Notre Dame Journal of Formal Logic
Volume IX, Number 1, January 1968

A NOTE ON THE ARITHMETICAL HIERARCHY

STEPHEN L. BLOOM

Introduction. The purpose of this paper is to give a new proof of this theorem:

there is a $\Sigma_2 \cap \Pi_2$ *predicate having no inverse image*[1] *under any function from* N *onto* N *in* Σ_1 *or in* Π_1.

Although this is a fact about the arithmetical hierarchy, the only known proof (so far as I know) veers through quantification theory. Kleene [1] has shown that every consistent formula of quantification theory has a model in the domain of natural numbers N in which the satisfying predicates are in $\Sigma_2 \cap \Pi_2$. In [2] an example is given of a formula **F** with one predicate variable P having no model with domain N when P is interpreted as a Σ_1 or Π_1 predicate. Since predicates of integers and their inverse images satisfy the same sentences of quantification theory without identity, we can conclude that the predicate which satisfies **F** has the property stated in the theorem.

This is a somewhat surprising result, since it shows that the arithmetical hierarchy is, in a sense, independent of the 'names' of the integers. In contrast, Putnam [3] has shown that every $\Sigma_2 \cap \Pi_2$ predicate has an inverse image under a certain function from N onto N in the smallest class of predicates containing the r.e. predicates and closed under truth functions.

Since the theorem is a fact of recursive function theory, it would be appropriate to have a proof which does not involve extra-disciplinary detours. We present such a proof here.

Proof of the theorem. The trick in our proof is to code enough predicates with one predicate S to guarantee its inverse images are not too simple.

Let S_1, S_3, S_5, be the following recursive predicates: $S_1(x) \longleftrightarrow x = 0$; $S_3(x) \longleftrightarrow x = 1$; $S_5(x,y) \longleftrightarrow y = x + 1$. Let $S_7(x)$ be a r.e. non-recursive predicate, and define $S_{i+1} \longleftrightarrow$ as $\sim S_i$, for $i = 1,3,5,7$. We let $S(x,y,z)$ be

1. Throughout the remainder of the paper, "inverse image" will mean "inverse image under an arbitrary function from N onto N". We use the notations Σ_n, Π_n as Davis [4] uses $\mathbf{P}_n$, $\mathbf{Q}_n$.

Received July 14, 1967

defined by

$$S(x,y,z) \longleftrightarrow (z = 1,2,3,4,7 \text{ or } 8 \textit{ and } S_z(x) \textit{ and } y = z) \textit{ or } \\ (z = 5 \text{ or } 6 \text{ and } S_z(x,y))$$

S is clearly $\Sigma_2 \cap \Pi_2$.

For each natural number n we define a predicate $V_n(x)$ which is true iff $x = n$. (This device was suggested by Marvin Minsky.)

$$V_0(x) \longleftrightarrow S_1(x);\ V_1(x) \longleftrightarrow S_3(x);$$
$$V_{n+1}(x) \longleftrightarrow (\exists y_1)(\exists y_2) \ldots (\exists y_n)\,[V_1(y_1) \,\&\, S_5(y_1,y_2) \,\&\, S_5(y_2,y_3) \,\&\, \ldots \,\&\, S_5(y_n,x)]$$

or, equivalently,

$$(1)\quad \longleftrightarrow (y_1)(y_2) \ldots (y_n)[[V_1(y_1) \,\&\, S_5(y_1,y_2) \,\&\, \ldots \,\&\, S_5(y_{n-1}, y_n)] \rightarrow {\sim}S_5(y_n,x)]$$

V_{n+1} is recursive, since it can be written in both existential and universal quantifier forms. Finally, suppose that f is a fixed function from N onto N and $a_0, a_1, \ldots a_8$ are numbers such that $f(a_i) = i$. Let $P(x,y,z) \longleftrightarrow S(f(x),f(y),f(z))$. P is the inverse image of S under f.

