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SOLUTION TO A COMPLETENESS PROBLEM OF LEMMON AND SCOTT

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The purpose of this paper is to confirm and prove a conjecture of Lemmon and Scott [1] concerning a characteristic frame condition for a very general modal axiom schema. Familiarity is assumed with the model-theoretic methods now widely employed in the study of modal logics. A detailed exposition may be found, e.g., in Segerberg [2].

We write $Q(p_1, \ldots, p_k)$ to indicate that the modal wff Q has $p_1, \ldots, p_k$ as its propositional variables. $Q(A_1, \ldots, A_k)$ denotes the wff obtained from Q by uniform substitution of A_i for p_i. $L^n Q$ ($M^n Q$) is the wff obtained by prefixing Q with a sequence of n necessity (possibility) operators. Q is *positive* if it contains at most the operators $\vee$, $\wedge$, L, and M, i.e., has no occurrence of negation or implication.

If R is a binary relation on a set, for each natural number n we denote by R^n the n'th Pierce product of R with itself. Thus

$$xR^0y \text{ iff } x = y$$

and

$$xR^{n+1}y \text{ iff } \exists z\, (xRz \,\&\, zR^ny).$$

Now let $\langle W, R\rangle$ be a normal modal frame and $Q(p_1, \ldots, p_k)$ a positive wff with k variables. For each k-tuple $\mathbf{n} = \langle n_1, \ldots, n_k\rangle$ of natural numbers and each k-tuple $\mathbf{t} = \langle t_1, \ldots, t_k\rangle$ of elements of W we define a condition $RQ(x, \mathbf{t}, \mathbf{n})$ on $\langle W, R\rangle$ by recursion on Q:

$$\begin{array}{rll} R_{p_i}(x, \mathbf{t}, \mathbf{n}) & \text{iff} & t_iR^{n_i}x \ (\forall_i \leqslant k) \\ RQ\wedge\psi(x, \mathbf{t}, \mathbf{n}) & \text{iff} & RQ(x, \mathbf{t}, \mathbf{n}) \,\&\, R\psi(x, \mathbf{t}, \mathbf{n}) \\ RQ\vee\psi(x, \mathbf{t}, \mathbf{n}) & \text{iff} & RQ(x, \mathbf{t}, \mathbf{n}) \text{ or } R\psi(x, \mathbf{t}, \mathbf{n}) \\ RLQ(x, \mathbf{t}, \mathbf{n}) & \text{iff} & \forall y(xRy \Rightarrow RQ(y, \mathbf{t}, \mathbf{n})) \\ RMQ(x, \mathbf{t}, \mathbf{n}) & \text{iff} & \exists y(xRy \,\&\, RQ(y, \mathbf{t}, \mathbf{n})). \end{array}$$

For each positive $Q(p_1, \ldots, p_k)$ and each pair $\mathbf{m} = \langle m_1, \ldots, m_k\rangle$ and $\mathbf{n} = \langle n_1, \ldots, n_k\rangle$ of k-tuples of numbers we have a *Lemmon-Scott axiom*

$$Q^{\mathbf{m}}_{\mathbf{n}} : M^{m1}L^{n1}p_1 \wedge \ldots \wedge M^{mk}L^{nk}p_k \rightarrow Q(p_1, \ldots, p_k).$$

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Corresponding to this is the condition

$$RQ^{\mathsf{m}}_{\mathsf{n}} : (\forall x, t_1, \ldots, t_k)(xR^{ml}t_1 \ \& \ldots \& \ xR^{mk}t_k \Rightarrow RQ(x, \mathbf{t}, \mathbf{n}))$$

We can now state the conjecture of [1] as

Theorem *Let K be the smallest normal modal logic and $Q^{\mathsf{m}}_{\mathsf{n}}$ a Lemmon-Scott axiom. Then the logic $KQ^{\mathsf{m}}_{\mathsf{n}}$ is complete for the class of frames satisfying $RQ^{\mathsf{m}}_{\mathsf{n}}$.*

The result is indeed wide ranging. The Lemmon-Scott axioms include as special cases the Hintikka schemata of [1] which in turn, as Segerberg [2] observes, "cover most of the 'ordinary' systems in the literature."

