$, MMK_3, $LMK_3 \ldots LC_4$, LLC_4, MLC_4	k
	its VA	1
	the VOG LLC_2; $LLC_2 \ldots LK_3$, LLK_3, $MLK_3 \ldots C_4$-MLC_4	k
	its VA Lp, Mp	k
	its other VA	1
	the VOG $LLC_2 \ldots MK_3$, MMK_3, $LMK_3 \ldots C_4$-MLC_4	k
	its VA	1
	the VOG+VA $MLC_2 \ldots C_4$; $MLC_2 \ldots MK_3$, MMK_3, $LMK_3 \ldots C_4$	1
	the VOG $MLC_2 \ldots LC_4$, LLC_4, MLC_4; $MLC_2 \ldots LK_3$, LLK_3, $MLK_3 \ldots C_4$-MLC_4	k
	its VA Lp, Mp	k
	its other VA	1
	the VOG $MLC_2 \ldots MK_3$, MMK_3, $LMK_3 \ldots LC_4$, LLC_4, MLC_4	k
	its VA	1
(784)	the VOG+VA LC_1: distribution as in **2.465**, except:	
(551)	the cv 2 turn into 3, the cv 3 turn into k	
(784)	the VOG+VA LLC_1: distribution as in **2.465**, except:	
(547)	the cv 2 and 3 turn into k	
(784)	the VOG+VA MLC_1: distribution as in **2.465**, except:	
(551)	the cv 2 turn into 1, the cv 3 into k	
2.466	$C_1\ C_2K_3pqr\ \ C_2K_3qpr$	1
(196)	all VA	1
	all VOG+VA C_2-$MLC_2 \ldots K_3$-LMK_3	1
(196)	all VOG+VA LC_1	1
(196)	all VOG+VA LLC_1	1
(196)	all VOG+VA MLC_1	1
2.467	$C_1\ C_2K_3pqr\ \ C_2K_3NrpqNp$	2
(196)	all VA	1
(147)	the VOG LK_3-LMK_3	k
	its VA Lp	k
	its VA Mp	3
	its other VA	1

		cv
	the *VOG* $LC_2 \ldots K_3$	3
	its *VA*	1
	the *VOG* LC_2, LLC_2, MLC_2, . . . LK_3, LLK_3, MLK_3	*k*
	its *VA* *Lp*, *Mp*	*k*
	its other *VA*	1
	the *VOG* LC_2, LLC_2, MLC_2, . . . MK_3, MMK_3, LMK_3	*k*
	its *VA*	1
	the *VOG* $LLC_2 \ldots K_3$	*k*
	its *VA*	1
	the *VOG+VA* $MLC_2 \ldots K_3$	1
(196) (140)	the *VOG+VA* LC_1: the *cv* 2 turn into 3, the *cv* 3 into *k*	
(196) (139)	the *VOG+VA* LLC_1: the *cv* 2 and 3 turn into *k*	
(196) (140)	the *VOG+VA* MLC_1: the *cv* 2 turn into 1, the *cv* 3 into *k*	

2.47 Valid variations of the (propositional-logical form of) modus ponens in SS1 and SS1M

2.471 *C K .Cpq -*

p	*MMq*	1
p	*q*, *Mq*, *LMq*	2
Lp	*q*, *Mq*, *MMq*, *LMq*	2
LLp	*q*, *MMq*	1
LLp	*Mq*, *LMq*	2
MLp	*q*, *Mq*, *MMq*, *LMq*	2

2.472 *C K .LCpq -*

p	*q*, *Mq*, *MMq*, *LMq*	2
Lp	*Lq*, *MLq*, *Mq*, *MMq*, *LMq*	1
Lp	*q*	2
LLp	*Lq*, *MLq*, *Mq*, *MMq*, *LMq*	1
LLp	*q*	2
MLp	*MLq*, *Mq*, *MMq*, *LMq*	1
MLp	*q*	2
MLp	*Lq*	3
Mp	*Mq*, *MMq*	1
Mp	*LMq*	3
LMp	*Mq*, *MMq*, *LMq*	1

2.473 *C K .LLCpq -*

p	*q*, *MMq*	1
p	*Mq*, *LMq*	2
Lp	*Lq*, *MLq*, *Mq*, *MMq*, *LMq*	1
Lp	*q*	2
LLp	*Lq*, *LLq*, *MLq*, *q*, *Mq*, *MMq*, *LMq*	1
MLp	*MLq*, *Mq*, *MMq*, *LMq*	1
MLp	*q*	2
MLp	*Lq*	3

			cv
	Mp	Mq, MMq	1
	Mp	LMq	3
	MMp	MMq	1
	LMp	Mq, MMq, LMq	1

2.474 $C_1 K_2 p_2 C_3 p_1 q_1 \ q_2$ 2

(196) only the VAG of p_2 and q_2 are calculated here (not these
(125) of p_1 and q_1); this is indicated by writing 'VAG_2' instead of 'VAG'

	all VAG_2	k
	the VOG LC_3, MLC_3; $LK_2 \ldots C_3$, MLC; $LLK_2 \ldots MLC_3$; $MLK_2 \ldots C_3$, MLC_3	2
	its VAG_2 LLp, MMp	k
	its other VAG_2	1
	the VOG $LK_2 \ldots LC_3$, LLC_3; $MLK_2 \ldots LC_3$, LLC_3	2
	its VAG_2	1
	the $VOG+VAG_2$ LLC_3; $LLK_2 \ldots C_3$, LC_3, LLC_3	1
	the VOG MK_2, MMK_2, $LMK_2 \ldots C_3$	k
	its VAG_2	k
	the VOG MK_2, MMK_2, $LMK_2 \ldots LC_3$, MLC_3	k
	its VAG_2 Lp	3
	its VAG_2 LLp, Mp, MMp	k
	its other VAG_2	1
	the VOG MK_2, MMK_2, $LMK_2 \ldots LLC_3$	k
	its VAG_2 Lp	3
	its VAG_2 Mp	k
	its other VAG_2	1
(196)	the $VOG+VAG_2$ LC_1: distribution as in **2.474,** except:	
(110)	the cv 2 turn into 3, the cv 3 into k	
(196)	the $VOG+VAG_2$ LLC_1: distribution as in **2.474,** except:	
(98)	the cv 2 and 3 turn into k	
(196)	the $VOG+VAG_2$ MLC_1: distribution as in **2.474,** except:	
(110)	the cv 2 turn into 1, the cv 3 into k	

3. Investigations on the validity or invalidity of axiom systems of propositional calculus in SS1 and SS1M

3.0 Classical propositional calculus

3.01 Whitehead-Russell (PMt) (restriction: Bernays (AUA)) Hilbert-Ackermann (GZT)

3.011	$C \ App \ p$	(cf. **2.311**)	1
3.012	$C \ q \ Apq$	(cf. **2.32**)	2
(196)	all VA		1
(160)	the VO LA, LLA, MLA		k
	its VA Lp		k
	its VA Mp		3
	its other VA		1
	the VO MA, LMA		2

				cv
		its *VA*		1
		the *VO+VA MMA*		1
		The *cv* of *VO+VA LC, LLC, MLC* can easily be obtained with a method like at the end of **2.474**		
3.013	*C Apq Aqp*		(cf. **2.33**)	1
(294)		all *VO+VA* (also the *VO+VA LC, LLC, MLC, MC, MMC, LMC*)		1
3.014	*C Cpq CArpArq*		(cf. **2.428** and **2.427**)	2
3.02	*Nicod*, (RNP)			
3.021	*N KpNp*		(cf. **2.14**)	1
(49)		all *VO+VA*		1
3.022	*C Cpq CCqNrCpNr*			2
(28)		all *VA*		1
(27)		the *VO LC*		*k*
		its *VA*		1
		the *VO+VA LLC, MLC*		1
3.03	Łukasiewicz, (UAK)			
3.031	*C Cpq CCqrCpr*		(cf. **2.461**)	2
3.032	*C CNpp p*		(cf. **2.417**)	1
3.033	*C p CNpq*			2
(7)		the *VO LC, LLC, MLC, MC, MMC, LMC* (without *VA*-var.)		1
3.04	Rosser. (LMt)			
3.041	*C p Kpp*		(cf. **2.21**)	1
(49)		all *VA*		1
		all *VO+VA LC—LMC*		1
3.042	*C Kpq p*			2
(7)		all *VA*		1
3.043	*C Cpq CNKqrNKrp*			2

3.06 Completeness of SS1 and SS1M. **3.061** First sense of completeness: A system is complete (under a given interpretation), if a decision procedure enables us to prove in the system all the logically true (or: valid) propositions, i.e. all the theorems, which its formation rules enable us to express in the system.

3.062 SS1 and SS1M are complete in this first sense (**3.061**). The interpretation (for *T* and *F*) is given already in the definition of the system **1.08**. The formation rules for SS1 and SS1M are determined by the definition of the system in **1.08**, and from **1.82** it is clear that the required decision procedure for SS1 and SS1M exists.

3.063 Second sense of completeness: A system is complete (under a given interpretation), if the deductive postulates and substitution rules (or: the definitions of the concept of consequence) enable us to prove from any valid formula (sentence) of the system all the logically true (or: valid) propositions i.e. all the theorems, which its formation rules enable us to express in the system[19].

3.064 SS1 and SS1M are complete in this second sense (**3.063**). This can be seen in the following way:

1. The interpretation (for T and F) is given in **1.08**; the formation rules are determined by the definition of the system in **1.08**. The concept of consequence is defined in **1.6.**

2. It is clear from **1.41** and **1.71** that a valid formula of SS1 or SS1M has a *cv* not higher than 3 (i.e. either 1 or 2 or 3). Thus the *cv* of the premiss (being valid) is either 1 or 2 or 3. For any conclusion, drawn from such a premiss, which is valid in SS1 or SS1M the same holds. Thus all deductive situations which are required to cover all the cases for a complete derivation of all the theorems (which the formation rules enable us to express) of SS1 and SS1M have the following property: the *cv* of the matrix of the premisses (taken together in a conjunction) is either 1 or 2 or 3 and the *cv* of the matrix of the conclusion (or of more conclusions taken together in a conjunction) is also either 1 or 2 or 3. But this property is satisfied at least by the material concept of consequence as defined in **1.6.** Thus all deductive situations which are necessary for the completeness defined in **3.063** are satisfied (at least) by the material concept of consequence.

Note: The concepts of consequence in SS1 and SS1M allow for proof of every theorem of SS1 and SS1M, not only from certain axioms, but from any other valid sentence (or: theorem) of the system. Thus one can also say: All the theorems of SS1 and SS1M can be derived from the principle of noncontradiction or from the tertium non datur or from the principle of identity *Epp* and so on. The kind of concept of consequence (material, strict, strong) which is used depends on the *cv* of the premiss and the conclusion as is clear from **1.611-1.613.** But nevertheless it should be remembered that in order to decide whether a sentence of SS1 or SS1M is a theorem (in SS1 or SS1M) or not it is not necessary at all to use derivation and the concept of consequence. This question can be answered always independently by calculating the *cv* of the matrix of the sentence in question (cf. **1.82**). In other words one could say: All the theorems of SS1 and SS1M follow from the empty set of sentences[20].

3.065 Third sense of completeness: A system is complete (under a given interpretation), if a decision procedure enables us to prove in the system all the valid propositions of the classical propositional calculus (CPC).

3.066 SS1 and SS1M are complete in this third sense (**3.065**). This can be shown in two steps: The intepretation (for T and F) is given in **1.08**; the decision procedure is described in **1.82.**

1. The sentences which the formation rules of the CPC enable us to express form a subset of the sentences which the formation rules of SS1, SS1M enable us to express. This can be shown by comparison of the definitions of the systems which are constructed by the matrix method. The definition of SS1 (and—if the interpretation for L and M is made—of SS1M) is given in **1.08.** The definition of CPC with the help of the matrix method is due to Tarski (UAK)[21]. It can be reformulated (making some nonessential changes of numbers and letters and taking instead of the operations C and N the operations A and N) as this: The classical propostional calculus (ordinary system of sentential calculus) is the set of all sentences which are satisfied by the matrix $Mat = \langle T,F,N,A \rangle$ where $T =$

$\{1\}$, $F=\{6\}$ and the operations N and A are defined by the following formulas:

$$N(1) = 6;\ N(6) = 1$$
$$A(1,1) = A(1,6) = A(6,1) = 1;\ A(6,6) = 6.$$

It is easily seen from the definition in **1.08** (which can be considered as the formation rules of SS1 and SS1M), and from the definition just given that the sentences (well-formed formulas) of the CPC form a subset of the sentences of SS1 or SS1M.

