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A NOTE ON UDEs IN AN n-VALUED LOGIC

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Professor Sobociński has shown that in a 2-valued logic there exists a function with four inputs which is a Universal Decision Element (**UDE**) and there exists no three-place function with this property [1]. J. C. Muzio has shown recently that in an n-valued logic there is a **UDE** with $n^2 + n + 1$ inputs. He has conjectured that in an n-valued logic there is a **UDE** with $n^2 + 1$ inputs [2]. This note exhibits a **UDE** with $n^2 + 2$ inputs.

Both this note and Muzio's result are obtained by a generalization of the operation called "conditioned disjunction" [3] or "the McCarthy conditional" [4]. Church defines:

$$[x, y, z] = \begin{cases} x, & \text{if } y = \mathbf{T}; \\ z, & \text{if } y = \mathbf{F}. \end{cases}$$

For $[x, y, z]$ McCarthy writes: $(y \rightarrow x, z)$. Note that the generalizations given here do not coincide with McCarthy's generalization to the three-valued case ([4], p. 54).

Let $X(n)$ be the space consisting of the values $0, 1, 2, \ldots, n - 1$. Over $X(n)$ define:

(A) $(x; y_0, y_1, \ldots, y_{n-1}) = y_i$ if $x = i$, $0 \leqslant i \leqslant n - 1$.

Muzio's construction consists of defining:

(B) $F = (x; c_0, c_1, \ldots, c_{n-1})$, where $c_i = (y, z_{i1}, z_{i2}, \ldots, z_{i(n-1)})$.

It is clear that if a two-place function fxy is defined by an n by n operation table of values $a_{ij}\,(0 \leqslant i, j \leqslant n - 1)$, then fxy may be obtained from the definition of F by the substitutions:

$$z_{ij} = a_{ij}\,(0 \leqslant i, j \leqslant n - 1).$$

The n repetitions of y in the definition (B) can be eliminated in a simple way. Let $N = n^2$. We define:

(C) $G = (x, y; z_0, \ldots, z_{N-1}) = z_i$, where $i = nx + y$.

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In an analogous manner to the above, *fxy* may be obtained from G by the substitution of the constants:

$$z_k = a_{ij}\text{, where } i = [k/n]\text{, and } j = k - n[k/n].$$

(Square brackets denote the greatest integer function.)

The improvement of (C) over Muzio's construction is small, since the n occurrences of the variable y are only formally distinct. It seems unlikely that the value $n^2 + 2$ represents a best result, since for the case $n = 2$ the best result is known to be 4, not 6.

REFERENCES

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