For a normal logic S the *canonical model* is the structure $\mathfrak{M}_S = \langle W_S, R_S, V_S\rangle$ where

$$W_S = \{x : x \text{ is an } S\text{-maximal set of wff}\}$$
$$x\, R_S\, y \text{ iff } \{A : LA \in x\} \subseteq y$$
$$V_S(p, x) = 1 \text{ iff } p \in x.$$

Any non-theorem of S is falsifiable in $\mathfrak{M}_S$, so in order to show that S is complete for a class of frames satisfying a certain condition, it suffices to show that the frame of $\mathfrak{M}_S$ satisfies that condition.

Lemma 1 ([1], Th. 2.7)

$$xR^n_S y \text{ iff } \{A : L^nA \in x\} \subseteq y \text{ iff } \{M^nA : A \in y\} \subseteq x.$$

Lemma 2 *If S is a normal logic, $Q(p_1, \ldots, p_k)$ positive, and $\vdash_S A_i \to B_i \ \forall i \leq k$, then $\vdash_S Q(A_1, \ldots, A_k) \to Q(B_1, \ldots, B_k)$.*

Proof: By induction on the length of Q, using the fact that from $A \to B$ we may infer in S the wff

$$A \wedge C \to B \wedge C, A \vee C \to B \vee C, LA \to LB, \text{ and } MA \to MB.$$

The key to our proof is

Lemma 3 *If S is any normal logic, $Q(p_1, \ldots, p_k)$ positive, and $\langle W, R\rangle$ is the canonical frame for S, we have*

$$RQ(x, \mathbf{t}, \mathbf{n}) \text{ iff } \{Q(A_1, \ldots, A_n) : L^{nl}A \in t_1, \ldots, L^{nk}A_k \in t_k\} \subseteq x.$$

Proof: In the interest of expository clarity we will present the inductive proof with $k = 1$, i.e., Q has a single variable p. The general case is only technically, and not conceptually, more complex.

(1) The basis of the induction is immediate from Lemma 1. Since $p(A) = A$, $R_p(x, \mathbf{t}, \mathbf{n})$ if tR^nx iff $\{p(A) : L^nA \in t\} \subseteq x$.

(2) Suppose the result holds for Q and ψ. Then $RQ \vee \psi(x, \mathbf{t}, \mathbf{n})$ only if either (i) $RQ(x, \mathbf{t}, \mathbf{n})$ or (ii) $R\psi(x, \mathbf{t}, \mathbf{n})$. The two cases are analogous and so we consider only (i). If $L^nA \in t$, by the induction hypothesis $Q(A) \in x$. Hence $Q(A) \vee \psi(A) = Q \vee \psi(A) \in x$. Thus $\{Q \vee \psi(A) : L^nA \in t\} \subseteq x$ as required.

On the other hand if not $RQ \vee \psi(x, \mathbf{t}, \mathbf{n})$, then neither $RQ(x, \mathbf{t}, \mathbf{n})$ nor

$R\psi(x, \mathsf{t}, \mathsf{n})$. By the induction hypothesis it follows that there are wff A, B such that L^nA, $L^nB \in t$ but $Q(A) \notin x$ and $\psi(B) \notin x$. Letting $C = A \wedge B$ we have $\vdash_S L^nC \leftrightarrow L^nA \wedge L^nB$, so $L^nC \in t$. By Lemma 2, $\vdash_S Q(C) \rightarrow Q(A)$ and $\vdash_S \psi(C) \rightarrow \psi(B)$, so $Q(C) \notin x$ and $\psi(C) \notin x$. Hence $Q(C) \vee \psi(C) = Q \vee \psi(C) \notin x$ and we have $\{Q \vee \psi(C) : L^nC \in t\} \not\subseteq x$.

(3) Assume the result for Q. If $RMQ(x, \mathsf{t}, \mathsf{n})$ then for some y, xRy and $RQ(y, \mathsf{t}, \mathsf{n})$. Then $L^nA \in t$ only if $Q(A) \in y$ (induction hypothesis) only if $MQ(A) \in x$ (Lemma 1).