2. Every sentence which is valid in the CPC is also (provable) valid in SS1 and SS1M. In other words: every sentence which has the value T (i.e. 1) in the CPC has the value T (i.e. either 1 or 2 or 3) in SS1 and SS1M. This can be shown by a comparison of the matrices of CPC and SS1 or SS1M. The matrices for N, A, K, C, E of SS1 (or SS1M) contain the matrices for N, A, K, C, E of CPC respectively: The matrix N of SS1 contains the matrix N of CPC in its first and last value; the other matrices of SS1 (or SS1M) (which have the form of a square) contain the other corresponding matrices of CPC in their corners:

Apq	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	2	2	1	2
3	1	2	3	1	3	3
4	1	2	1	4	4	4
5	1	1	3	4	5	5
6	1	2	3	4	5	6

Kpq	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	2	3	4	6	6
3	3	3	3	6	5	5
4	4	4	6	4	5	6
5	5	6	5	5	5	6
6	6	6	6	6	6	6

Cpq	1	2	3	4	5	6
1	1	2	3	4	5	6
2	1	1	3	4	5	5
3	1	2	1	4	4	4
4	1	2	3	1	3	3
5	1	2	2	2	1	2
6	1	1	1	1	1	1

Thus SS1 turns into CPC if one drops in **1.08** the operation L and the values 2,3,4 and 5.

			cv
3.1	Positive Implicational Calculus		
3.11	Łukasiewicz (Prior, (Flg) Appendix)		
3.111	*C p Cqp*		2
(7)	all *VA*		1
3.112	*C CpCqr CCpqCpr*		2
(28)	all *VA*		1
(27)	the *VO LC*		*k*
	its *VA*		1
	the *VO+VA LLC, MLC*		1
3.12	Hilbert, (GLM)		
3.121	*C CpCpq Cpq*		1
(28)	all *VA*		1
(27)	the *VO LC*		*k*
	its *VA*		1
	the *VO+VA LLC, MLC*		1
3.122	*C Cpq CCqrCpr*	(cf. **2.461**)	2
3.123	*C p Cqp*	(cf. **3.111**)	2

3.2 Weak Positive Implicational Calculus

3.21 Church, (WTI)

			cv
3.211	C $CpCpq$ Cpq	(cf. **3.121**)	1
3.212	C Cqr $CCpqCpr$		2
(28)	all VA		1
(27)	the VO LC		k
	its VA		1
	the $VO+VA$ LLC, MLC		1
3.213	C $CpCqr$ $CqCpr$		2
(28)	all VA		1
(19)	the VO LC, LLC, MLC		k
	its VA Lp, Mp		k
	its other VA		1

3.3 Full Intuitionistic Calculus

3.31 Heyting, (FRI)

3.311	C p Kpp	(cf. **2.21**)	1
3.312	C Kpq Kqp	(cf. **2.23**)	1
(196)	all VA		1
	all $VO+VA$		1
3.313	C Cpq $CKprKqr$	(cf. **2.427**)	2
3.314	C $KCpqCqr$ Cpr	(cf. **2.46**)	2
3.315	C p Cqp	(cf. **3.111**)	2
3.316	C $KpCpq$ q	(cf. **2.47**)	2
3.317	C p Apq	(cf. **2.32**)	2
3.318	C Apq Aqp	(cf. **2.33**)	1
(196)	all VA		1
	all $VO+VA$		1
3.319	C $KCprCqr$ $CApqr$	(cf. **2.463**)	1
3.320	C Np Cpq		2
(7)	all VA		1
3.321	C $KCpqCpNq$ Np		1
(196)	all VA		1
(190)	the VO LK, MLK		2
	its VA		1
	the $VO+VA$ LLK		1
	the VO MK, MMK, LMK		k
	its VA Lp		k
	its VA Mp		3
	its other VA		1
	the VO LC, LC . . . MK, MMK, LMK		3
	its VA		1
	the $VO+VA$ LC . . . LK, LLK, MLK; all $VO+VA$ of LLC and MLC		1

3.4 Modal Systems

3.41 Lewis S1, (SLg)

3.411	LC Kpq Kqp	(cf. **3.312**)	1
(147)	all VA		1

				cv
		all *VO+VA*		1
3.412	*LC*	*Kpq p*	(cf. **3.042**)	3
(7)		all *VA*		1
3.413	*LC*	*p Kpp*	(cf. **2.21** and **3.041**)	1
(42)		all *VA*		1
		the *VO+VA LLC—LMC* (without *K*-var.)		1
3.414	*LC*	*KKpqr KpKqr*	(cf. **2.24**)	3
(147)		all *VA*		1
(134)		the *VO LK, MK, MMK, LMK*		*k*
		the *VO LLC . . . K, LK, MK, MMK, LMK;*		
		MLC . . . LK, MK, MMK, LMK		*k*
		the *VO LC . . . LLK, MLK; LLC . . . LLK, MLK;*		
		MLC . . . K, LK, MLK		1
		the *VA* of all mentioned *VO*		1
3.415	*LC*	*KLCpqLCqr LCpr*	(cf. **2.46**)	1
(147)		all *VA*		1
(144)		the *VO LC . . . MK, MMK, LMK*		*k*
		the *VO+VA LLC, MLC . . . K-LMK;*		
		the *VO+VA LC . . . LK-MLK*		1
3.416	*LC*	*p Mp*		3
(21)		all *VA*		1
(20)		the *VO LLC*		*k*
		its *VA*		1
		the *VO+VA MLC*		1
3.42	Lewis S2, (SLg)			
3.421	Axioms **3.411-3.416** of S1			
3.432	*LC*	*MKpq Mp*	(cf. **2.28**)	1
(42)		all *VA*		1
		all *VO+VA LLC . . . LMC*		1
3.43	Lewis S3, (SSL)			
3.431	Axioms **3.411-3.416, 3.422**			
3.432	*LC*	*LCpq LCMpMq*		*k*
	In SS1 the following are valid:			
	LC	*LCpq CMpMq*	(cf. **2.455-2.459**)	1
	LC	*LLCpq LCMpMq*	(cf. **2.455-2.953**)	
3.44	Lewis S4, (SLg)			
3.441	Axioms **3.411-3.416, 3.422, 3.432**			
3.442	*LC*	*MMp Mp*		*k*
	In SS1 the following is valid:			
	LC	*Mp MMp*	(cf. **2.197**)	1
3.45	Lewis S5, (SLg)			
3.451	Axioms **3.411-3.416, 3.422, 3.432, 3.442**			
3.452	*LC*	*Mp LMp*		*k*
	In SS1 the following is valid:			
	LC	*LMp Mp*	(cf. **2.196**)	
3.46	Gödel, (IIA)			
3.461	*C*	*Lp p*	(cf. **2.194**)	2

				cv
(49)		all *VA*		1
(48)		the *VO LC*		3
		the *VO LLC*		*k*
		the other *VO MLC*, *MC*, *MMC*, *LMC*		1
		the *VA* of all *VO*		1
3.462	C_1	$LC_2pq\ C_3LpLq$		1
(84)		all *VA*		1
(78)		the *VOG* $LC_2 \ldots LC_3$, MLC_3		*k*
		its *VA*		1
		the *VOG+VA* $LLC_2 \ldots C_3$, LC_3, LLC_3, MLC_3		1
		the *VOG* $MLC_2 \ldots C_3$		3
		its *VA*		1
		the *VOG* $MLC_2 \ldots LC_3$, LLC_3, MLC_3		*k*
		its *VA*		1

3.5 Strict and Strong Implication

3.51 Ackermann (strong implication) (LSI) (LSS)

3.5101	*LC*	*p p*		1
(42)		all *VA* and *VO+VA*		1
3.5102	*LC*	*LCpq LCLCqrLCpr*	(cf. **2.461, 3.031**)	*k*
(21)		all *VA*		1
(20)		the *VO+VA LLC*, *MLC*		1
3.5103	*LC*	*LCqr LCLCpqLCpr*	(cf. **3.212**)	*k*
(21)		all *VA*		1
(20)		the *VO+VA LLC*, *MLC*		1
3.5104	*LC*	*LCpLCpq LCpq*		*k*
(21)		all *VA*		1
(20)		the *VO+VA LLC*, *MLC*		1
3.5105	*LC*	*Kpq p*	(cf. **3.042**)	3
(7)		all *VA*		1
3.5106	*LC*	*Kpq q*		3
(7)		all *VA*		1
3.5107	LC_1	$K_3LC_2pqLC_2pr\ \ LC_2pK_3qr$	(cf. **2.462**)	1
(147)		all *VA*		1
(118)		the *VOG+VA* $LLC_2 \ldots K$; $MLC_2 \ldots K$		1
		Continuation as in **2.462**, (except the two lines at end)		
3.5108	*LC*	*p Apq*	(cf. **2.32**)	3
(147)		all *VA*		1
(117)		the *VO LA*, *LLA*, *MLA*		*k*
		its *VA Lp*, *Mp*		*k*
		the *VO MA*, *LMA*		3
		its *VA*		1
		the *VO+VA MMA*		1
		the *VO LLC . . . A*, *MA*, *LMA*		*k*
		its *VA*		1

		cv
	the *VO LLC . . . LA, LLA, MLA; MLC . . . LA, LLA, MLA*	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VO+VA LLC . . . MMA; MLC . . . A, MA, MMA, LMA*	1
3.5109 *LC*	*q Apq* (cf. **3.012**)	3
(147)		
(117)	the same distribution of *cv* as in **3.5108**	
3.5110 *LC*	*KLCprLCqr LCApqr* (cf. **2.463**)	1
(1029)	distribution of *cv* as in **2.463,** beginning from:	
(806)	the *VO+VA LC . . . K*	
3.5111 *LC*	*KpAqr AqKpr*	3
(1029)	all *VA*	1
(794)	the *VO LC . . . LA, LLA, MLA*	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VO LC . . . MA, MMA, LMA*	*k*
	its *VA*	1
	the *VO LC . . . LK, LLK, MLK . . . A, LA, LLA, MLA*	*k*
	its *VA*	1
	the *VO LC . . . LK, LLK, MLK . . . MA, MMA, LMA*	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VO LC . . . MK, MMK, LMK . . . A*	*k*
	its *VA Lp, Mp*	*k*
	its other *VA*	1
	the *VO LC . . . MK, MMK, LMK . . . LA, LLA, MLA, MA, MMA, LMA*	*k*
	its *VA*	1
	the *VO LLC . . . K . . . A*	*k*
	the other *VO+VA* of *LLC . . .* have the same distribution of *cv* as these of *LC . . .*	
	the *VO MLC . . . K . . . A*	1
	the other *VO+VA* of *MLC . . .* have the same distribution of *cv* as these of *LC . . .*	

