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A FORMAL METASYSTEM FOR FREGE'S SEMANTICS

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1 *Aims* This paper* uses a formal metasystem to clarify Frege's semantics, and it aims at both exegesis and proposed revisions. The point is to preserve Frege's basic insights while shedding some undesirable features related to these three problems:

(1) The diagonal paradoxes to which his theory is subject.
(2) The inability of a proper name or definite description to refer to a concept (*begriff*) in Frege's precise technical sense. This is the well-known paradox that for Frege "The concept *horse* is not a concept."
(3) The role of the notion of sense.

I must argue in appropriate places that my use of a formal metasystem will beg no important questions, but I should first say what I take Frege's basic insights to be. (For simplicity's sake I limit the discussion to objects and *one*-place, *first*-level functions; but functions of higher degree and/or level could be added by a straightforward extension.)

2 *Outline of Frege's Semantics* Frege opposed psychologism. For him the meaning of language was a matter of how it related to the world in a way not in its essence "routed through a mind". Now a paradigm for this is the relation of *naming* holding between a word and some one particular thing. This motivates the central role of *denotation* in Frege's theory, for that notion is just a generalization of this name relationship. Thus, Frege's first basic insight is that meaning arises from the relation between words and objective, nonpsychological things.

His second insight is a kind of corollary to the first. It is that language must somehow share the structure of the reality it describes. This leads him to posit a basic metaphysical type of *thing* for each basic type of linguistic *expression.* For example, by an *object* Frege means whatever can be named by a proper name. (Here I follow his use of 'proper name' ('*Eigenname*')[1] to mean either a proper name in the usual sense or a definite description.)

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1. I shall use single quotation marks to form metalinguistic names ('example') and double ones for all other purposes, including the "apologetic" one ("example").

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It turns out that Frege needs just one more linguistic type: an expression formed by *removing* a proper name from a meaningful linguistic whole. Such expressions are *incomplete* in view of their blank spaces; hence, the nonlinguistic entities corresponding to them must be likewise incomplete. Frege uses 'unsaturated' to describe both an incomplete expression and the entity corresponding to it, but I shall follow Montgomery Furth [1] in using 'complete' and 'incomplete' for expressions and reserving 'saturated' and 'unsaturated' for entities corresponding to the expressions. Thus, complete expressions (proper names) refer to saturated entities (objects), and incomplete expressions refer to unsaturated entities. Frege took this "referring to" relation to be the same in each case and called it *denotation.* So denotation is just the usual name relation generalized to include the case of incomplete expressions and unsaturated entities.

But what are unsaturated entities? First notice that the idea of an incomplete expression depends on that of a "meaningful linguistic whole," for they are simply what remain of such wholes after deleting one or more proper names. Now one kind of linguistic whole for Frege is a compound proper name, a definite description. The phrase 'the father of George Washington' is a compound proper name, and removing the proper name 'George Washington' from it gives the incomplete expression 'the father of . . .'. By analogy with mathematical phrases like 'the square of . . .' Frege uses the term 'function' for the unsaturated entities denoted by these expressions. But the unsaturatedness of functions is *nothing more* than an analogue to the incompleteness of incomplete expressions. To call an entity unsaturated is to say that it must be "filled" with an object to itself "become" an object, just as the blank space of an incomplete expression must be filled by a proper name to form a compound proper name. Frege therefore has an operation of "filling" for both linguistic expressions and the entities they denote.

But another kind of "meaningful linguistic whole" is a complete sentence, such as 'George Washington crossed the Delaware'. The incomplete expression '. . . crossed the Delaware' must therefore denote a function, and this function too must "become" some sort of object when "filled" with some object like George Washington. Frege decided that the object it became was one of the two truth-values: The TRUE and The FALSE. Such functions are called *concepts*, and the expressions denoting them—that is, incomplete expressions formed from sentences—are *predicates*. Thus, sentences must be complete expressions denoting truth-values.

Now this is not the whole of Frege's view of sentences, for he also regarded them as the only kind of complete expressions which can be *asserted.* Frege's theory of assertion is not a topic of this paper, but we can mention it to forestall the possible charge that Frege has unduly distorted our notion of sentences by saying that they denote their truth-values. They are not *mere* denoters of truth-values but rather those which, unlike 'The TRUE' or 'the truth-value of my last remark', can be asserted.

By taking sentences as denoters of this sort Frege could incorporate a very streamlined theory of *inference* into his account of the denotations of compound complete expressions. For a theory of inference says that certain sentences are true whenever certain other ones are; and if truth-values are denotations of sentences, then a theory explaining how denotations in general are related will also explain how the truth-values of different sentences are related.

