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TWO IDENTITIES FOR LATTICES, DISTRIBUTIVE LATTICES AND MODULAR LATTICES WITH A CONSTANT

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In his paper [3] J. A. Kalman has defined lattices using two identities and six variables. We shall define lattices using two identities and five variables in Theorem 1. In Theorem 2 we shall give an axiom system for lattices with 0 consisting of two identities. J. Sholander's axiom system for distributive lattices with 0 contains three identities (*cf.*, [5]), but our axiom system in Theorem 3 consists of two identities. In Theorem 4 we shall give a definition for distributive lattices with 1 in the Croisot-Sobociński style (*cf.*, [1] and [7]). Finally, as axiom system for modular lattices with 0 shall be given in Theorem 5. In the remarks, axiom systems for lattices, distributive lattices and modular lattices with two constants are given by three identities.

Theorem 1. *Any algebraic system* $\langle A; \cdot; +\rangle$ *with two binary operations* $\cdot$ *and* $+$, *which satisfies the following two identities*

L1. $a = ba + a$
L2. $((ab)c + d) + e = ((bc)a + e) + (b + d)d$

is a lattice

Proof: We can prove it as Kalman has shown in [3] (*cf.*, Theorem 2 in this paper).

Theorem 2. *Any algebraic system* $\langle A; \cdot; +, 0\rangle$ *with two binary operations* $\cdot$ *and* $+$, *and with a constant* 0, *which satisfies the following two identities*

L1. $a = ba + a$
L2'. $(((0 + a)b)c + d) + e = ((bc)a + e) + (b + d)d$

is a lattice with 0.

Proof:

3. $c + a = (((0 + a)b)c + c) + a = ((bc)a + a) + (b + c)c = a + (b + c)c$ [*L1, L2', L1*]
4. $c + a = a + (bc + c)c = a + cc$ [3, *L1*]

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5. $a = aa + a = a + (aa)(aa) = (aa)(aa) + aa = aa$ [$L1$, 4, 4, $L1$]
6. $c + a = a + cc = a + c$ [4, 5]
7. $a + a = aa + a = a$ [5, $L1$]
8. $(b + c)c = (b + c)c + (b + c)c = c + (b + c)c = (b + c)c + c = c$ [7, 3, 6, $L1$]
9. $(((0 + a)b)c + d) + e = ((bc)a + e) + (b + d)d = ((bc)a + e) + d$ [$L2'$, 8]
10. $(d + a) + e = (a + d) + e = (aa + d) + e = (((0 + a)a)a + d) + e$
$= ((aa)a + e) + d = (a + e) + d = d + (a + e)$ [6, 5, 8, 9, 5, 6]
11. $0 + a = 0 + ((a + a) + a) = ((0 + a) + a) + a$
$= (((0 + a)(0 + a))(0 + a) + a) + a$
$= (((0 + a)(0 + a))a + a) + a = a + a = a$ [7, 10, 5, 9, $L1$, 7]
12. $a0 = (0 + a)0 = (a + 0)0 = 0$ [11, 6, 8]
13. $((ab)c + d) + e = (((0 + a)b)c + d) + e = ((bc)a + e) + (b + d)e$ [11, $L2'$]

We can prove the remaining part of this proof as Kalman has shown in [3].

Remark 1. We define lattices with 1 as the dual of postulates in Theorem 2.

*L*1.* $a = (b + a)a$
*L*2'.* $(((1a + b) + c)d)e = (((b + c) + a)e)(bd + d)$

Remark 2. If the system $\langle A; \cdot; +; 0; 1\rangle$ satisfies $L1$, $L2'$, and

L3. $a1 = a$,

then it is a lattice with 0 and 1 (*cf.*, [5]).

Theorem 3. *Any algebraic system* $\langle A; \cdot; +; 0\rangle$ *with two binary operations and* +, *and with a constant* 0, *which satisfies the following two identities*

P1. $a = a(a + b)$
P2'. $a(b + c) = c(a + 0) + b(a + 0)$

is a distributive lattice with 0.

Proof:

3. $a = a(a + a) = a(a + 0) + a(a + 0) = a + a$ [$P1$, $P2*$, $P1$]
4. $a = a(a + a) = aa$ [$P1$, 3]
5. $ab = a(b + b) = b(a + 0) + b(a + 0) = b(a + 0)$ [3, $P2'$, 3]
6. $a = a(a + 0) = (a + 0)(a + 0) = a + 0$ [$P1$, 5, 4]
7. $a(b + c) = c(a + 0) + b(a + 0) = ca + ba$ [$P2'$, 6]
8. $a0 = a0 + 0 = a0 + 00 = 0(0 + a) = 0$ [6, 4, 7, $P1$]

We can prove the remaining part of this proof as Sholander has shown in [5].