Note: The corresponding variations of *C KpAqr AqKpr* (*C* not varied), (343 variations, 271 valid) differ from the above distributions for *LC . . .* only in the following respect: the following formulas are not valid in the strict (*LC . . .*) of **3.5111,** but valid in the material form (*C . . .*):

the three *VA Mp* of *CKpLAqrLAqKpr, C . . . LLA, C . . . MLA*;

the three *VA Lp* of *CMK pAqrAqMKpr, C . . . MMk . . . A, C . . . LMK . . . A.*

3.5112 *LC*	*LCpq LCNqNp* (cf. **2.415**)	1

				cv
(21)	*LC*	all *VA*		1
		all *VO+VA LLC, MLC*		1
3.5113	*LC*	*p NNp*	(cf. **2.12**)	1
(21)		all *VA*		1
		all *VO+VA LLC, MLC*		1
3.5114	*LC*	*NNp p*	(cf. **2.12**)	1
(21)		all *VA*		1
		all *VO+VA LLC, MLC*		1
3.5115	*LC*	*KpNq NLCpq*		3
3.52	Schmidt (Strict Implication) (AZM) (VAL)			
3.5201	*LC*	*Kpq Kqp*	(cf. **3.411**)	1
3.5202	*LC*	*KKpqr KpKqr*	(cf. **3.414**)	3
3.5203	*LC*	*Kpq p*	(cf. **3.412**)	3
3.5204	*LC*	*p Kpp*	(cf. **3.413**)	1
3.5205	*LC*	*KpLCpq q*	(cf. **3.472** and **3.474**)	3
3.5206	*LC*	*KLCpqLCqr LCpr*	(cf. **2.46**)	1
3.5207	*LC*	*LCpq LCNqNp*	(cf. **3.5112**)	1
3.5208	*LC*	*LCKpqr LCKpNrNq*		*k*
3.52081	C_1	$C_2K_3pqr\ C_2K_3pNrNq$		2
(196)		all *VA*		1
(147)		the *VOG* $C_2 \ldots LK_3\text{-}LMK_3$		*k*
		its *VA Lp*		*k*
		its *VA Mp*		3
		its other *VA*		1
		the *VOG* $LC_2 \ldots K_3$		3
		its *VA*		1
		the *VOG* $LC_2 \ldots LK_3\text{-}MLK_3$; $LLC_2 \ldots LK_3\text{-}MLK_3$; $MLC_2 \ldots LK_3\text{-}MLK_3$		*k*
		its *VA Lp, Mp*		*k*
		its other *VA*		1
		the *VOG* $LC_2 \ldots MK_3\text{-}LMK_3$; $LLC_2 \ldots K_3$, $MK_3\text{-}LMK_3$; $MLC_2 \ldots MK_3\text{-}LMK_3$		*k*
		its *VA*		1
		the *VOG+VA* $MLC_2 \ldots K_3$		1
(196)		the *VOG+VA* LC_1: distribution as in **3.52081**, except:		
(139)		the *cv* 2 turn into 3, the *cv* 3 into *k*		
(196)		the *VOG+VA* LLC_1: distribution as in **3.52081**, except:		
(138)		the *cv* 2 and 3 turn into *k*		
(196)		the *VOG+VA* MLC_1: distribution as in **3.52081**, except:		
(139)		the *cv* 2 turn into 1, the *cv* 3 into *k*		
3.5209	*LC*	*p NNp*	(cf. **2.12**)	1
3.5210	*LC*	*NNp p*	(cf. **2.12**)	1
3.5211	*LC*	*LCpKqr Cpq*		2

			cv
3.53	Lemmon (Strict Implication, fragment S5) (APS)		
3.531	*LC*	*LCpq LCLCqrLCpr* (cf. **2.461**)	*k*
	In SS1 and SS1M the following is valid:		
	LLC	*LLCpq LLCLLCqrLLCpr*	1
3.532	*LC*	*LCLCLCrpqLCrp LCrp*	3
3.5321	C_1	$C_2C_2C_3rpqC_3rp\ \ C_3rp$	1
(120)		all *VA*	1
(103)		the *VOG+VA* LC_3, LLC_3, MLC_3	1
		the *VOG* LC_2, LLC_2, $MLC_2 \ldots C_3$	*k*
		its *VA* *Lp*, *Mp*	*k*
		its other *VA*	1
		the *VOG* LC_2, LLC_2, $MLC_2 \ldots LC_3$	3
		its *VA*	1
		the *VOG+VA* LC_2, LLC_2, $MLC_2 \ldots LLC_3$, MLC_3	1
(120)		the *VOG+VA* LC_1: distribution as in **3.5321,** except:	
(100)		the *cv* 3 turn into *k*	
(120)		the *VOG+VA* LLC_1: distribution as in LC_1	
(100)			
(120)(100)		the *VOG+VA* MLC_1: distribution as in LC_1	
3.533	*LC*	*LCrp LCqLCrp*	*k*
		In SS1 (and SS1M) the following are valid:	
	C	*LCrp CqLCrp*	2
	LC	*LCrp CqLCrp*	3
	LLC	*LLCrp LLCqLLCrp*	1
3.5331	C_1	$C_3rp\ \ C_2qC_3rp$	2
(120)		all *VA*	1
(100)		the *VOG* LC_3	2
		its *VA*	1
		the *VOG+VA* LLC_3, MLC_3	1
		the *VOG* LC_2, LLC_2, $\ldots C_3$; $LLC_2 - MLC_2$	*k*
		its *VA* *Lp*, *Mp*	*k*
		its other *VA*	1
		the *VOG* LC_2, LLC_2, $\ldots LC_3$; $MLC_2 \ldots C_3$	*k*
		its *VA*	1
		the *VOG+VA* $LC_2 \ldots LLC_3$, MLC_3; $LLC_2 \ldots LLC_3$; $MLC_2 \ldots LC_3$, LLC_3, MLC_3	1
(120)		the *VOG+VA* LC_1: distribution as in **3.5331,** except:	
(100)		the *cv* 2 turn into 3	
(120)		the *VOG+VA* LLC_1: distribution as in **3.5331,** except:	
(98)		the *cv* 2 turn into *k*	
(120)		the *VOG+VA* MLC_1: distribution as in **3.5331,** except:	
(100)		the *cv* 2 turn into 1	
3.534	*LC*	*p p* (cf. **2.11**)	1

4. The intuitionistic calculus SS1I

4.1 With the help of the operation LM of SS1 (or SS1M) one can turn the system SS1 (or SS1M) into the intuitionistic calculus SS1I by the following definitions (which are formulated as logically necessary equivalences). As it can be seen the definientia for the intuitionistic operations are solely taken from SS1:

4.11 Intuitionistic Negation (N') $LLE\ N'p\ NLMp$ Df.

4.12 Intuitionistic Disjunction (A') $LLE\ A'pq\ Apq$ Df.
(the intuitionistic disjunction is identical with the disjunction of SS1)

4.13 Intuitionistic Conjunction (K') $LLE\ K'pq\ KLMpLMq$ Df.

4.14 Intuitionistic Implication (C') $LLE\ C'pq\ CLMpLMq$ Df.

4.15 Intuitionistic Equivalence (E')
$LLE\ E'pq\ ELMpLMq$ Df.
$LLE\ E'pq\ K'C'pqC'qp$

All the other properties of SS1 (and SS1M) remain unchanged[22]

4.2 The strong validity of Heytings axioms (FRI) (Int) in SS1I

			cv
4.201	$C'\ p\ K'pp$	(cf. **3.31**)	1
	unabbreviated: $C\ LMp\ LMKLMpLMp$		1
4.202	$C'\ K'pq\ K'qp$		1
4.203	$C'\ C'pq\ C'K'prK'qr$		1
4.204	$C'\ K'C'pqC'qr\ C'pr$		1
4.205	$C'\ q\ C'pq$		1
4.206	$C'\ K'pC'pq\ q$		1
4.207	$C'\ p\ A'pq$		1
4.208	$C'\ A'pq\ A'qp$		1
4.209	$C'\ K'C'prC'qr\ C'A'pqr$		1
4.210	$C'\ N'p\ C'pq$		1
4.211	$C'\ K'C'pqC'pN'q\ N'p$		1

4.3 The validity and invalidity of important other formulas in SS1I

4.301 $A\ p\ N'p$ tertium non datur k

The tertium non datur is a contingent sentence in SS1I, its matrix is: 1 2 3 4 1 1

4.302	$N'\ N'A\ p\ N'p$	1
4.303	$A\ Np\ N'N'p$	2
4.304	$A\ Np\ N'Np$	k
4.305	$A\ N'p\ N'N'p$	1
4.306	$A\ N'p\ N'Np$	k
4.307	$C\ AN'pN'Np\ ANpN'Np$	1
4.308	$C\ AN'pN'Np\ ApN'p$	1
4.310	$C\ p\ N'N'p$	2
4.311	$C\ N'N'p\ p$	k

Although the sentence **4.311** is not valid (k) in SS1I the same sentence with intuitionistic implication C' is valid in SS1I as it is seen from **4.313**

4.312	$C\ N'Np\ p$	2
4.313	$E'\ p\ N'N'p$	1

			cv
4.314	$E\ N'p\ N'N'N'p$		1
4.315	$E'\ N'p\ N'N'N'p$		1

4.316 The following sentences have identical matrices and are therefore substitutable for one another:
$N'N'N'p$, $N'NN'p$, $N'NNp$, $NN'N'p$, $NNN'p$. And: $N'N'Np$, $NN'Np$. Instead of saying two formulas have identical matrices one could also write down an equivalence theorem with the *cv* 1.

4.317	$C\ N'N'N'p\ N'N'Np$	(C')	1

'(C')' indicates that the same formula is valid also with C' (this refers always only to the main-connective of the formula)

4.320	$C\ K'pN'p\ q$	(C')	1
4.321	$C\ KpN'p\ q$		2
4.322	$C'\ KpN'p\ q$		1

4.323 The following sentences have identical matrices:
$K'pN'p$, $K'N'pN'N'p$, $K'N'pNNp$, $K'N'pN'Np$, $KN'pNNp$, $KN'pN'Np$.

4.324	$C\ K'pNp\ q$	(C')	k

The results stated in **4.320** and **4.324** are important. They show that the system SS1I is able to distinguish both of the controversial views of intuitionists concerning the principle "ex falso quodlibet". This principle holds when formulated as in **4.320** but it does not hold (is not valid in SS1I) when it is formulated as in **4.324,** i.e. with classical negation.

4.325	$E\ K'pNp\ K'NpN'N'p$	(E')	1
4.326	$C\ KNpN'N'p\ K'pNp$		2
4.327	$C\ N'p\ C'pq$	(C')	1
4.330	$E\ N'Apq\ K'N'pN'q$	(E')	1
4.331	$E\ AN'pN'q\ N'K'pq$	(E')	1

This seems to differ from the usual intuitionistic calculus in which the second law of De Morgan, **4.331,** only holds with an implication. However if one takes a classical conjunction instead of a intuitionistic one **4.331** holds only with an implication (for **4.330** the change from intuitionistic to classical conjunction does not make any difference):

4.332	$C\ AN'pN'q\ N'Kpq$	(C')	1
4.333	$C\ N'Kpq\ AN'pN'q$	(C')	k
4.335	$E\ C'pq\ C'N'qN'p$	(E')	1

SS1I differs from many intuitionistic systems in respect to the formula **4.335.** In these systems only an implication instead of the equivalence holds. But also this property of intuitionistic systems is explainable in SS1I. This can be seen from the following formulas (more explicitly from the validity of **4.336** and **4.337** and the invalidity of **4.338** and **4.339**):

4.336	$C\ C'pq\ C'N'qNp$	(C')	1
4.337	$C\ C'pq\ CN'qNp$	(C')	1
4.338	$C\ C'N'qNp\ C'pq$	(C')	k
4.339	$C\ CN'qNp\ C'pq$	(C')	k

Here also SS1I is able to distinguish between different intuitionistic systems like in **4.320** and **4.324, 4.331** and **4.332.**

			cv
4.340	$C\ q\ C'pq$	(C')	2
4.341	$C\ AN'pq\ C'pq$	(C')	2
4.350	$E\ N'N'K'pq\ K'N'N'pN'N'q$	(E')	1

4.351 The formulas $N'N'K'pq$, $NNK'pq$, $K'pq$, $N'NK'pq$, $KN'N'pN'N'q$ and $K'N'N'pN'N'q$ have identical matrices

4.352	$C\ K'N'pN'q\ N'K'pq$	(C')	1
4.353	$C\ N'K'pq\ K'N'pN'q$	(C')	k
4.355	$E\ AN'N'pN'N'q\ N'N'Apq$	(E')	1

Also **4.355** differs from the usual intuitionistic systems. However the following formulas show again a more detailed differentiation in SS1I:

4.356	$C\ ANNpNNq\ N'N'Apq$	(C')	2
4.357	$C\ N'N'Apq\ ANNpNNq$	(C')	k
4.358	$C\ AN'NpN'Nq\ N'N'Apq$	(C')	2
4.359	$C\ N'N'Apq\ AN'NpN'Nq$	(C')	k
4.360	$C\ N'Apq\ AN'pN'q$	(C')	1
4.361	$C\ AN'pN'q\ N'Apq$	(C')	k

4.4 A consistency proof for SS1I can easily be obtained in a similar way as in 1.7 for SS1. **4.5** A decision procedure can be described for SS1I in a similar way as in 1.8 for SS1. **4.6** A completeness proof with the result that SS1I is complete in the first and second sense of completeness, defined in **3.061** and **3.063**, can easily be obtained in an analogous way to that in **3.062** and **3.064**.

5. Syllogistic. **5.1** Oskar Becker (UMo) introduced an interpretation of the modal calculus which he called "statistical interpretation (Deutung) of the modal calculus" and which was anticipated by Thomas Aquinas in his (PMo). According to this interpretation the sentence 'it is necessary that p' is interpreted as a universal sentence and the sentence 'it is possible that p' as an existential sentence. The definitions which underlie this interpretation are the following:

5.11 $Lp = \text{df. } (x)Px$[23]

5.12 $NLp = \text{df. } \sim(x)Px$

5.13 $NLNp = \text{df. } \sim(x)\sim Px$

5.131 $\sim(x)\sim Px \equiv (Ex)Px$

5.132 $NLNp = \text{df. } Mp$

5.133 $Mp = \text{df. } (Ex)Px$

5.14 $LNp = \text{df. } (x)\sim Px$

5.2 In the following the 24 syllogisms of the assertoric categorical syllogism (CS) will be reinterpreted with the help of the definitions above and with the further definitions **5.21-5.24.** The corresponding 24 modal-sentences which are the result of this intepretation are then calculated for their validity or invalidity in SS1 and SS1M.