However, my concern is with the more general theory, not just its application to sentences. Its main principle is that the denotation of a compound proper name depends *only* on those of its contained proper name(s) and incomplete expression. (Frege's whole outlook suggests this principle; for without it there would be a mystery, of a rather psychologistic sort, as to how a compound complete name could come by its denotation.) Thus, replacing a contained name by another with the same denotation preserves the denotation of the whole compound. This is Frege's famous principle of the substitutivity of identicals.

3 *The Formal Meta-Fregean System* $\mathfrak{M}$ I now construct a more formal rendition of Frege's theory. Define two set-theoretic *structures*, in the sense of ordinary first-order semantics, to represent language and the world (Frege's "realm of reference"):

$\mathfrak{L} \overset{df}{=} \langle \mathsf{E}, \mathsf{N}, \mathsf{I}, C \rangle$, where:

- E = the set of all *e*xpressions.
- N = the set of complete expressions (proper *n*ames).
- I = the set of *i*ncomplete expressions.
- C = the *c*ompletion operation, a mapping from $\mathsf{I} \times \mathsf{N}$ into N.

$\mathfrak{R} \overset{df}{=} \langle \mathsf{R}, \mathsf{O}, \mathsf{F}, \mathcal{G} \rangle$, where:

- R = the set of all denotations (*r*eferences).
- O = the set of *o*bjects.
- F = the set of *f*unctions.
- $\mathcal{G}$ = the *g*iving operation, a mapping from $\mathsf{F} \times \mathsf{O}$ into O.

The binary operations in $\mathfrak{L}$ and $\mathfrak{R}$ correspond to Frege's notion of "filling". In $\mathfrak{L}$ this is the *completion* operation C; given an incomplete expression and a complete one, this operation inserts the latter into the former to give a compound complete name. In $\mathfrak{R}$ we have the *giving* operation $\mathcal{G}$; for any function and any object this operation gives the value of the function for that object as argument.

We also have the denotation mapping δ, a *homomorphism* from $\mathfrak{L}$ to $\mathfrak{R}$:

$$\delta:\ \mathfrak{L} \xrightarrow{\text{homo}} \mathfrak{R}$$

That is:

(a) For each $e \in \mathsf{E}$, $\delta(e) \in \mathsf{R}$. [Every expression denotes a referent.]
(b) For each $n \in \mathsf{N}$, $\delta(n) \in \mathsf{O}$. [A proper name denotes an object.]
(c) For each $i \in \mathsf{I}$, $\delta(i) \in \mathsf{F}$. [An incomplete expression denotes a function.]

(d) For each $i \in \mathsf{I}$ and $n \in \mathsf{N}$, $\delta(C(i,n)) = \mathcal{G}(\delta(i), \delta(n))$.

[The denotation of any compound complete expression $C(i,n)$ is a function of its components' denotations, $\delta(i)$ and $\delta(n)$.]

Equation (d) here incorporates the principle of the substitutivity of identicals. For i an incomplete expression and n a complete one, $\delta(i)$ is the function i denotes, $\delta(n)$ the object n denotes, and $C(i,n)$ the compound expression formed from i and n. The equation then says that the denotation of this compound name is a function of the denotations of i and n; in fact, the operation $\mathcal{G}(\ ,\ )$ *is* that (meta)function determining this. $\delta(C(i,n))$ just *is* $\mathcal{G}(\delta(i), \delta(n))$. Thus, if:

(1) $\delta(n) = \delta(n')$

then:

(2) $\mathcal{G}(\delta(i), \delta(n)) = \mathcal{G}(\delta(i), \delta(n'))$

and we have, by equation (d) above:

(3) $\delta(C(i, n)) = \delta(C(i, n'))$.

That the inference from (1) to (3) holds in general is the more explicit statement of the substitutivity principle.

This formalizes all of Frege's theory that was presented in the previous section. But we should also note that, though functions and objects are fundamentally different from each other, Frege does consider every function to have a "value-range" (*Wertverlauf*) which *is* an object. For a concept this value-range is in effect the *extension,* the set of all objects giving the value TRUE when used as the concept's argument. We incorporate this with the value-ranging metafunction ν, which maps F into O; for every function $f \in \mathsf{F}$, $\nu(f)$ is its value-range. So we finally define *the meta-Fregean system* $\mathfrak{M}$ as:

$$\mathfrak{M} \overset{df}{=} \langle \mathfrak{L}, \mathfrak{N}, \delta, \nu \rangle.$$

I can foresee three possible objections to such a metasystem. In the first place it *relies* on many of the very notions Frege's theory tries to explain: set, function, predicate, etc. Secondly, it makes a sharp distinction between object-language and metalanguage, something Frege never did. Thirdly, it cavalierly disregards the unsaturatedness of functions by treating them as objects; it forms sets of them, calls them values of the (meta)function δ, etc. I shall answer these objections in turn.