Lemma. If P (or ${\sim}P$) is r.e., *there is a recursive function g such that $g(n) = u$ implies $f(u) = n$.*

Proof of lemma. Define $g(0) = a_0$, $g(1) = a_1$. Suppose $n + 1$ is given and P is r.e. (If ${\sim}P$ is r.e., we use a similar argument.) We want to define $g(n + 1)$. For any u,

$$\begin{aligned} f(u) = n + 1 &\longleftrightarrow V_{n+1}(f(u)) \\ &\longleftrightarrow (\exists y_1) \ldots (\exists y_n)\,[V_1(y_1) \,\&\, S_5(y_1,y_2) \,\&\, \ldots \,\&\, S_5(y_n,f(u))] \\ &\longleftrightarrow (\exists v_1) \ldots (\exists v_n)\,[V_1(f(v_1)) \,\&\, S_5(f(v_1),f(v_2)) \\ &\qquad \&\, \ldots \,\&\, S_5(f(v_n),f(u))] \end{aligned}$$

since f is onto. Using the definitions of S and P we can write this last line as

$$(2)\quad (\exists v_1) \ldots (\exists v_n)\,[P(v_1,a_3,a_3) \,\&\, P(v_1,v_2,a_5) \,\&\, \ldots \,\&\, P(v_n,u,a_5)]$$

But we can use (1) and the same procedure to obtain as an equivalent form of (2)

$$(3)\quad (v_1) \ldots (v_n)\,[[P(v_1,a_3,a_3) \,\&\, P(v_1,v_2,a_5) \,\&\, \ldots \,\&\, P(v_{n-1},v_n,a_5)] \rightarrow \\ {\sim}P(v_n,u,a_6)]$$

From (2), (3) and the fact that P is r.e., we see that the predicate $\hat{u}V_{n+1}(f(u))$ can be written in both one-quantifier forms and is therefore recursive. If we now define $g(n + 1)$ to be the least u such that (3) holds, g will be recursive. (Notice that such a u always exists, since f is assumed to be onto.) This concludes the proof of the lemma.

We can now finish the proof of the theorem. Recall that $S_7(x) \longleftrightarrow P(u,a_7,a_7)$ and ${\sim}S_7(x) \longleftrightarrow P(u,a_8,a_8)$, where u is any number such that $f(u) = x$. If P (or ${\sim}P$) were r.e., by the lemma we could find such a u as a recursive function of x, so that S_7 would be recursive, contrary to hypothesis. Thus P cannot be in Σ_1 or Π_1 and S is a predicate having the desired property.

Since this argument "relativizes", we have the following corollaries.

Corollary 1. For any set B of natural numbers, there is a predicate in $\Sigma_2^B \cap \Pi_2^B$ *having no inverse image in* $\Sigma_1^B \cup \Pi_1^B$.

Corollary 2. For any set B of natural numbers and any $n \geq 1$, *there is a predicate in* $\Sigma_{n+1}^B \cap \Pi_{n+1}^B$ *having no inverse image in* $\Sigma_n^B \cup \Pi_n^B$.

The proof of corollary 2 uses corollary 1 and the facts that $\Sigma_k^{A(m-k)} = \Sigma_m^A$, $\Pi_k^{A(m-k)} = \Pi_m^A$, (See Davis [4], p. 159). Using the above methods, we can also prove

Corollary 3. There is a predicate $S(x,y,z)$ *in* $\Sigma_2 \cap \Pi_2$ *with the property that for any functions f, g and h of N onto N, the predicate* $\hat{x}\hat{y}\hat{z}\, S(f(x), g(y), h(z))$ *is not in* $\Sigma_1 \cup \Pi_1$.

REFERENCES

[1] Kleene, S. C., *Introduction to Metamathematics*, New York, 1952; p. 394.

[2] Putnam, H., Arithmetic models for consistent formulae of quantification theory, *The Journal of Symbolic Logic*, vol. 22 (1957), pp. 110-111.

[3] Putnam, H., Trial and error predicates, *The Journal of Symbolic Logic*, vol. 30 (1965).

[4] Davis, M., *Computability and Unsolvability*, New York (1958), McGraw Hill.

Stevens Institute of Technology
Hoboken, New Jersey