5.21	$SaP = \text{df. } LCsp$	universal affirmative form
5.22	$SeP = \text{df. } LCsNp$	universal negative form
5.23	$SiP = \text{df. } MKsp$	particular affirmative form
5.24	$Sop = \text{df. } MKsNp$	particular negative form

			cv
5.25	First figure		
5.251	*C KLCqpLCsq LCsp*	Barbara	1
5.252	*C KLCqNpLCsq LCsNp*	Celarent	1
5.253	*C KLCqpMKsq MKsp*	Darii	1
5.254	*C KLCqNpMKsq MKsNp*	Ferio	1
5.255	*C KLCqpLCsq MKsp*	Barbari	*k*
5.256	*C KLCqNpLCsq MKsNp*	Celaront	*k*
5.26	Second figure		
5.261	*C KLCpNqLCsq LCsNp*	Cesare	1
5.262	*C KLCpqLCsNq LCsNp*	Camestres	1
5.263	*C KLCpNqMKsq MKsNp*	Festino	1
5.264	*C KLCpqMKsNq MKsNp*	Baroco	1
5.265	*C KLCpNqLCsq MKsNp*	Cesaro	*k*
5.266	*C KLCpqLCsNq MKsNp*	Camestrop	*k*
5.27	Third figure		
5.271	*C KLCqpLCqs MKsp*	Darapti	*k*
5.272	*C KLCqNpLCqs MKsNp*	Felapton	*k*
5.273	*C KMKqpLCqs MKsp*	Disamis	1
5.274	*C KLCqpMKqs MKsp*	Datisi	1
5.275	*C KMKqNpLCqs MKsNp*	Bocardo	1
5.276	*C KLCqNpMKqs MKsNp*	Ferison	1
5.28	Fourth figure		
5.281	*C KMKpqLCqs MKsp*	Dimaris	1
5.282	*C KLCpNqMKqs MKsNp*	Fresison	1
5.283	*C KLCpqLCqNs LCsNp*	Camenes	1
5.284	*C KLCpqLCqs MKsp*	Bamalip	*k*
5.285	*C KLCpqLCqs MKsNp*	Camenop	*k*
5.286	*C KLCpNqLCqs MKsNp*	Fesapo	*k*

5.29 In **5.24-5.28** the 24 forms of CS are reinterpreted as sentential modal forms of SS1M. As one can see from the *cv* there are 15 of the 24 modal forms which are strongly valid in SS1M. The other 9 are not valid but contingent in SS1M. The 15 sentential modal forms are exactly these which can be derived from a subsystem of the axiomatized CS[24] which arises from the full system (of axiomatized CS) when the axiom *PiP* is dropped[25]. A full axiomatization of CS one can get by taking some basic laws of propositional calculus as axioms and adding Barbara and Datisi or Ferio as syllogistic axioms[26] and as further syllogistic axioms *PaP* and *PiP*. From these axioms one can derive all the 24 forms of CS. However if one drops the axiom *PiP* only 15 forms can be derived because the laws of the logical square—except the laws of contradictorical opposition and some laws of conversion—are no longer valid. The corresponding modal sentences of exactly these 15 forms are strongly valid in SS1M. According to Hilbert-Ackermann (GZT) p. 62 ss. the axiom *PiP* represents the existential presuppositions of the Aristotelian syllogistic system which are not made in most of the systems of modern logic since Frege[27].

Shepherdson, (ISy) p. 143, has shown that the function 'i' in the axiom PiP of the full axiomatized CS (in which all 24 forms are valid) is not a truth-function. However 'i' can be interpreted as a truth-function in the subsystem of CS which arises from CS if PiP is dropped[28].

5.3 In an analogous way one could investigate the Aristotelian and Scholastic modal syllogistic. As it is clear from what has been said at the beginning (**0.2** and **0.4**) there must be two ways of doing this: First one can interpret the modal sentence with modality de dicto and secondly with modality de re. Of the latter one can distinguish two kinds. Thus one gets three groups of modal sentences each one consisting of the four Aristotelian propositions modified by modalities of a certain kind:

5.31 Modality de dicto

5.311 $L(SaP) = \text{df. } L(LCsp)$
$M(SaP) = \text{df. } M(LCsp)$
$LL(SaP) = \text{df. } LL(LCsp)$
$MM(SaP) = \text{df. } MM(LCsp)$
$ML(SaP) = \text{df. } ML(LCsp)$
$LM(SaP) = \text{df. } LM(LCsp)$

5.312 $L(SeP) = \text{df. } L(LCsNp)$
etc.

5.313 $L(SiP) = \text{df. } L(MKsp)$
etc.

5.314 $L(SoP) = \text{df. } L(MKsNp)$
etc.

5.32 Modalities de re I (cf. **0.2**)

5.321 $L(SaP) = \text{df. } LCLsLp$
$M(SaP) = \text{df. } LCMsMp$
etc.

5.322 $L(SeP) = \text{df. } LCLsLNp$
etc.

5.323 $L(SiP) = \text{df. } MKLsLp$
etc.

5.324 $L(SoP) = \text{df. } MKLsLNp$
etc.

5.33 Modalities de re II[29]

5.331 $L(Sap) = \text{df. } LCsLp$
$M(SaP) = \text{df. } LCsMp$
etc.

5.332 $L(SeP) = \text{df. } LCsLNp$
etc.

5.333 $L(SiP) = \text{df. } MKsLp$
etc.

5.334 $L(Sop) = \text{df. } MKsLNp$

According to Bocheński, (AFL) p. 61s., the number of modal syllogisms which are built analogous to the first, second and third figure (cf. **5.25-5.276**) are 95. The number of modal syllogisms which follow from that with the help of the law *CpMp* are 7. 35 other syllogisms can be obtained by the help of the law *Cont* (*SaP*) ≡ *Cont* (*SeP*) and *Cont* (*SiP*) ≡ *Cont* (*SoP*). But the corresponding modal-interpretations (according to 5.11-5.24) of these two laws of contingency ('*Cont*' stands for 'contingent') are not valid in SS1M. Thus the number of the remaining modal syllogisms are 102; interpreted as modalities de dicto and de re I and II this gives 306 forms. The investigations on the validity of these forms of modal syllogisms in SS1M are not yet finished by the author.

6. The Epistemic System SS1E. **6.11** What has been said in **1.03, 1.04** and **1.071** holds also for SS1E. **6.12** The letters ***'a', 'b', 'c', 'd'*** . . . are used in SS1E as personal variables (designating arbitrary human persons).
6.13 ***'aWp'*** stands for 'the person ***a*** knows that ***p*** (is the case)'
'NaWp' stands for 'it is not the case that the person ***a*** knows that ***p*** (is the case)'
'aWNp' stands for 'the person ***a*** knows that not-***p*** (is the case)'
'aWApq' stands for 'the person ***a*** knows that either ***p*** or ***q*** (is the case)' etc.
'aW°p' stands for 'the person ***a*** knows whether ***p*** (is the case)'
'aWaWp' stands for 'the person a knows that the person ***a*** knows that ***p*** (is the case)'
'aW(bWp)' stands for 'the person ***a*** knows that the person ***b*** knows that ***p*** (is the case)'
Note: ***aWbWp*** is also a sentence of SS1E but has a matrix different from ***aW(bWp)***; cf. **6.29** and **6.537.**
'L***aWp'*** stands for 'the person ***a*** necessarily knows that ***p*** (is the case)'; cf. **6.63.**
6.14 At first sight it seems reasonable to interpret '*aWp*' in SS1 just as *MLp*. One can see at once that a number of interesting theorems for a logic of knowledge result from this interpretation. Under these the following are important:

6.141 *C aWp p*
6.142 *C aWp aWaWp*
6.143 *C aWp NaWNp*
6.1431 *C aWNp NaWp*
6.144 *N KaWp NaWp*
6.145 *A aWp NaWp*
6.146 *E aWKpq KaWpaWq*
6.147 *C AaWpaWq aWApq* (the converse does not hold)
6.1471 *C AaWpaWNp aWApNp*
aW°p = df. *AaWp aWNp*
6.148 *C aWp aW°p*
6.1481 *C aWNp aW°p*
etc.

6.149 But the important difficulties of such an intepretation are the following ones:
1. The modal system SS1M cannot be used independently any more; thus if one wants a logic of knowledge where also modalities can be used one cannot use *MLp* as a modal statement any longer.

2. When using the modal system SS1M in the logic of knowledge one gets theorems which one certainly does not want. Thus *CaWpLp* holds materially, *CLpaWp* holds strongly etc.

6.15 A much better base for a logic of knowledge one can obtain if *aWp* has a matrix which in this sense is independent of SS1M that it is not defined with the help of its modal operations. Thus *aWp* could have the matrix: 3 4 3 5 5 6 (where *p* has the basic matrix 1 2 3 4 5 6). This gives much better results: All the theorems 6.141-6.1481 hold and even much more. However though the first point (of 6.149) is removed the second remains in some of its modifications. More explicitly: though the sentences *CaWpLp* and *CLpaWp* are no longer valid there are other counterexamples like these: *C NaWMp LNp* (if it is not the case that *a* knows that possibly *p* then necessarily not-*p*) is a materially valid sentence of SS1 if *aWp* has the matrix 3 4 3 5 5 6. Even if some of the counterexamples can be removed if one uses the intuitionistic negation *N'* (cf. **4.11**) instead of those *N* which occur just before *aW* . . . , the above mentioned counterexample (and others) remain[30].

6.16 The counterexamples stated in **6.149** and **6.15** suggest a complete new matrix for the sentence ***aWp***: This matrix should be independent of SS1M in the mentioned sense (cf. **6.149**) and it should not lead to counterexamples analogous to point 2. in **6.149.** A matrix which satisfies these purposes can be constructed for a (deductive) system which is an extension of SS1 and may be called SS1E. The system SS1E is defined in what follows.

6.20 Formula of the system SS1E. **6.201** **1.071** holds in SS1E. **6.202** If ***'p'*** is a formula then ***'Np'***[31], ***'Lp'***, ***'LLp'***, ***'Mp'***, ***'MMp'***, ***'MLp'***, ***'LMp'*** and ***'aWp'*** ***'bWp'***, . . . ***'aWLp'*** . . . , ***'LaWp'*** . . . , ***'aWaWp,'*** ***'aWbWp'***, ***'aW(bWp)'***, ***'aW°p'***, ***'bW°p'*** . . . are formulas (cf. **6.24, 6.27, 6.28**). **6.203** If ***'p'*** and ***'q'*** are formulas then ***Apq, Kpq, Cpq, Epq*** are formulas (cf. **6.25, 6.26**).

6.21 The basic matrix of any sentence of SS1E is: o 0 1 2 3 4 5 6 7 8.

6.22 o, 0, 1, 2, 3 are (different) truth-values for true, 4, 5, 6, 7, 8 are (different) truth-values for false.

6.23 Definition of the system SS1E. The system SS1E can be defined as the set of all sentences which are satisifed by the matrix ***Mat*** = $<$***T, F, N, A, L, aW, bW***$>$ where ***T*** = {o, 0, 1, 2, 3}, ***F*** = {4, 5, 6, 7, 8} and the operations ***N, A, L, aW,*** and ***bW*** are defined by the definitions in **6.24, 6.25, 6.27, 6.281** and **6.282.**

6.24 Iff ***p*** has the basic matrix then ***Np*** has the matrix: 8 7 6 5 4 3 2 1 0 o

6.25 Iff ***p*** and ***q*** have the basic matrix then ***Apq*** has the matrix:

Apq	o	0	1	2	3	4	5	6	7	8
o	o	1	1	1	1	1	1	1	1	1
0	1	0	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1
2	1	1	1	2	2	2	1	2	2	2
3	1	1	1	2	3	1	3	3	3	3
4	o	0	1	2	1	4	4	4	7	8
5	o	0	1	1	3	4	5	5	7	8
6	o	0	1	2	3	4	5	6	7	8
7	o	1	1	2	3	7	7	7	7	8
8	1	0	1	2	3	8	8	8	7	8

One can easily see that this matrix of ***Apq*** (of SS1E) contains the matrix of *Apq* of SS1 as a part. By giving the matrices for ***Np*** and ***Apq*** the system is defined as a propositional calculus (cf. **1.111**).