In using contemporary notions of set and function, I am simply trying to enhance communication. That is, I believe I rely on no more in my first-order metalanguage than does Frege in using German to explain his Begriffsschrift. For instance, use of the metafunction δ corresponds quite strictly to the use of a phrase like 'the denotation of . . .' in a natural language. The former relies on a prior understanding of the notion of a function only to the same extent the latter does.

The second objection could be met in the same way by saying that even

Frege is using a metalanguage, German, to discuss an object-language, his Begriffsschrift. But here I can raise more than just this *tu quoque* point. So far I have not precluded the possibility that the metafunctions δ, ν, $\mathcal{C}$, and $\mathcal{G}$ are themselves members of the set of all functions **F**.[2] And all of the expressions in **E** might well be members of **O**, the set of objects. So I countenance the *possibility* of the identity of object-language and metalanguage here. An assertion to the contrary would be the conclusion of an argument, not an essential presupposition of the basic framework.

The third objection is the most serious one. Frege insisted that saturated and unsaturated entities are fundamentally different from each other and that this prevents a complete expression from denoting an unsaturated entity. For instance, 'the concept horse' is a definite description and hence a complete expression; it therefore cannot denote the unsaturated entity that it appears to. This is the paradox that for Frege "The concept *horse* is not a concept." In the same way no concept could really be the value of any function, such as my denotation mapping δ; for only objects can be values of functions. Here a *tu quoque* reply will not do at all, for Frege calls even his own use of definite descriptions as function names illegitimate. He sees that it is also unavoidable for the exposition of his theory, but he regards it as the mere giving of "hints" as to what he means and requests that his readers "meet him halfway" on this point.

A distinction between language levels might appear to help me here, in that a member of **F** could be saturated on the metalinguistic level while unsaturated for the object-language. But this would undo my earlier claim that my framework is neutral on the issue of the identity of metalanguage and object-language. Besides, the difficulty is the nature of the entities involved, not the language for them. A determined Fregean could argue that an entity retains its saturation status regardless of what language is describing it. I therefore think my best strategy is to seize on Frege's remark that he is just giving "hints" rather than using language in a strictly correct way. For I could regard **F** in structure $\mathfrak{R}$ as not the set of functions themselves but as a large enough set of arbitrary objects to *represent* all functions. The other devices, such as δ, then just show the proper structure of Frege's theory rather than directly map saturated things onto unsaturated ones. The fundamental cleavage between objects and functions would then amount to the sets **O** and **F** being disjoint. In fact, the possibility of our system's either affirming or denying this disjointness is what makes it neutral on this issue in the same way it was about metalanguage. (If **O** and **F** are disjoint and **F** does consist of objects, **O** too must be a representative selection of objects rather than all of them.)

4 *Applications* I can now use this machinery with a clear conscience as I approach my three main problems. And at *this* point I turn revisionary for

2. Since $\mathcal{C}$ and $\mathcal{G}$ are two-place functions, and since $\mathcal{G}$ and ν can take first-level functions as arguments, they could really be in **F** only after we had enlarged the system to handle more than just one-place, first-level functions.

the first time. Up to now the attempt has been to be completely faithful to Frege, but from now on the question is how to eradicate various defects though still remain true to Frege's basic motivation.

(a) *Russell's Paradox* The first problem indeed cries out for revision of Frege, for it is Russell's well-known proof of his inconsistency. If every concept has an extension (value-range) that is itself an object, then so must the concept denoted by '. . . is an extension not containing itself'. But then that extension must both contain and not contain itself. Specifically, if f is the concept denoted by '. . . is an extension not containing itself', then $\mathcal{G}(f, o)$ is The FALSE for any object o that is *not* a value-range; and if $\nu(g)$ is the value-range of any function g:

$$\mathcal{G}(f, \nu(g)) = \text{TRUE} \leftrightarrow \mathcal{G}(g, \nu(g)) \neq \text{TRUE}.$$

Then, putting in f for g:

$$\mathcal{G}(f, \nu(f)) = \text{TRUE} \leftrightarrow \mathcal{G}(f, \nu(f)) \neq \text{TRUE}$$

and we have, in effect, Russell's famous contradiction (actually, something like Grelling's version of it).