Remark 3. We define distributive lattices with 1 as the dual of postulates in Theorem 3:

*P*1.* $a = a + ab$
*P*2'.* $a + bc = (c + a1)(b + a1)$

Remark 4. If the system $\langle A; \cdot; +; 0, 1\rangle$ satisfies *P1*, *P2'*, and

P3. $a1 = a,$

then it is a distributive lattice with 0 and 1 (*cf.*, [5]).

Theorem 4. *Any algebraic system* $\langle A; \cdot; +; 1\rangle$ *with two binary operations* $\cdot$ *and* $+$, *and with a constant* 1, *which satisfies the following two identities*

D1'. $a = a(b + 11)$
D2'. $a(bb + c1) = ca + ba$

is a distributive lattice with 1.

Proof:

3.	$a = a(bb + 11) = 1a + ba$	[*D1'*, *D2'*]
4.	$1 = 11 + b1$	[3]
5.	$a1 = a(11 + 11) = a$	[4, *D1'*]
6.	$1 = 11 + b1 = 1 + b$	[4, 5]
7.	$a + 1 = (a + 1)(bb + 11) = 1(a + 1) + b(a + 1) = 1(a + 11) + b(a + 11)$ $= 1 + b = 1$	[*D1'*, *D2'*, 5, *D1'*, 6]
8.	$a(bb + c) = a(bb + c1) = ca + ba$	[5, *D2'*]

We can prove the remaining part of this proof as Sobociński has shown in [7].

Remark 5. We define distributive lattices with 0 as the dual of postulates in Theorem 4.

*D*1'.* $a = a + b(0 + 0)$
*D*2'.* $a + (b + b)(c + 0) = (c + a)(b + a)$

Remark 6. If the system $\langle A; \cdot; +; 0; 1\rangle$ satisfies *D1'*, *D2'*, and

D3. $a + 0 = a,$

then it is a distributive lattice with 0 and 1 (*cf.*, [6]).

Theorem 5. *Any algebraic system* $\langle A; \cdot; +; 0\rangle$ *with two binary operations* $\cdot$ *and* $+$, *and with a constant* 0, *which satisfies the following two identities*

M1. $(a + bb)b = b$
M2'. $((0 + a)b)c + ad = (da + cb)a$

is a modular lattice with 0.

Proof:

3.	$a = (da + aa)a = ((0 + a)a)a + ad$	[*M1*, *M2'*]
4.	$aa = (((0 + a)a)a + aa)a = a$	[3, *M1*]
5.	$(a + b)b = (a + bb)b = b$	[4, *M1*]
6.	$a = ((0 + a)a)a + ad = aa + ad = a + ad$	[3, 5, 4]
7.	$a + a = a + aa = a$	[4, 6]

8. $ad = ad + ad = ((0 + a)a)d + ad = (da + da)a = (da)a$ [7, 5, *M2'*, 7]

9. $a(a + b) = ((a + b)a)a = ((aa + bb)a)a = (((0 + a)b)b + aa)a = a$ [8, 4, *M2'*, *M1*]

10. $(a + b)a = (a(a + b))(a + b) = a(a + b) = a$ [8, 9, 9]

11. $(0 + a) + a = ((0 + a)(0 + a))(0 + a) + aa = (aa + (0 + a)(0 + a))a$
$= (a + (0 + a))a = a$ [4, *M2'*, 4, 10]

12. $0 + a = (0 + a) + (0 + a)a = (0 + a) + a = a$ [6, 5, 11]

13. $0a = 0(0 + a) = 0$ [12, 9]

14. $(ab)c + ad = ((0 + a)b)c + ad = (da + cb)a$ [12, *M2'*]

We can prove the remaining part of this proof as Kolibiar has shown in [4].

Remark 7. We define modular lattices with 1 as the dual of postulates in Theorem 5:

*M*1*. $a(b + b) + b = b$
*M*2'*. $((1a + b) + c)(a + d) = (d + a)(c + b) + a$

Remark 8. If the system $\langle A; \cdot; +; 0; 1\rangle$ satisfies *M1*, *M2'*, and

M3. $a1 = a$,

then it is a modular lattice with 0 and 1.

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