6.26 The operations ***Kpq***, ***Cpq*** and ***Epq*** are defined as in SS1: ***Kpq*** = ***NANpNq***, ***Cpq*** = ***ANpq*** and ***Epq*** = ***KCpqCqp***. Iff ***p*** and ***q*** have the basic matrix (of SS1E) ***Kpq*** and ***Cpq*** have the matrices:

Kpq	o	0	1	2	3	4	5	6	7	8
o	o	0	o	o	o	4	5	6	7	6
0	o	0	0	0	0	4	5	6	6	8
1	o	0	1	2	3	4	5	6	7	8
2	o	0	2	2	3	4	6	6	7	8
3	o	0	3	3	3	6	5	6	7	8
4	4	4	4	4	6	4	5	6	6	6
5	5	5	5	6	5	5	5	6	6	6
6	6	6	6	6	6	6	6	6	6	6
7	6	6	6	6	6	6	6	6	7	6
8	6	6	6	6	6	6	6	6	6	8

Cpq	o	0	1	2	3	4	5	6	7	8
o	1	0	1	2	3	8	8	8	7	8
0	o	1	1	2	3	7	7	7	7	8
1	o	0	1	2	3	4	5	6	7	8
2	o	0	1	1	3	4	5	5	7	8
3	o	0	1	2	1	4	4	4	7	8
4	1	1	1	2	3	1	3	3	3	3
5	1	1	1	2	2	2	1	2	2	2
6	1	1	1	1	1	1	1	1	1	1
7	1	0	1	1	1	1	1	1	1	1
8	o	1	1	1	1	1	1	1	1	1

6.27 Iff ***p*** has the basic matrix (of SS1E) then ***Lp*** has the matrix: o 0 1 3 6 6 6 6 7 8. As it is clear from **1.151** with the help of the matrices of *Np*, *Apq* and *Lp* the whole system of modal logics SS1M can be established. Thus the non-epistemic base of the epistemic system SS1E which includes SS1 (or: SS1M, if the modal interpretation is taken) can be defined with the help of the extended matrices of *Np*, *Apq* and *Lp* given in **6.24, 6.25** and **6.27**. A definition like the one in **1.08** for SS1 could be given for the non-epistemic base of SS1E in the like manner. The table of the modal operations in SS1E is analogous to that of **1.21**

LLp	***Lp***	***MLp***	***p***	***LMp***	***Mp***	***MMp***
o	o	o	o	o	o	o
0	0	0	0	0	0	0
1	1	1	1	1	1	1
6	3	1	2	1	1	1
6	6	6	3	1	1	1
6	6	6	4	1	1	1
6	6	6	5	6	4	1
6	6	6	6	6	6	6
7	7	7	7	7	7	7
8	8	8	8	8	8	8

The table shows that in SS1E not only the values 1 and 6 remain unchanged when modal operations are applied (as in SS1M) but also the values o, 0, 7, 8 are not touched by the application of modal operations. The operations ***LC, LLC, LE*** and ***LLE*** can be defined with the help of **6.27** and **6.26** in analogy to **1.22-1.26**.

6.28 The matrices for ***aWp*** and for ***bWp*** (where '*a*' and '*b*' are personal variables, for which different names of different persons may be substituted) complete the definition of SS1E: Iff *p* has the basic matrix of SS1E then:

6.281 *aWp* has the matrix: 5 0 1 2 3 6 6 6 7 8
6.282 *bWp* has the matrix: o 5 1 2 3 6 6 6 7 8

6.29 The following table shows some epistemic matrices of SS1E:

p	*Np*	*aWp*	*NaWp*	*aWNp*	*NaWNp*	*bWp*	*NbWp*	*bWNp*	*NbWNp*	*aWaWp*	*aWbWp*
o	8	5	2	8	o	o	8	8	o	6	5
0	7	0	7	7	0	5	2	7	0	0	6
1	6	1	6	6	1	1	6	6	1	1	1
2	5	2	5	6	1	2	5	6	1	2	2
3	4	3	4	6	1	3	4	6	1	3	3
4	3	6	1	3	4	6	1	3	4	6	6
5	2	6	1	2	5	6	1	2	5	6	6
6	1	6	1	1	6	6	1	1	6	6	6
7	0	7	0	0	7	7	0	5	2	7	7
8	o	8	o	5	2	8	o	o	8	8	8

6.3 Truth and Consequence in SS1E.

6.31 A sentence (formula) is (materially or strictly or strongly) logically true (or: valid) in SS1E iff from its negation both ***p*** and ***Np*** are (materially or strictly or strongly) derivable. What is (materially or strictly or strongly) derivable from a certain sentence is determined by the matrix of ***Cpq*** (cf. **6.26** and **6.27**).

6.311 If the matrix of a sentence contains exclusively values between o and 3 then this sentence is logically true (or: valid) in SS1E. If the ***cv*** (cf. **1.42-1.44**) of the matrix is 1 the sentence is strongly, if the ***cv*** is 2 it is strictly and if the ***cv*** is 3 it is materially logically true.

6.312 The converse of **6.311** does not hold. Thus **6.311** is included in **6.31**. The reason for the more complicated form of **6.31** in contradistinction to **1.41** is this: there are formulas in SS1E which are logically false (because ***p*** and ***Np*** are derivable from it) even though their matrices do not contain exclusively values between 4 and 8 (but also some lower values between o and 3). An example is ***KaWpbWNp*** (its matrix contains 100 values) which is logically false because ***p*** and ***Np*** are derivable from it, although its matrix contains also some values between o and 3. That means that ***aWp*** and ***bWNp*** are contraria in SS1E, i.e. they can both be false but cannot both be true. The negation of ***KaWpbWNp***, namely ***ANaWpNbWNp*** is therefore a logically true (or: valid) sentence in SS1E.

6.32 A sentence (formula) is (materially or strictly or strongly) logically false in SS1E iff both ***p*** and ***Np*** are (materially or strictly or strongly) derivable from it. What is (materially or strictly or strongly) derivable from a certain sentence of SS1E is determined by the matrix of ***Cpq*** (cf. **6.26** and **6.27**).

6.33 A sentence is contingent in SS1E iff it is neither logically true nor logically false in SS1E.

6.34 The characteristical value of validity (***cv***) of a sentence of SS1E is the highest value between o and 8 which occurs in its matrix.

6.35 The sentences of **1.52** where the letters '*L*' and '*M*' are replaced by the corresponding bold face letters hold in SS1E.

6.36 In SS1E the relations of the matrices of the atomic formulas to one another are not so simple as in SS1. In SS1 the sentences *p*, *q*, *r* have the same basic matrix 1 2 3 4 5 6 and the relation is such that for instance *Kpq* has a matrix of 36, *KKpqr* has a matrix of 216 values. In other words: if the number of different atomic sentences is *n* then the number of values of the matrix of the compound sentence is 6^n in SS1. The matrices of the modal variations of the atomic sentences (*VA*) are viewed as belonging to the basic matrix of the corresponding atomic sentence which is not varied. Thus the matrix of *KpLp* has 6 values and the matrix of *KKpLqLp* has 36 values.

6.361 The relations of the matrices of the atomic formulas to one another are a little more complicated in SS1E. First of all there is no change refering to the atomic sentences ***p, q, r*** . . . : If the number of different atomic sentences is *n* then the number of values of the matrix of the compound sentence is 10^n in SS1E. All other cases can be decided by the help of two simple rules. Such other cases are for instance: ***KaWpbWp*** (the matrix has 100 values), ***KaWpp*** (the matrix has 10 values), ***KaWpaWNp*** (the matrix has 100 values), ***KaWbWpbWp*** (the matrix has 10 values), ***KaWpNaWp*** (the matrix has 10 values) etc. The two rules are:

6.362 R1: The matrix of a compound sentence consisting of two sentences ***r*** and ***s*** has 10 values iff ***r*** and ***s*** bear the same truth-functional relation to each other as ***p*** and ***Np.***

6.363 R2: The matrix of a compound sentence consisting of two sentences ***r*** and ***s*** has 10 values iff ***r*** and ***s*** bear the same truth-functional relation to each other as either ***p*** and ***aWp*** or ***p*** and ***bWp.***

6.364 Every case which does not satisfy either R1 or R2 has to be decided according to **6.361:** the matrix of a compound sentence containing different atomic sentences is 10^n.

6.365 R1 and R2 must not be applied both to one and the same sentence or to one and the same pair of sentences. Thus in the formula ***K AaWpaWCpq NaWpaWp*** and ***NaWp*** have as a compound sentence according to R1 a matrix with only 10 values; ***p*** and ***q*** have different matrices each consisting of 10 values such that the whole formula has a matrix of 1000 values.

6.37 Consequence in SS1E

6.371 A conclusion ***q*** follows (materially or strictly or strongly) from a premiss ***p*** iff from the negation of ***Cpq***, i.e. from ***NCpq*** (or: from ***KpNq***) both ***p*** and ***Np*** are (materially or strictly or strongly) derivable. The question whether ***p*** and ***Np*** are (materially or strictly or strongly) derivable has to be decided by the matrix of ***Cpq*** (cf. **6.26** and **6.27**).

6.372 The laws **1.611** a) and b) are included in **6.371. 6.373 1.611**a) holds unchanged. **6.374 1.611**b) can be replaced by the following extension of it: For all coordinated pairs of values of the matrices of ***p*** (premiss) and ***q*** (conclusion): the value of the matrix of ***p*** is one of the three values 6 or 7 or 8—or the value of the matrix of ***q*** is one of the three values 1 or 0 or o (or both cases hold). **6.375 1.612** holds unchanged. **6.376 1.613** holds unchanged.

6.4 Decision procedure, Consistency and Completeness of SS1E. **6.41** A sentence of SS1E is a theorem of SS1E iff it is (materially or strictly or strongly) logically true (cf. **6.31**). **6.42** There exists a decision procedure i.e. an answer to the question whether or not any sentence of SS1E is a theorem (of SS1E) for any sentence of SS1E; it is afforded by calculating the matrix of the formula ***CNfp*** and of the formula ***CNfNp*** (where ***f*** is any sentence of SS1E) for its ***cv***. If the ***cv*** of the matrix of both formulas ***CNfp*** and ***CNfNp*** is between o and 3 then ***f*** is a theorem of SS1E, if the ***cv*** of either ***CNfp*** or ***CNfNp*** is higher than 3 (i.e. 4 or 5 or 6 or 7 or 8) than ***f*** is not a theorem of SS1E.

6.421 A part of **1.82** is included in **6.42**. That means: If for any formula ***f*** of SS1E the ***cv*** is between o and 3 ***f*** is a theorem of SS1E; if for any formula ***f*** of SS1E the lowest value of its matrix is between 4 and 8 then ***f*** is not a theorem of SS1E, but provable false.

6.43 SS1E is consistent. This can be seen from **1.72, 1.73, 6.41** and **6.42**:

Case 1: The ***cv*** of a formula ***f*** of SS1E is between o and 3. Then ***f*** is a theorem of SS1E. The matrix of the negation of ***f*** has then only values between 4 and 8 (cf. **6.24**) i.e. ***Nf*** is therefore not a theorem (**6.421**) of SS1E.

Case 2: The matrix of a formula ***f*** of SS1E contains at least one value between o and 3 and at least one value between 4 and 8. Then ***f*** is a theorem iff the ***cv*** of the matrix of ***CNfp*** and of ***CNfNp*** is between o and 3. But ***Nf*** is then not a theorem for the ***cv*** of ***Cfp*** and of ***CfNp*** is then higher than 3 (cf. **6.42**). This is so because exactly these values between 4 and 8 in the matrix of ***Nf*** which make it possible to derive ***p*** and ***Np*** from ***Nf*** (i.e. which cause the ***cv*** of ***CNfp*** and of ***CNfNp*** to be between o and 3) are values between o and 3 in the matrix of ***f*** and thus must cause the ***cv*** of ***Cfp*** and of ***CfNp*** to be higher than 3.

6.44 SS1E is complete in the first, second and third sense of completeness, defined in **3.061, 3.063** and **3.065.** This is clear from the fact that SS1E contains the system SS1 (and SS1M) as a subsystem. This again can easily be seen by comparing the defining matrices of both systems.