Now it would be arbitrary to deny that there is such a function f, for we cannot fault the phrase '. . . is an extension not containing itself' as an example of the kind of incomplete expressions which motivated Frege's original notion of unsaturated entities. Thus, we must either give up the principle that each concept has an extension or else concede that some such extensions are not objects. The first choice simply rejects the notion of an extension in its full logical generality and would violate the whole spirit of Frege's enterprise. The second approach is the Von Neumann-Bernays approach to set theory, in which some extensions (the proper classes) are ineligible for membership in extensions (or as arguments for functions). But Frege hesitates at this step (in his own discussion of Russell's paradox[3]) because of his intuition that there is nothing "predicative" about extensions; they are saturated, not unsaturated, and hence are objects. How, then, can they be denied the right of membership in extensions or argumenthood of functions?

Our metasystem can resolve this dilemma, for it can distinguish between an object in the sense of a suitable argument for a function and an object in the more metaphorical sense of "saturated entity". An extension need not be an object in the first sense while remaining one in the second sense, which we may call a "saturant". That is, we add a set $\mathbf{O}'$ of all saturants as a proper *superset* of $\mathbf{O}$, the set of all objects in the first sense; and the value-ranging metafunction ν maps the set of functions $\mathbf{F}$ into $\mathbf{O}'$ rather than $\mathbf{O}$. Thus, we keep $\mathbf{O}$ as the set of all possible arguments for functions, but it now lacks the extensions of some of the functions in $\mathbf{F}$. The set $\mathbf{O}'$ includes both all arguments and all extensions; but, unlike $\mathbf{O}$, it

3. See [1], p. 128.

is not an item of the realm of reference $\mathfrak{R} = \langle \mathsf{R}, \mathsf{O}, \mathsf{F}, \mathcal{G}\rangle$, It is simply another element in the meta-Fregean system $\mathfrak{M}$, now defined as $\langle \mathfrak{L}, \mathfrak{R}, \delta, \nu, \mathsf{O}'\rangle$.

In what sense, then, can O' be considered a set of "saturants"? Simply in that, though not all its members are *named* by a complete expression (i.e., are not $\delta(n)$ for some $n \in \mathsf{N}$), they are all so name*able*. That is, O' can serve as the set of all objects for a *new* realm of reference $\mathfrak{R}'$, which is part of a new meta-Fregean system $\mathfrak{M}'$, complete with its own $\mathfrak{L}'$, ν', etc. Metaphysically, then, each member of O' is saturated; it *can* be an argument of a function, but perhaps only after a step of semantic ascent to a wider language system has been taken. In short, consistency for the whole scheme, when we try to stay as close to Frege's original ideas as possible, simply *demands* this kind of extendibility. And since we can always put E (the domain of expressions of our original $\mathfrak{L}$) into O', we can always make the new language a *metalanguage* for the old one.

Frege avoided metalanguages, probably because he regarded his fundamental linguistic categories as basic to *any* language and therefore just as applicable to the language he used (German) as the one he discussed (Begriffsschrift). But this universality may instead be put in this way: any metalanguage formed as above has the same general form as the original language. That is, we indeed have an $\mathfrak{L}'$, δ', ν'—in short, a whole $\mathfrak{M}'$—which meets all requirements for a meta-Fregean system. Thus, the most Fregean way out of Russell's paradox also shows the legitimacy of a semantic ascent to a metalanguage. We therefore have a (potential) infinite hierarchy of object-languages and metalanguages linked by the various value-ranging metafunctions in this way:

$$
\begin{array}{lll}
\vdots & & \vdots \\
 & & \nwarrow \nu'' \\
\mathfrak{M}'' = \ldots & & \ldots, \langle \mathsf{R}'', \mathsf{O}'', \mathsf{F}'', \mathcal{G}''\rangle, \ldots \\
 & & \nwarrow \nu' \\
\mathfrak{M}' = \langle \mathfrak{L}', \mathfrak{R}', \delta', \nu', \mathsf{O}''\rangle & = \langle\langle \mathsf{E}', \mathsf{N}', \mathsf{I}', \mathcal{C}'\rangle, & \langle \mathsf{R}', \mathsf{O}', \mathsf{F}', \mathcal{G}'\rangle, \delta', \nu', \mathsf{O}''\rangle \\
 & & \nwarrow \nu \\
\mathfrak{M} = \langle \mathfrak{L}, \mathfrak{R}, \delta, \nu, \mathsf{O}'\rangle & = \langle\langle \mathsf{E}, \mathsf{N}, \mathsf{I}, \mathcal{C}\rangle, & \langle \mathsf{R}, \mathsf{O}, \mathsf{F}, \mathcal{G}\rangle, \delta, \nu, \mathsf{O}'\rangle
\end{array}
$$