Conversely suppose

$$\{MQ(A) : L^nA \in t\} \subseteq x. \tag{*}$$

$$\text{Let } y_0 = \{A : LA \in x\} \cup \{Q(B) : L^nB \in t\}.$$

If y_0 is not S-consistent, then since $\{A : LA \in x\}$ is closed under finite conjunctions, it follows that there are wff $A, B_1, \ldots, B_n$ such that $LA \in x$, $L^nB_i \in t$ $\forall_i \leq n$ and

$$\vdash_S A \rightarrow \sim(Q(B_1) \wedge \ldots \wedge Q(B_n)).$$

Hence

$$\vdash_S LA \rightarrow L \sim (Q(B_1) \wedge \ldots \wedge Q(B_n)).$$

But $LA \in x$ so we obtain (since $\vdash_S L \sim \alpha \rightarrow \sim M \alpha$):

$$\sim M(Q(B_1) \wedge \ldots \wedge Q(B_n)) \in x. \tag{**}$$

Now let $B = B_1 \wedge \ldots \wedge B_n$. Then (*cf.* [2]) $L^nB \in t$ so by (*) $MQ(B) \in x$. But Lemma 2 gives $\vdash_S Q(B) \rightarrow Q(B_i)$ $\forall_i \leq n$, so $\vdash_S Q(B) \rightarrow Q(B_1) \wedge \ldots \wedge Q(B_n)$ whence $\vdash_S MQ(B) \rightarrow M(Q(B_1) \wedge \ldots \wedge Q(B_n))$ and thus $M(Q(B_1) \wedge \ldots \wedge Q(B_n)) \in x$, which is impossible, given (**) and the consistency of S.

We therefore conclude that y_0 is S-consistent and so has an S-maximal extension y. From the definition of R, the induction hypothesis, and the construction of y_0, we find that xRy and $RQ(y, \mathsf{t}, \mathsf{n})$ whence $RMQ(x, \mathsf{t}, \mathsf{n})$ as required.

(4) The inductive cases for $\wedge$ and L are quite straighforward and left to the reader to verify.

Corollary *If* $KQ^{\mathsf{m}}_{\mathsf{n}} \subseteq S$ *then the canonical frame* $\langle W, R\rangle$ *for* S *satisfies* $RQ^{\mathsf{m}}_{\mathsf{n}}$.

Proof: Let $xR^{ml}t_1$ & . . . & $xR^{mk}t_k$. If $L^{ni}A_i \in t_i$ $\forall_i \leq k$ then by Lemma 1 $M^{mi}L^{ni}A_i \in x$ $\forall_i \leq k$ and so

$$M^{ml}L^{nl}A_1 \wedge \ldots \wedge M^{mk}L^{nk}A_k \in x.$$

But x contains every substitution-instance of $Q^{\mathsf{m}}_{\mathsf{n}}$ and is closed under detachment, so $Q(A_1, \ldots, A_k) \in x$. Thus $\{Q(A_1, \ldots, A_k) : \forall_i \leq k, L^{ni}A_i \in t_i\} \subseteq x$ so by Lemma 3, $RQ(x, \mathsf{t}, \mathsf{n})$. Q.E.D.

In particular the canonical frame for $KQ^{\mathsf{m}}_{\mathsf{n}}$ satisfies $RQ^{\mathsf{m}}_{\mathsf{n}}$ and, as explained in the remarks prior to Lemma 1, this establishes the Theorem of this paper.

With regard to soundness, the reader may wish to prove for himself the following result and apply it to show that any frame satisfying $RQ_{\mathsf{n}}^{\mathsf{m}}$ validates $Q_{\mathsf{n}}^{\mathsf{m}}$.

Lemma 4 *In any normal model* $\langle W, R, V\rangle$, *and for any positive* $Q(p_1, \ldots, p_k)$, *if* $RQ(x, \mathbf{t}, \mathbf{n})$ *and* $V(L^{n_i}A_i, t_i) = 1$, $\forall_i \leq k$, *then* $V(Q(A_1, \ldots, A_k), x) = 1$.

REFERENCES

[1] Lemmon, E. J., and Dana Scott, *Intensional Logic*. Preliminary draft of initial chapters by E. J. Lemmon, July 1966 (mimeographed).

[2] Segerberg, Krister, *An Essay in Classical Modal Logic,* Uppsala University Press, Uppsala (1971).

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