6.5 Theorems of SS1E. **6.51** Criteria of consistency[32]. ***'MMp'*** stands for '***p*** is consistent' or '***p*** is logically possible'.

			cv
6.511	*C MMAWp MMKaWpp*		1
6.512	*C MMaWp MMp*		1

The converse of **6.512** does not hold

6.52 Theorems for "knowing that"

6.521 *C aWp p* 1

By **6.521** the conditions for some strong concept of knowledge are laid down. For this kind of knowledge the case that somebody knows that ***p***, where ***p*** is false at the same time, is excluded[33].

6.522	*Nk aWp NaWp*	principle of non-contradiction	1
6.523	*A aWp NaWp*	tertium non datur	1
6.524	*C aWp NaWNp*		1
6.525	*C aWNp NaWp*		1
6.526	*C aWp aWaWp*		2
6.527	*C aWNp aWaWNp*		2

Note: **6.526** and **6.527** show the reflexive character of knowledge: if ***a*** knows that ***p*** then he knows that he knows that ***p***. However if ***a*** does not know that ***p*** it does not always follow that he knows this (that he does not know it); therefore ***C NaWp aWNaWp*** is not a theorem of SS1E[34].

6.528	*C aWaWp aWp*	1
6.529	*C aWaWNp aWNp*	1

6.53 Distribution of epistemic operations.

6.531	*C KaWpaWq aWKpq*	1
6.532	*C AaWpaWq aWApq*	1
6.5321	*C AaWpaWNp aWApNp*	1

Note: The converse of **6.532** and **6.5321** does not hold. This is important. **6.5321** says that somebody who knows whether ***p*** is the case (cf. **6.541**) also knows that either (***p*** or not-***p***) is the case. In other words: One can accept that somebody knows the tertium non datur (i.e. knows that ***p*** or not-***p*** is the case) without being committed to assert that he must also know (separately) either that ***p*** is the case or that not-***p*** is the case. But if he knows the latter then he must know the former too. On the other hand there is also no reason to deny the theorem **6.523** as one version of the tertium non datur in epistemic logic. The important thing concerning the concept "provable" (which may be interpreted here with the epistemic operator ***aW*** . . .) and the intuitionistic views about it seems—as far as propositional logic is concerned—that **6.532** and **6.5321** hold only as implications (as in SS1E) but not as equivalences.

6.533 *C aWCpq CaWpaWq* 2

The converse of **6.533** is not valid in SS1E. The importance of this fact is similar to that of **6.532** and **6.5321**

6.534	*E KNaWpNaWq NAaWpaWq*	1
6.5341	*E NKaWpaWq ANaWpNaWq*	1
6.535	*E KNaWpNbWp NAaWpbWp*	1
6.5351	*E NKaWpbWp ANaWpNbWp*	1

6.534-6.5351 state that the laws of De Morgan are valid for the epistemic logic SS1E. ***cv***

6.536	*E KaWaWpaWbWp aWKaWpbWp*		2
6.5361	*C KaWaWpaWbWp aWKaWpbWp*		1
6.537	*E aW(bWp) KaWaWpaWbWp*		1 Df.

6.537 states the definition of "the person ***a*** knows that the person ***b*** knows that ***p*** is the case" symbolized as ***'aW(bWp)'***. It is important to observe the difference between ***aW(bWp)*** which is defined by **6.537** and ***aWbWp*** which is defined in **6.29.** The matrix of ***aWbWp*** has 10 values whereas the matrix of *aW(bWp)* has 100.

6.5371	**C** ***aW(bWp) KaWpbWp***		1
6.5372	**C** ***aW(bWp) aWbWp***		3
6.5373	**C** ***aW(bWp) bWp***		3
6.5374	**C** ***aW(bWp) aWp***		1
6.538	***E KaWaWNpaWbWNp aWKaWNpbWNp***		2
6.5381	**C** ***KaWaWNpaWbWNp aWKaWNpbWNp***		1
6.539	***E KaWpaWq KaWqaWp***	law of commutation	1
6.5391	***E AaWpaWq AaWqaWp***		1

The analogous laws hold also for ***KaWpbWp*** and ***AaWpbWp.***

6.54 Theorems for "knowing whether"

6.541 ***E*** $aW^{o}p$ ***AaWpaWNp*** 1 Df.

6.541 states the definition of "***a*** knows whether ***p*** is the case", symbolized as '$aW^{o}p$'. Note: $aW^{o}p$ has a matrix with 100 values of the form ***Apq*** where ***p*** is ***aWp*** and ***q*** is ***aWNp.***

6.542	***E*** $aW^{o}p$ $aW^{o}Np$	1
6.543	**C** ***aWp*** $aW^{o}p$	2
6.5431	**C** ***aWNp*** $aW^{o}p$	2
6.544	**C** $NaW^{o}p$ ***NaWp***	2
6.5441	**C** $NaW^{o}p$ ***NaWNp***	2

6.55 Theorems for "knowing whether" with two persons

6.551	***E*** $aW(bW^{o}p)$ ***AaW(bWp)aW(bWNp)***	1 Df.
6.5510	***E*** $aW(bW^{o}p)$ ***AKaWaWpaWbWpKaWaWNpaWbWNp***	1

6.551 states the definition of "***a*** knows that ***b*** knows whether ***p*** is the case". There is also an important weaker form of "***a*** knows that ***b*** knows whether ***p*** is the case" which is defined in **6.552.** The difference between $aW(bW^{o}p)$ and $aWbW^{o}p$ is seen from the consequences of both forms stated in **6.5511-6.5521.**

6.552	***E*** $aWbW^{o}p$ ***aWAbWpbWNp***	1 Df.
6.5511	**C** $aW(bW^{o}p)$ $aWbW^{o}p$	3
6.5512	**C** $aW(bW^{o}p)$ $aW^{o}p$	2
6.5513	**C** $aW(bW^{o}p)$ $bW^{o}p$	3
6.5521	**C** $aWbW^{o}p$ $bW^{o}p$	1

It is easily observed that the difference between $aW(bW^{o}p)$ and $aWbW^{o}p$ is that from the former $aW^{o}p$ is derivable but not from the latter. An example for $aW(bW^{o}p)$ may be: The professor (a) knows that his student (b) knows whether p is the case. In this case it is presupposed that also the professor knows whether p is the case; thus $aW^{o}p$ must be derivable from $aW(bW^{o}p)$. An example for $aWbW^{o}p$ may be: The student (a) knows that his professor (b) knows whether p is the case. Here it is not presupposed that the student knows also whether p is the case; thus $aW^{o}p$ is not derivable from $aWbW^{o}p$.

cv

6.553 $E\ aW^{o}bWp\ AaW(bWp)aWNbWp$ 1 Df.

6.553 states the definition of "a knows whether b knows that p is the case" which seems natural when looking at the definiens. However one can observe by looking for consequences of $aW^{o}bWp$ that this is a weak interpretation of "a knows whether b knows that p"; for neither $bW^{o}p$ nor $aW^{o}p$ is derivable from it. The essential point for the weakness lies in the interpretation of $aWNbWp$ which is a part of the above definiens; $aWNbWp$ is determined by a matrix in SS1E such that it does not imply that aWp. Although we use such a weak interpretation of "a knows whether b knows that p" when it is not presupposed that either aWp or $aW^{o}p$ is the case, we sometimes use a stronger interpretation. Thus, when we say "a knows that it is not the case that b knows that p" we often presuppose that a knows that p. This sense of "a knows that it is not the case that b knows that p" is interpreted by $aW(NbWp)$ which is defined in 6.554. With the help of $aW(NbWp)$ one gets a stronger interpretation of "a knows whether b knows that p" which is defined in 6.555.

6.554 $E\ aW(NbWp)\ KaWaWpKaWNbWpaWNbWNp$ 1 Df.

6.555 $E\ aW^{o}(bWp)\ AaW(bWp)aW(NbWp)$ 1 Df.

6.5550 $E\ aW^{o}(bWp)\ A\ KaWaWpaWbWp\ KaWaWpKaWNbWpaWNbWNp$ 1

(cf. 6.555, 6.537, 6.554). That 6.555 is stronger than 6.553 can be seen from its consequences:

6.5551 $C\ aW^{o}(bWp)\ aWp$ 3

6.5552 $C\ aW^{o}(bWp)\ aW^{o}p$ 3

6.556 $E\ aW^{o}(bW^{o}p)\ A\ AaW(bWp)aW(bWNp)\ aW(NbWp)$ 1 Df.

6.5560 $E\ aW^{o}(bW^{o}p)\ A\ AKaWaWpaWbWpKaWaWNpaWbWNp$
$KaWaWpKaWNbWpaWNbWNp$ 1

(cf. 6.556, 6.537, 6.5510, 6.554). 6.556 states the definition of "a knows whether b knows whether p is the case". 6.5561 shows that $aW^{o}(bW^{o}p)$ is a strong form whereas $aW^{o}bW^{o}p$ is a weak form which is implied by $aW^{o}(bW^{o}p)$.

6.5561 $C\ aW^{o}(bW^{o}p)\ aW^{o}p$ 3

6.557 $E\ aW^{o}bW^{o}p\ A\ aWbW^{o}p\ aWNbW^{o}p$ 1 Df.

6.5570 $E\ aW^{o}bW^{o}p\ A\ aWAbWpbWNp\ aWNAbWpbWNp$
(cf. 6.541) 1

6.5571 $E\ aW^{o}bW^{o}p\ A\ aWAbWpbWNp\ aWKNbWpNbWNp$
(cf. 6.534) 1

6.56 Implicational relations between the forms of "knowing that" and "knowing whether"

			cv
6.561	$C\ aWp\ NbWNp$		1
6.5611	$C\ aWNp\ NbWp$		1
6.562	$C\ aW(bWp)\ aWKaWpbWp$		1
6.563	$C\ aWKaWpbWp\ aW(bW^{\circ}p)$	(cf. **6.551**)	2
6.5631	$C\ aW(bWp)\ aW(bW^{\circ}p)$	(cf. **6.537**)	3
6.5632	$C\ aW(bW^{\circ}p)\ aWbW^{\circ}p$	(cf. **6.551, 6.552**)	2
6.5633	$C\ aWbW^{\circ}p\ aW^{\circ}bW^{\circ}p$	(cf. **6.557, 6.5570**)	2
6.564	$C\ aW(bW^{\circ}p)\ aW^{\circ}(bW^{\circ}p)$	(cf. **6.556, 6.5560**)	2
6.5641	$C\ aW^{\circ}(bW^{\circ}p)\ aW^{\circ}bW^{\circ}p$		3
6.565	$C\ aWKaWpbWp\ aW^{\circ}(bWp)$	(cf. **6.555, 6.5550**)	2
6.5651	$C\ aW(bWp)\ aW^{\circ}(bWp)$		2
6.5652	$C\ aW^{\circ}(bWp)\ aW^{\circ}bWp$	(cf. **6.553, 6.537**)	2
6.5653	$C\ aW^{\circ}(bWp)\ aW^{\circ}(bW^{\circ}p)$		2
6.566	$C\ aWKaWpbWp\ aWbWp$		2
6.5661	$C\ aWbWp\ aW^{\circ}bW^{\circ}p$		2

6.6 Epistemic operations and modalities. There is no difficulty to deal with sentences like ***aWLp, aWLLp, aWMLp, aWMp, aWMMp, aWLMp*** in SS1E. It is clear that the following theorem must hold:

6.61	$C\ aWCLLpLp\ CaWLLpaWLp$	2

The analogous cases with the other modal operations yield to similar theorems by the principle **6.533.**

6.62 However it is not so simple to define "it is necessary that ***a*** knows that ***p***". First of all it seems that "it is necessary that a knows that ***p***" (symbolized as '**L*aWp***') must be independent of "***a*** knows that necessarily ***p***" (***aWLp***) in this sense that neither the former is in general derivable from the latter nor the latter from the former. A second condition for an adequate interpretation of **L*aWp*** seems to be the following:
If one says that a person necessarily knows some proposition ***p*** (say the principle of non-contradiction) then it is implied that this is known also by others. Thus if the person ***a*** necessarily knows that ***p*** is the case then it follows (or it is presupposed) that any person ***b*** (under normal conditions) also knows that ***p*** is the case. A third condition is that **L*aWp*** should be defined with the help of the modal operation ***Lp*** because in order to bring out that somebody necessarily knows something it is not sufficient to state that any or that every person knows it too.—These three conditions are satisfied by the following definition:

6.63	$E\ LaWp\ KaWpbWLLp$		1 Df.
6.631	$C\ LaWp\ aWLp$	(first condition)	***k***
6.631	$C\ aWLp\ LaWp$	(first condition)	***k***
6.6312	$C\ aWLLp\ LaWp$	(first condition)	***k***
6.6313	$C\ LaWp\ aWMLp$	(first condition)	***k***
6.632	$C\ LaWp\ bWp$	(second condition)	1

			cv
6.633	C LaWp $bWLLp$	(third condition)	1

Note: By the definition in **6.63** the statement '***a*** necessarily knows that ***p***' is interpreted in such a way that it is true only if ***p*** is a logically necessary statement and if in addition to that some person ***b*** knows this (that ***LLp*** holds). However from L***aWp*** it does not follow that ***a*** would know that ***LLp*** (or even ***Lp***) holds. This fact can be explained as follows: If one says of some person that he necessarily knows that ***p***, then one does not always imply that this person knows what that means: thus it seems reasonable to assert that we can truly say of a person that he necessarily knows the principle of non-contradiction without claiming that this person (who may be philosophically or logically uneducated) knows also (perhaps by reflection on his knowledge and by the help of philosophical and logical studies) that what he knows (namely that ***p***) does logically necessarily hold. Therefore it is allowed by **6.63** to say that a person ***a*** necessarily knows that ***NKpNp*** (perhaps because one thinks that this is a principle common to all men—whatever anthropological explanation one may give) although this person ***a*** does not know that ***LLNKpNp***, i.e. that ***NKpNp*** is a logically necessary law.