(b) *The Paradox of Unsaturation* Now the solution above depended on the wedge we drove between objects in the sense of arguments for functions and objects in the sense of saturants. But how could a really "Fregean" solution do this? Is not the distinction between what is or is not an argument the *same* as the distinction between what is or is not saturated? This raises my second main problem: the unsaturatedness of functions and the paradox about the concept horse. Here too I am revisionary, for I think Frege has made one very clear-cut mistake. He tries to make the one-place metapredicate '. . . is unsaturated' do work that is really cut out for the three-place metapredicate 'The function . . . gives object . . . as value for object . . . as argument'. That is, what captures the function-argument relationship is the giving operation $\mathcal{G}$ and its syntactic correlate

C, the completion operation. These notions give the substitutivity of identicals as an immediate consequence of the homomorphic character of the denotation mapping. By contrast, the notion of unsaturation does no real work in the theory and is dispensible; the essence of a function is not unsaturation but rather its capacity to serve as the first argument in the giving operation.

This is in part a change of metaphor. Frege seems to think of a function as a kind of Protean monster which, when fed an object like a pill, *becomes* some other object. I see a function as a machine with an input chute and an output chute; an argument inserted on the former makes the value of the function for that argument appear on the latter. The giving operation is just an embodiment of this machinelike character of functions. But if we wish we can *name* a function rather than use the giving operation to "turn it on". We can tag a function with a complete expression when we say, for instance, that it is continuous or is a concept and still be talking about the same entity we could *invoke* (not name) by using an incomplete expression and the completion operation. So we do preserve the absolute cleavage between complete and incomplete *expressions*, since only in that way can we recognize a phrase as decomposable into a complete and an incomplete part, and only in that way is the completion operation a clear notion. We can therefore use the complete expression 'squaring' to say:

(1) Squaring is a continuous function.

but *not* the incomplete expression 'square of . . .' to say:

(2) Square of is a continuous function.

for, as Frege would insist, (2) is ill-formed. However, 'Squaring' in (1) does denote (in the "tagging" way) the *same* function that 'square of . . .' does (in the "invoking" way) in:

(3) The square of 5 is 25.

Although this departs from Frege, it really expresses more faithfully the "ur-Fregean" idea that language and the world share a structure. For the syntactic relation of filling does, after all, involve three things: the incomplete expression, the complete one used to fill it, and the resulting compound one. Even here to describe the incomplete expression as itself *becoming* a complete one is not strictly accurate; but that way of putting it led Frege to think of functions as *becoming* objects and hence as possessing a weird quality of incompleteness called unsaturation. He thereby failed to see that a really direct structural resemblance between language and reality requires a three-place attribute in the latter, not a one-place one. We are robbing Frege to pay the ur-Frege.

By banishing unsaturation we remove the *metaphysical* obstacle to letting both complete and incomplete expressions denote (some) functions; both functions and objects are saturants. (There is still a difference: only functions have the machineline nature described above; and if each function were also an object (denotation of a complete name), we could re-create the

Russell paradox.) But there is also a *structural* obstacle. To allow some functions to have both complete and incomplete denoters is to allow sets F and O to overlap, for O consists of all denotations of complete names. But we have just seen that sets N and I, the complete and incomplete expressions, must remain disjoint. δ is therefore no longer a homomorphism. To see this in detail consider the four components of the homomorphism condition for δ:

(a) δ: $\mathsf{E} \rightarrow \mathsf{R}$ (δ maps E into R).

[Every expression has a denotation.]

(b) For each $e \in \mathsf{E}$, $e \in \mathsf{N} \leftrightarrow \delta(e) \in \mathsf{O}$.

[An expression is complete *if* and only if it denotes an object.]

(c) For each $e \in \mathsf{E}$, $e \in \mathsf{I} \leftrightarrow \delta(e) \in \mathsf{F}$.

[An expression is incomplete *if* and only if it denotes a function.]

(d) For each $e_1, e_2, e_3 \in \mathsf{E}$, $\mathcal{C}(e_1, e_2) = e_3 \leftrightarrow \mathcal{G}(\delta(e_1), \delta(e_2)) = \delta(e_3)$.

[Three expressions stand in the completion relation *if* and only if their denotations stand in the giving relation.]

Parts (b), (c), and (d) no longer hold. Specifically, the right-to-left direction of their biconditionals fail (the *if* parts in the English renditions). For if f is denoted by both complete expression n and incomplete one i, then it is both $\delta(n)$, and $\delta(i)$ and hence in both O and F; thus, $\delta(i) \in \mathsf{O}$ where $i \notin \mathsf{N}$, and $\delta(n) \in \mathsf{F}$ where $n \notin \mathsf{I}$. Similarly, if $\mathcal{G}(f, \delta(n)_1)) = \delta(n_2)$ then, though $\mathcal{C}(i, n_1) = n_2$, we do not have $\mathcal{C}(n, n_1) = n_2$, since '$\mathcal{C}(n, n_1)$' is not even defined. So n is an expression such that $\mathcal{G}(\delta(n), \delta(n_1)) = \delta(n_2)$ although it is not true that $\mathcal{C}(n, n_1) = n_2$. This contradicts (d) above.

However, the weakening of (b) and (c) is no real flaw. Each member of O still has *some* denoter which is complete; f (both a function and an object) was denoted by i, but also by n. An object is therefore still something denoted by a complete expression, and its having an incomplete one as well would just show that we now explain an expression's role not merely in terms of what it denotes but also in terms of the completion operation. In the same way a function is still something with at least one *in*complete denoter. Denoting now amounts to "naming" only for complete expressions; for incomplete ones it is really the broader "referring to" idea we called invoking above—the turning on of the "machine" (function) it denotes. This is entirely in accord with Frege's insistence that only in the context of a sentence does an expression have meaning. Only by seeing which completion-operation role a function-denoter plays can we see whether it is "turning on" the function or just tagging it.

The weakening of (d) is also innocuous, for here too whenever $\mathcal{G}(f, \delta(n_1)) = \delta(n_2)$, then f has *some* incomplete denoter i such that $\mathcal{C}(i, n_1) = n_2$, regardless of whether it also has a complete denoter. This means that the giving relation is never "left hanging"; it never holds among three things which lack denoting expressions related in the appropriate way by the completion relation. It is therefore still in effect a homomorphic image of the completion relation. The only change is that there can be "unused" (complete) denoters of a function that is being invoked by an incomplete denoter.

Thus, δ need not be quite a homomorphism to sustain the required relationships. But the most tidy way to put this is to retain the original definition of a meta-Fregean system, in which O and F are disjoint, and define an *augmented* meta-Fregean system as a meta-Fregean one into whose N have been put some complete names of functions in F, which in turn are added to O. This ensures that there are no "orphan" functions, functions with complete denoters but without incomplete ones; for that is vital to our above account of why the weakened morphism condition on δ is sufficient for our purposes.

An ordinary meta-Fregean system is all we would need if we do not wish to *discuss* any of the functions we want to invoke. In such a case it is quite proper to point out the ill-formedness of such expressions as:

(4) Square of is a continuous function.
(5) Is a horse is a concept.

But we can augment the system by adding appropriate complete terms for the functions and say:

(6) Squaring is a continuous function.
(7) Being a horse is a concept.

Indeed, the usual use-mention-confusing way of saying:

(5′) 'Square of' is a continuous function.

may be seen as a kind of elliptical version of:

(5″) The denotation of 'square of' is a continuous function.

in which the metalinguistic denotation function is explicitly used to provide a complete expression by *mentioning* an incomplete one. But, as (6) and (7) show, we need not take the full step to metalinguistic devices to legitimately incorporate complete function names into a meta-Fregean system. This helps show the distinction between our two problems about functions: Russell's paradox (solved by appeal to additional meta-Fregean systems) and the naming of functions (solved by the augmentation of a single meta-Fregean system).

To describe this augmentation solution more formally, consider an arbitrary meta-Fregean system $\mathfrak{M} = \langle \mathfrak{L}, \mathfrak{N}, \delta, \nu, \mathsf{O}' \rangle$. We first weaken the homomorphism condition for δ in the philosophically innocuous way described above. That is, we weaken slightly what it is to call $\mathfrak{M}$ meta-Fregean in the augmented sense by replacing the biconditionals in parts (b), (c), and (d) of the morphism condition on δ by conditionals and adding the claim that there are no orphan functions. We may then simply replace each function $f \epsilon \mathsf{F}$ by its value-range $\nu(f)$ and still have (an augmented) meta-Fregean system. Specifically, we define a new system $\mathfrak{M}^*$ as $\langle \mathfrak{L}, \mathfrak{N}^*, \delta^*, \nu^*, \mathsf{O}' \rangle$, where:

1. All unstarred items are parts of the original $\mathfrak{M}$.
2. $\mathfrak{R}^* \overset{df}{=} \langle \mathsf{R}, \mathsf{O}, \nu[\mathsf{F}], \mathcal{G}^* \rangle$, where $\mathcal{G}^*(\nu(f), o) \overset{df}{=} \mathcal{G}(f, o)$.
3. $\delta^*(e) \overset{df}{=} \begin{cases} \delta(e), \text{ for } e \in \mathsf{N}. \\ \nu(\delta(e)), \text{ for } e \in \mathsf{I}. \end{cases}$
4. ν^* is the identity mapping from $\nu[\mathsf{F}]$ onto itself.