6.64	C LaWp aWp	3
6.641	C LaWp NL$aWNp$	1
6.642	C L$aWNp$ NLaWp	1
6.643	C aWLaWp LaWp	1

Note: The converse of **6.643** does not hold; i.e. L***aWp*** does not have the reflexive character which ***aWp*** has. The reason for this should be clear from what has been said in the note of **6.633.** Nevertheless the following statement holds materially.

6.644	C LaWp $aWaWp$	3
6.646	E NKLaWpLaWq ANL$aWpN$LaWq	1
6.647	E KNL$aWpN$LaWq NALaWpLaWq	1
6.648	E NKLaWpLbWp ANL$aWpN$LbWp	1
6.649	E KNL$aWpN$LbWp NALaWpLbWp	1

Note: The forms which are analogous to the laws **6.531-6.533** (where '***aW***' is replaced by 'L***aW***') do not hold in SS1E.

6.65	C LaWp $KaWpbWp$	1
6.651	C $KaWLLpbWLLp$ LaWp	1
6.652	C $KaWLLpbWLLp$ aWLaWp	*k*
6.653	C aWLaWp $aW(bWp)$	2

Cf. the consequences of ***aW(bWp)***, **6.562-6.5652**

6.66	E MaWp $KpMMaW^{o}p$	1 Df.

6.66 states the definition of 'it is possible that a knows that ***p***'. According to this definition for the truth of 'M***aWp***' it is a necessary and sufficient condition that both, ***p*** is the case and '***a*** knows whether ***p***' is consistent (i.e. $MMaW^{o}p$).

6.661	C MaWp p	1

6.661 says that one presupposes that *p* is the case (is true) if one says that some person possibly knows that *p*. This condition seems to be adequate to the common use of 'it is possible that *a* knows that *p*'. The converse of **6.661** does not hold. Thus it is not claimed in the system SS1E that all what is true can possibly be known by men.

		cv
6.662	*C aWp* M*aWp*	2

The converse of **6.662** does not hold of course.

		cv
6.663	*C* L*aWp* M*aWp*	3
6.6631	*C* L*aWp aW*M*aWp*	3
6.664	*C aW*M*aWp* M*aWp*	1

The converse of **6.664** does not hold; i.e. it may be true to say that it is possible that somebody knows that *p*, although this person does not know this (cf. **6.643** and the note).

		cv
6.6641	*C aW*M*aWp aWp*	*k*
6.6642	*C bW*M*aWp bWp*	1

Note: The forms which are analogous to **6.641, 6.642** and **6.646-6.649** (where 'L*aWp*' is replaced by 'M*aWp*') do not hold.

			cv
6.665	*C aW°(bWp)* M*aWp*	(cf. **6.5551, 6.662**)	3
6.67	*C aWLp aWp*		2
6.671	*C aWLp NaWLNp*		1
6.6711	*C aWLp NaWNLp*		1
6.6712	*C aWLNp NaWLp*		1
6.6713	*C aWNLp NaWLp*		1
6.672	*E NKaWLpaWLq ANaWLpNaWLq*		1
6.6721	*E KNaWLpNaWLq NAaWLpaWLq*		1
6.673	*E NKaWLpbWLp ANaWLpNbWLp*		1
6.6731	*E KNaWLpNbWLp NAaWLpbWLp*		1

The analogous forms of **6.672-6.6731** with the other modalities (where '*L*' is replaced by ***'LL', 'ML', 'M', 'MM', 'LM'***) hold too. The analogous laws of **6.531-6.533** hold with all ***VA(aWLp*** . . .) strongly. Further laws can be easily proved with the help of the matrices.

6.68 There is another sense of "the person *a* necessarily knows that *p*" (named by: **'L'*aWp*'**) than that defined in **6.63.** This other sense is used if one makes assertions about his own psychic (mental) phenomena at the present time. More explicitly: It is this assertion which states that one has (is aware of) now this or that psychic (mental) phenomenon. The mental phenomenon or psychic action may be any one, for example an action of representing, imagining, guessing, doubting, judging, asserting, desiring, enjoying, unpleasing, loving, hating . . . etc. This judgement or assertion with which I state that I have (now) this or that mental action is called "judgement of introspection"[35]. Now it seems that one could say in some perfectly good sense that a person who asserts something about his own mental phenomena by a judgement of introspection not only knows that he has this mental action but necessarily knows that.

Comparing the three conditions of **6.62** with conditions for this new concept of "necessarily knowing that ***p***" one can easily see that the first condition of **6.62** must hold here too. But on the other hand the second condition of **6.62** cannot be required here. For a person ***b*** cannot know directly something about the mental phenomena of a person ***a***. Also the third condition is not defensible because this what is known by introspection (the mental phenomena) is not necessarily the case (as it is if one knows the principle of non-contradiction) neither logically necessary nor empirically necessary. The new requirement here seems to be the following: if the person ***a*** knows that he has a certain mental action then the person ***a*** necessarily knows that he has this action. This requirement is satisfied by the definition

6.681 ***E*** L'***aWr aWr*** ***cv*** 1 Df.

where '***r***' is a sentence which has such a form that the following sentences are concrete instances of this form:

'***a*** represents ***x***'	'***a*** wants that ***p***'
'***a*** guesses that ***p***'	'***a*** loves ***x***'
'***a*** judges that ***p***'	'***a*** enjoys ***x***'
'***a*** affirms that ***p***'	etc.

From this determination of ***r*** it is clear that statements like ***CaWLraWr***—though they are true—need not to violate the above mentioned requirements because in all such statements the antecedens is viewed to be false. This is so if one agrees that mental phenomena—the occurrence of which is expressed assertively in the sentence '***r***'—neither occur with physical or natural necessity nor with logical necessity.

7. Tense-Logic based on SS1. **7.1** In his (TCT) and (PTL) Prior gives axioms for a Tense-Logic. In (PTL) he also mentions interpretations of time-operations by modal operations. For this purpose he uses the system T of Feys with the additional axiom of Geach for the system S4.3, *ALCLpqLCLqp*, and the system S4 of Lewis (cf. **3.44**). It is shown in the following that all axioms of Prior's system GH1 are valid in SS1 (more accurately: SS1M) if one interprets the time-operations by modal-operations of SS1M in the way of **7.5**.

7.2 Definitional abbreviations
7.21 '*Pp*' for 'it has been the case that *p*'
7.22 '*Hp*' for 'it has always been the case that *p*'
7.23 '*Fp*' for 'it will be the case that *p*'
7.24 '*Gp*' for 'it will always be the case that *p*'
7.3 Definitions
7.31 *F* = df. *NGN*
7.32 *P* = df. *NHN*
7.4 Rules

7.41 *RG*: If $\vdash \alpha$ then $\vdash G\alpha$

As it is clear from **1.53** this rule does not hold in SS1M. Instead of this rule the more detailed statements of **1.53** are valid.

7.42 *MI* (the Mirror Image rule): *In any thesis we may replace P by F, G by H and vice versa, throughout.*

7.5 Interpretation. **7.51** In the following *Pp* and *Fp* are interpreted by *LMp* in SS1 (SS1M); *Hp* and *Gp* are interpreted by *MLp* in SS1 (SS1M). This satisfies the definitions **7.31** and **7.32.** That it is not necessary to have different interpretations for *Pp* and *Fp* on the one hand and for *Hp* and *Gp* on the other is clear by the rule *MI* of the system GH1.

7.52 In the following I want to give reasons for choosing the operations *ML* and *LM* of SS1M in order to interpret the time-operations *P*, *H*, *F* and *G*.

At first it seems clear that *LLp* and *MMp* (as the strongest form of necessity and the weakest form of possibility in SS1M) are not suitable for the following reason: that which is logically necessary should be viewed as more generally valid than that which is valid for all the time; for the time of which we speak is—according to the theory of relativity—bound to our (factual) universe whereas a logically necessary statement is valid in all possible worlds (Leibniz) but not only in our universe. On the other hand: that which is logically possible is less general than that which is valid for at least one time-stretch *t*; for if one says that it is logically possible that a certain event occurs, this statement may be consistent even if this event never does occur (i.e., even if there is no time-stretch *t* where the event occurs).

Secondly it remains to give reasons for not having chosen the operations *L* and *M* for an interpretation of *P*, *H*, *F* and *G*. The stimulation for such a reason the author got from the essay (UDN) of Popper in which a definition of natural (or: physical) necessity is given. Poppers definition is the following: "A statement may be said to be naturally or physically necessary if, and only if, it is deducible from a statement function which is satisifed in all worlds that differ from our world, if at all, only with respect to initial conditions" (LSD) p. 433.

If one would try now to interpret natural necessity by 'for all times *t*, it yields that . . .' or by 'it was always the case and it will always be the case that . . .' (*KHpGp*) then one arrives at the following conclusion: the statements 'for all times *t*, it yields that . . .' and 'it was always the case and it will always be the case that . . .' are also valid for the more general initial conditions which hold in the (factual) universe; as for example for the totality of mass or energy which is in the universe. From this consideration and from Popper's definition of natural necessity it seems to follow that natural necessity is stronger and more general than that the validity of which is determined by universal time-operations or time-quantification. If therefore natural necessity is represented in SS1M by *L* then the statements 'for all *t*, it yields that', '*Hp*' and '*Gp*' must be interpreted with an operation weaker than *L*. The conditions for such an operation which—when applied to *p*—should give a statement weaker than *Lp* but (and this seems clear) stronger than *p* are exactly satisfied by the operation *ML* of SS1 (SS1M). Analogous reasons can be given for interpreting *Pp* and *Fp* by *LMp* of SS1 (SS1M).

			cv
7.6	The system T of Feys in SS1 (SS1M)		
7.61	$C\ Lp\ p$	(cf. **3.461**)	2
7.62	$C\ LCpq\ CLpLq$	(cf. **3.462**)	1
7.63	Rule: *If* $\vdash \alpha$ *then* $\vdash L\alpha$		

This rule (more accurately: their corresponding formula of the calculus SS1 (SS1M) which is in the object language and not in the meta language as the rule) is not in general valid in SS1 (or SS1M). Only the more detailed statements of **1.53** are valid in SS1 and SS1M.

7.64	Additional axiom of Geach for S4.3		
7.641	$A\ LCLpq\ LCLqp$		3
7.65	Additional axiom of Hintikka for S4.3		
7.651	$C\ KMpMq\ AMKpMqMKqMp$		1
7.7	Axioms of GH1 interpreted in SS1M		
7.71	$C\ GCpq\ CGpGq$		
7.711	$C\ MLCpq\ CMLpMLq$		1
7.72	$C\ Gp\ NGNp$		
7.721	$C\ MLp\ NMLNp$		1
7.73	$C\ Gp\ GGp$		
7.731	$C\ MLp\ MLMLp$		1
7.74	$C\ GGp\ Gp$		
7.741	$C\ MLMLp\ MLp$		1
7.75	$C\ GCpq\ C\ GCpGq\ CGCFpqCFpGq$		
7.751	$C\ MLCpq\ C\ MLCpMLq\ CMLCLMpqCLMpMLq$		1
7.76	$C\ NHNGp\ p$		
	$C\ NGNGp\ p$	(with *MI*)	
7.761	$C\ NMLNMLp\ p$		2
7.77	$C\ p\ C\ Gp\ CHpGHp$		
	$C\ p\ C\ Gp\ CGpGGp$	(with *MI*)	
7.771	$C\ p\ C\ MLp\ CMLpMLMLp$		1

7.78 As Prior remarks in a footnote (PTL) p. 153, Lemmon showed that the axiom 7 (**7.77**) can be derived from the axioms 1, 5 and 6 (**7.71, 7.75** and **7.76**) with the help of *RG* and *MI* (i.e. axiom 7 is not an independent axiom). In the same footnote Prior says that independency-proofs for all the other 6 axioms have been given by Berg and Hacking.

7.79 Validity of the axioms of GH1 in SS1M. If one takes the interpretation of **7.5** and if one assumes the validity of the rule *MI* then all the seven axioms of GH1 (and all the six independent axioms of GH1) are valid in SS1M. The axioms 1-5 are strongly valid, the axiom 6 is strictly valid in SS1M. If the rule *MI* is dropped oniy axioms 1-5 are (strongly) valid, whereas axioms 6 and 7 are no longer interpretable in the above sense.

7.8 It is perhaps worth noting what happens if one takes the modal interpretations for *Hp* and *Gp* which are mentioned by Prior (PTL) p. 153:

(1) Hp is interpreted as p and Gp is interpreted as Lp (in the absence of the rule MI) of the system T of Feys plus Geach's axiom S4.3. In this case—as Prior states—all axioms of GH1, except axiom 3, are valid. If we take instead of the system T (and Geach's axiom) the system SS1M the axioms 1, 2, 4 and 5 are strongly valid in SS1M; axiom 3 is not valid in SS1M (but contingent) under this interpretation. Axiom 6 and axiom 7 which are the only axioms containing Hp are not representable in SS1M if one interprets Hp as p (in the absence of the rule MI). However if one assumes the rule MI to be valid then axiom 6 is strictly valid and axiom 7 is strongly valid in SS1M.
(2) Hp and Gp are interpreted as Lp of SS1M. Then the axioms 3 and 7 are not valid (contingent) in SS1M, all other axioms are valid in SS1M.
(3) Hp is interpreted as Pp. This interpretation violates definition **7.32.** If one accepts definition **7.31** and interprets Gp as MLp and Pp as LMp of SS1M then all axioms of GH1 are valid in SS1M. This is also the case if Pp is interpreted as MLp of SS1M.
7.81 Not all of the interpretations of Pp, Hp, Fp and Gp which are stated in **7.8** satisfy the reasons given in **7.52** for the interpretation in **7.51.** Thus although there may be interpretations of **7.8** and some similar ones under which all the axioms of GH1 become valid statements of SS1M, they need not satisfy the reasons given in **7.52.** It is interesting to observe that under interpretation (3) the axioms of GH1 are valid in SS1M no matter if interpreting Pp as LMp or as MLp of SS1M. If Pp is interpreted as LMp then—for $CpLMp$ is valid in SS1M and $Hp = Pp$—the statement 'if p is true then it has always been the case that p' must be valid. But this seems odd for p may be only contingently true and not necessarily. One may conclude: The equating of Hp with Pp seems counterintuitive if Pp is interpreted as LMp in SS1M. If one looks for the other case of equating Hp and Pp, taking Pp as MLp then—for $CMLpp$ is a theorem of SS1M—the statement 'if it has been the case (at some time) that p then p is the case' must be valid. But also this statement seems to be counterintuitive because the truth of p (even if it is a contingent truth) is not restricted to a certain time. These considerations (**7.81**) seem to substantiate the interpretation of Pp, Hp, Fp and Gp which has been given in **7.51** and which satisfies the conditions of **7.52.**

SUMMARY

The following summary was given by Prof. K. R. Popper (in a letter to the author from April 1967) summarizing three talks of the author on the topics of this paper at the University of London in February 1967.

"(1) You have given a demonstrably consistent method of introducing the modalities "logically necessary (possible, impossible, . . .)" and "physically necessary (possible, impossible)" into propositional logic. (These modalities may perhaps also be differently interpreted.)
(2) You have, furthermore, given a method of introducing, in addition to the modalities, an epistemic logic. This is a much discussed problem; and although I

personally believe that it is a mistake to expect that epistemic logic is of special interest for epistemology, the solution of the problem you have given is transparent and probably the best that can be expected.
(3) You have done all this with the help of a very simple and straightforward idea: that of introducing a new couple of (positive and negative) truth values for each of the new levels (level of physical modality; level of logical modality; epistemic level) which you introduce.
(4) By doing all this you have at the same time given the first useful and philosophically interesting interpretation of many-valued formal systems of which I know; and you have given reason to expect that only $2n$-valued systems can be expected to furnish philosophically interesting interpretations."

NOTES

1. Works and essays on Aristotle's modal logics: A. Becker (TND), (ATM), O. Becker (UMo), Bocheński (FLg) 15.01-15.23, (AFL) p. 55-62, McCall (AMS), Feys (SMA), Hintikka (IIn), (NUT), (FSF), Kneale (DLg) p. 81-96, Łukasiewicz (ASS), (CPA), Patzig (ASy), Prior (TMo), Weingartner (VFW).

2. Under the "dictum" they understood an expression which originates from a statement and begins with "that": the dictum of the statement "Socrates runs" is "that Socrates runs". *Cf.* Thomas Aquinas (PMo), Bocheński (LTh), (FLg), 15.13, 17.12-17.17, 29.09-29.14, (NHP), Kneale (MDR), Prior (MDR).

3. *Cf.* Bocheński (AFL) p. 57ff. and A. Becker (ATM). *Cf.* the references under footnote 2.

4. *Cf.* Alexander Aphrodisiensis (AAP) p. 183, 42 ff. Epictetus (DAD) II, 19,1. Bocheński (FLg) p. 132, Kneale (DLg) p. 117-128. Mates (SLg) p. 36-41. For the master-argument of Diodorus: Prior (DMe), Hintikka (AMD).

5. For Quotations from these works see Bocheński (FLg) 33.04-33.05, 29.10-29.11, 33.09-33.19, 29.12.

6. For a detailed bibliography see Feys (MoL).

7. *Cf.* also Hermes (TAM).

8. The system as it is developed in this paper lacks precision in some (unessential) points. For instance, when it is said that propositional variables are considered as formulas to which truth-values can be assigned. Most precisely speaking one should give truth-values only to quantified propositional functions i.e. to full propositions as it is done for instance in the system of Leśniewski (GZN), (BFM) and Tarski (PTL). The reasons for not having established a propositional calculus with quantification are the following: 1. This lack of precision does not touch the validity of any statement of the system. 2. The addition of quantification would rather complicate the system and would not help for a better understanding of it. 3. There are methods of interpreting quantification for propositional logic in SS1 but the scope of the paper is too restricted to discuss this in detail. The following remark may suffice to indicate the idea: the matrix of the proposition "for all p, it yields that p" is interpreted as the conjunction (1.12) of the values of p; i.e. if p has the basic matrix, the matrix of this conjunction consists of the only value 6. Similarly the matrix of the proposition "for some p, it yields that p" is interpreted as the disjunction (1.11) of the values of p; i.e.

if p has the basic matrix the matrix of this disjunction consists of the only value 1. See: Weingartner (KBB). 4. In almost all of the systems of propositional calculus there is the same lack of precision, i.e. these systems are not quantified. Also for the sake of simplicity no metalanguage (with certain metalinguistic symbols) is introduced. The main reason is that the system SS1 does not contain rules, except a kind of substitution rule. As it is shown later (**1.62**) there is no need for deduction rules as in other systems which are built on the axiomatic method or on the method of natural deduction.

9. 'iff' is to be understood as "if and only if".

10. The concept of consequence class owes its origin to Tarski. It is formulated and defined in his fundamental studies (FBM), (FMD) and (GZS).

11. *Cf.* Kleene (IMM) p. 129.

12. *Cf.* Kleene (IMM) p. 136f.

13. In order to be able to read more easily the formulas I left some space between the main-connective and the rest of the formula and between those two parts of the formula which are connected by the main-connective.

    The number in brackets under the current number of the formulas (in this case: 7) is the number of variations proved. If the number of the variations which have been proved, does not equal the number of the valid variations (of the proved ones) in SS1 or SS1M—but is smaller—then this number of the valid variations is written (in brackets) under the number of the proved variations as in **2.24**. The number of proved (decided) formulas in chapters 2, and 3, are about 18,300; of these about 14,400 have been proved valid in SS1 (SS1M).

14. The *VO LLK* of **2.24** is *CLLKLLKpqrLLKpLLKqr*.

15. '*k*' stands for 'contingent'; i.e. the *cv* is either 4 or 5 or 6 and the matrix contains at the same time at least one value between 1 and 3.

16. The *VA\Mp* of the *VO MA, MMA, LMA* of **2.311** are: *CMAMpMpMp*, *CMMAMpMpMp* and *CLMAMpMpMp*.

17. The *VO + VA LC ... MLK* of **2.427** are: *LCLCpqLCMLKrpMLKrq* and its *VA*-variations.

18. From here on—if not explicitly the contrary is said—all the operations *C* are varied only with *L, LL* and *ML*, but no longer with *M, MM* and *LM*.

19. *Cf.* Kleene (IMM) p. 131.

20. *Cf.* Tarski (FBM) Theorem 3*.

21. Definition 5.

22. As it is well-known Gödel (IIA) gave an interpretation of the intuitionistic calculus with the help of laws of the system S4 of Lewis.

23. O. Becker (UMo) p. 16ff.

24. A full axiomatization of CS was first given by Łukasiewicz in (ASS). A different one is due to Bocheński (KSy). *Cf.* Lorenzen (FLg) ch. I. Lorenzen gives an interesting new interpretation of syllogistic.

25. *Cf.* Bocheński (KSy) in: Bocheński (LPS) p. 32f. Or: (CSy) p. 28ff.

26. *Cf.* Łukasiewicz (ASS) p. 88; Bocheński (KSy) in Bocheński (LPS) p. 24 and 31. Or: (CSy) in: (LPSe) p. 22 and 28.

27. *Cf.* Brentano (PES) II, p. 78ff. and 176f., Prior (FLg) p. 164ff., Scholz (MUn) p. 330f., Weingartner (VFW) p. 61ff., Juhos (EZM) p. 71ff.

28. *Cf.* Bocheński (KSy) in: Bocheński (LPS) p. 33. Or: (LPSe) p. 29f.

29. *Cf.* Bocheński (AFL) p. 57.

30. Note: Hintikka, in his (KBe) p. 59, says that the statements 'it is not the case that *a* knows that *p*' and 'it is possible for all that *a* knows that not-*p*' are interchangeable in his system; the same holds for the two statements 'it is not the case that it is possible, for all that *a* knows, that *p*' and '*a* knows that not-*p*' i.e. they are also interchangeable in his system. If one interprets 'it is possible for all that *a* knows that not-*p*' by *aWMNp* (perhaps Hintikka would not agree with that interpretation) then *NaWp* and *a WMNp* are interchangeable in the system proposed in **6.12** but not in the system proposed in **6.13** and not in SS1E. If one interprets further 'it is not the case that it is possible, for all that *a* knows that *p*' with *NaWMp* then again *aWNp* and *NaWMp* are interchangeable in the system proposed in **6.12** but not in the system proposed in **6.13** and not in SS1E. If however the statement 'it is not the case that it is possible for all that *a* knows that *p*' is interpreted with *aWNMp* then in both systems of **6.12** and **6.13** and in SS1E only the implication *C aWNMp aWNp* holds but no interchangeability.—From all this it seems that the concept of knowledge, which Hintikka has in mind, is not so understood as to include in its negation a kind of not-knowing which one may call ignorance. Because in the case of ignorance (put for not-knowing) the laws of interchangeability of Hintikka's system do not seem to hold.

31. For the operation signs of the extented system SS1E bold-face letters are used to keep in mind the difference between the operations of SS1 and SS1E.

32. These criteria of consistency are due to Hintikka (KBe) p. 16ff.

33. For a discussion in defense for using such a concept of knowledge see Hintikka (KBe) p. 48f. and Weingartner (CoA) footnote 59, in: Weingartner (DAE) p. 310 and the discussion (DAE) p. 403f.

34. For a discussion of connected problems see Hintikka (KBe) ch. 5.

35. For a discussion of problems concerning introspection see Weingartner (CoA), in: Weingartner (DAE) p. 303 ff. and the discussion (DAE) p. 401f. *Cf.* Brentano (PES) Bd. I p. 128, 178, 196.

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