Functions are now just their value-ranges, and the set of these is what we henceforth mean by F. F is now just a subset of O' rather than what is mapped into O' by ν, and we may therefore drop ν as an independent theoretical notion. Also, we need not mention O' as a part of $\mathfrak{M}$; the "next level" system $\mathfrak{M}'$ still has O' as its set of objects is now related to $\mathfrak{M}$ in that O' is a proper superset of F rather than what ν maps F into.

But, temporarily retaining the unneeded baggage of ν and O', we see that $\mathfrak{M}^*$ as defined above is indeed an augmented meta-Fregean system. For instance, the new denotation mapping is still the appropriate kind of near-homomorphism; for all we have done is replace each $f \in \mathsf{F}$ with some saturant from O' and redefined $\mathcal{G}$, δ, and ν so as to preserve structural relationships. The only such relationship not so preserved is that some saturants used as replacements may have already had complete denoters in N, and this is just the change permitted by the notion of an augmented meta-Fregean system. We have therefore given a consistency proof akin to those in axiomatic set theory: from an arbitrary model we construct another having the properties we wish to prove consistent with the original theory. In this case the properties are that functions be saturated and that they in fact be none other than their value-ranges.

We can now drop the unneeded baggage and redefine a meta-Fregean system as $\mathfrak{M} = \langle \mathfrak{L}, \mathfrak{R}, \delta \rangle$, where each component is as discussed above (in the sense of an *augmented* meta-Fregean system). The realm of reference $\mathfrak{R}$ now has a domain of saturants and no unsaturated entities. But we still distinguish between objects and functions, even though it is possible for some functions to be objects as well. That is, $\mathsf{O} \neq \mathsf{F}$ (and neither $\mathsf{O} \subset \mathsf{F}$ nor $\mathsf{F} \subset \mathsf{O}$), though $\mathsf{O} \cap \mathsf{F} \neq \emptyset$. For only functions have the machinelike character described above (that value-ranges can have such a character is in effect what our consistency proof showed), and (though now only to avoid Russell's paradox) only objects are candidates for arguments of functions.

(c) *The Role of Senses* With the realm of reference now clearer, the meta-Fregean system can help explain the realm of sense, at least in outline. We incorporate this realm into the system with an intermediate structure $\mathfrak{S}$ for senses:

$\mathfrak{S} \overset{df}{=} \langle \mathsf{S}, \mathsf{S}_\mathrm{N}, \mathsf{S}_\mathrm{I}, \mathcal{F}_\mathrm{S} \rangle$, where:
- S = the set of *s*enses.
- S_N = the set of *s*enses of proper *n*ames.
- S_I = the set of *s*enses of *i*ncomplete expressions.
- $\mathcal{F}_\mathrm{S}$ = the "*f*illing" operation for *s*enses, which maps $\mathsf{S}_\mathrm{I} \times \mathsf{S}_\mathrm{N}$ into S_N.

We also add the "sense of" mapping σ, which is a homomorphism from $\mathfrak{L}$ to $\mathfrak{S}$, and the "reality" mapping ρ, a near-homomorphism from $\mathfrak{S}$ to $\mathfrak{R}$

(in the same sense that δ is a near-homomorphism from $\mathfrak{L}$ to $\mathfrak{R}$). ρ maps each sense onto the denotation of any expression that has that sense, and the denotation mapping is therefore just the composition of ρ and σ: for each $e \in \mathbf{E}$, $\delta(e) = \rho(\sigma(e))$. This is the principle that an expression's denotation is determined by its sense. And the substitutivity principle for senses is just the statement that σ is a homomorphism. So our final definition of the meta-Fregean system $\mathfrak{M}$ is:

$$\mathfrak{M} \overset{df}{=} \langle \mathfrak{L}, \mathfrak{S}, \mathfrak{R}, \sigma, \rho, \delta \rangle,$$

where:

1. σ: $\mathfrak{L} \xrightarrow{\text{homo}} \mathfrak{S}$ [Substitutivity principle for senses.]
2. ρ: $\mathfrak{S} \xrightarrow[\text{homo}]{\text{near}} \mathfrak{R}$ [The realm of sense shares some structure with the realm of reference.]
3. δ: $\mathfrak{L} \xrightarrow[\text{homo}]{\text{near}} \mathfrak{R}$ [Substitutivity for references.]
4. $\delta = \rho \circ \sigma$ [Sense determines reference.]

Notice that σ is a homomorphism while ρ is just a near-homomorphism. That is, $\mathbf{S}_N$ and $\mathbf{S}_I$, like $\mathbf{N}$ and $\mathbf{I}$ in $\mathfrak{L}$, are disjoint; for the difference between a complete and an incomplete denoter of a function is surely in part a matter of sense. Note also that the near-homomorphic character of ρ is a consequence of the homomorphism of σ and the near-homomorphism of δ. It is therefore in the nature of a discovery that this new realm of entities $\mathfrak{S}$, which is pinned onto $\mathfrak{L}$ by the purely conventional homomorphism σ, nevertheless shares structure with the objective realm of reference $\mathfrak{R}$ (assuming, of course, that $\mathfrak{L}$ is an extensional language—i.e., that δ is indeed a near-homomorphism). This is an explication of Frege's view that senses are something "objective" despite the purely conventional nature of the choice of symbols to express them.

So far this machinery shows only that the role of senses is to define *finer* equivalence classes of expressions than can denotations; the relation of sharing a sense is a stronger one than is sharing a denotation. For a possible interest in such finer equivalence classes, we could add a relation of *assertion* (or disposition to assert) to the meta-Fregean system that could hold between persons and sentences. Then the cash value of the equivalence relation "sentences . . . and . . . have the same sense" would be a matter of people's disposition to make assertions. Frege's doctrine of assertion would presumable be a guide here.

But the wider value of this framework is the many *kinds* of intermediaries senses can represent. In fact, there could be a *number* of intermediary structures between language and the realm of reference, each defining equivalence classes of expressions which are of interest for some independent reason. These could define either progressively weaker relations as we move from language to the world or ones whose strengths were not comparable, except that they all be stronger than the "same denotation" one. We would therefore have:

$$\begin{tikzcd}
 & & \mathfrak{S}^*_{21} \arrow[r, "\rho_{21}"] & \cdots & \mathfrak{S}^*_{n1} \arrow[dr, "\rho_{n1}"] & \\
\mathfrak{L} \arrow[r, "\sigma_1"] \arrow[dr, "\sigma_2"'] & \mathfrak{S}_{11} \arrow[ur, "\rho_{11}"] \arrow[r, "\rho_{12}"] & \mathfrak{S}^*_{22} \arrow[r, "\rho_{22}"] & \cdots & \mathfrak{S}^*_{n2} \arrow[r, "\rho_{n2}"] & \mathfrak{R} \\
 & \mathfrak{S}_{12} \arrow[r, "\rho_{13}"] & \mathfrak{S}^*_{23} \arrow[r, "\rho_{23}"] & \cdots & \mathfrak{S}^*_{n3} \arrow[ur, "\rho_{n3}"'] &
\end{tikzcd}$$

[We use '*' with the $\mathfrak{S}^*_{ij}$ above in view of the doubt over whether such intermediary structures should really be regarded as *senses* or as some other sort of way station on the journey to the realm of reference.]

With one change the different reality mappings ρ_{ij} above would in effect be different principles for selecting the different "possible worlds" (realms of reference) so much in vogue in current work in the semantics of modal and other nonstandard systems of logic. Specifically, we relax the condition that there be only one realm of reference $\mathfrak{R}$ and thereby countenance different denotation mappings $\delta_i = \rho_{ni} \circ \rho_{n-1i} \circ \ldots \circ \rho_{1i} \circ \sigma_i$. This in a way adopts Hintikka's suggestion in [2] that Frege's notion of sense be replaced by one of "multiple reference", reference in different possible worlds. But by decomposing such multiple denotation mappings into their σ_i and ρ_{ij} components, we could hope to distinguish the different factors at work in such multiple denotations: the purely lexical factor of what a symbol "means", the different factors arising for the different modalities (including the propositional attitude ones of Hintikka [2]), and the differences due to the different people whose attitudes may be discussed. This should provide a way to bring out explicitly what is common to all statements of belief, for example, and also what is *different* for different people's beliefs. Only by such interpersonal comparisons can we do justice to the full variety of our propositional attitude talk, and only by distinguishing between what is objective (interpersonal) and what is not can we capture what Frege was aiming at with his notion of sense.

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