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A CATEGORICAL EQUIVALENCE OF PROOFS

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0 *Introduction.** An intuitionist proof of a sequent $B \rightarrow A$ is essentially a "function," and in this paper we shall study certain properties of the class of such functions. In order to gain sufficient generality, we shall adopt a "multilinear" point of view and take a propositional subsystem of Gentzen's calculus **LJ** as a starting point.

Gentzen's *Hauptsatz* states that for every provable sequent $\Gamma \rightarrow A$, the class $\{P\}$ of **LJ** proofs of $\Gamma \rightarrow A$ contains at least one cut-free representative. We can regard $\{P\}$ as an *equivalence* class with respect to the relation E_0 on proofs in **LJ** defined by $P\mathsf{E}_0Q$ iff P and Q are proofs of the same sequent $\Gamma \rightarrow A$. The question which arises naturally in category theory is to what extent, if at all, E_0 can be refined to an equivalence relation E for a definite propositional fragment of **LJ**, denoted simply by "**L**" below, such that the following are true:

(i) Each E-class has a cut-free representative;
(ii) E separates the structural and operational rules of **L** conservatively;
(iii) If P and Q are two cut-free proofs of the sequent $\Gamma \rightarrow A$, then $P\mathsf{E}Q$ iff P and Q are equi-general, where P and Q are equi-general, roughly speaking, if the terms in the initial sequents of P can be made as distinct as those in Q and conversely without destroying P and Q as *proofs of the same sequent* (but not necessarily of $\Gamma \rightarrow A$).

(i) and (ii) will of course establish immediately certain invariance properties of well-known logical theorems, whereas an *effective* notion of "equi-generality" is needed in order to preserve the decidability of **L**.

Whilst the questions raised in (i), (ii), and (iii) are of logical interest in their own right, the motivation for studying the particular deductive system **L** lies in the fact that **L** constitutes, as is easily deducible from

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results of Lambek [4] (implicit also in Lawvere [7]), the generating system for cartesian closed multicategories, equivalent in a categorical sense to cartesian closed categories, whose significance for a "categorical foundation" of mathematics has been amply demonstrated by Lawvere [6]. (iii) entails a solution of the "coherence problem" for such categories by giving an effective description of the canonical maps in "free" cartesian closed multicategories. The deeper algebraic significance of coherence questions in connection with the generality of proofs will however be discussed elsewhere. The results obtained in this paper are independent of the theory of multicategories and a proper Gentzen system results from **L**, as defined below, if "multicategory" is read as "discrete multicategory."

1 *The Deductive System* **L**. We begin by recalling the definition of a multicategory, *cf.* [4]:

Definition: A *multicategory* $\mathfrak{M}$ consists of a class of "objects" together with a class of "multimaps" $f: A_1, \ldots, A_n \to B$. Among the multimaps is an "identity map" $1_A: A \to A$ for each object A.

For obvious reasons we call the domain of f (which may be empty) the "antecedent" and the codomain the "succedent" of the expression $A_1, \ldots, A_n \to B$. Multimaps are composed by "substitution," using essentially the cut rule for **LJ**. Substitution satisfies, furthermore, the following conditions:

$$\text{(i)} \qquad \frac{A \xrightarrow{1_A} A \quad \Gamma, A, \Delta \xrightarrow{f} B}{\Gamma, A, \Delta \to B} = \Gamma, A, \Delta \xrightarrow{f} B$$

$$\text{(ii)} \qquad \frac{\Gamma \xrightarrow{f} A \quad A \xrightarrow{1_A} A}{\Gamma \to A} = \Gamma \xrightarrow{f} A$$

$$\text{(iii)} \qquad \frac{\dfrac{\Gamma \xrightarrow{f} A \quad \Delta, A, \Theta \xrightarrow{g} B}{\Delta, \Gamma, \Theta \to B} \qquad \Phi, B, \Psi \xrightarrow{h} C}{\Phi, \Delta, \Gamma, \Theta, \Psi \to C} = \frac{\Gamma \xrightarrow{f} A \qquad \dfrac{\Delta, A, \Theta \xrightarrow{g} B \quad \Phi, B, \Psi \xrightarrow{h} C}{\Phi, \Delta, A, \Theta, \Psi \to C}}{\Phi, \Delta, \Gamma, \Theta, \Psi \to C}$$

$$\text{(iv)} \qquad \frac{\Gamma \xrightarrow{f} A \qquad \dfrac{\Delta \xrightarrow{g} B \quad \Theta, A, \Phi, B, \Psi \xrightarrow{h} C}{\Theta, A, \Phi, \Delta, \Psi \to C}}{\Theta, \Gamma, \Phi, \Delta, \Psi \to C} = \frac{\Delta \xrightarrow{g} B \qquad \dfrac{\Gamma \xrightarrow{f} A \quad \Theta, A, \Phi, B, \Psi \xrightarrow{h} C}{\Theta, \Gamma, \Phi, B, \Psi \to C}}{\Theta, \Gamma, \Phi, \Delta, \Psi \to C} .$$

Remark: All notions and notations not explicitly defined in this paper coincide with those in [9].

For the remainder of this paper, we shall assume that $\mathfrak{M}$ is a fixed, but arbitrary multicategory, and we stipulate the "terms," "formulae" (sequents), "axioms," and "rules of inference" of **L** (= **L**($\mathfrak{M}$)) as follows:

Definition: The *terms* of **L** are (i) the objects of $\mathfrak{M}$; (ii) I; (iii) if A and B are terms, then so are $A \wedge B$ and $A \supset B$. The terms in (i) will be called "atomic" and will be denoted by X_1, X_2, etc.

Definition: The *formulae* of **L** are expressions of the form $A_1, \ldots, A_n \to B$, where $A_1, \ldots, A_n$, and B are terms.

Definition: The *axioms* of L are (i) $\mathsf{L}f\colon \Gamma \to A$, whenever $f\colon \Gamma \to A$ is a multimap in $\mathfrak{M}$, and $\mathsf{L}f \neq \mathsf{L}g$, whenever $f \neq g\colon \Gamma \to A$; (ii) $\to \mathsf{I}$; (iii) $A \to A$. The axioms in (i) will be called the "eigenaxioms" of L.

Definition: The *rules of inference* of L are the following:

Structural rules

$$(\mathrm{R}_0)\ \frac{\Gamma \to A \quad \Delta, A, \Theta \to B}{\Delta,\ \Gamma, \Theta \to B}$$

$$(\mathrm{R}_1)\ \frac{\Gamma, \Delta \to A}{\Gamma, B, \Delta \to A}$$

$$(\mathrm{R}_2)\ \frac{\Gamma, B, B, \Delta \to A}{\Gamma, B, \Delta \to A}$$

$$(\mathrm{R}_3)\ \frac{\Gamma, A, B, \Delta \to C}{\Gamma, B, A, \Delta \to C}$$

Operational rules

$$(\mathrm{R}_4)\ \frac{\Gamma, A, B, \Delta \to C}{\Gamma, A \wedge B, \Delta \to C}$$

$$(\mathrm{R}_5)\ \frac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \wedge B}$$

$$(\mathrm{R}_6)\ \frac{\Gamma \to A \quad \Delta, B, \Theta \to C}{\Delta,\ \Gamma, A \supset B, \Theta \to C}$$

$$(\mathrm{R}_7)\ \frac{A,\ \Gamma \to B}{\Gamma \to A \supset B}\ .$$

The structural rules (R_0)-(R_3) will also be called "cut," "thinning," "contraction," and "interchange," respectively.

Remark: The reader will recognize these rules as mild variants of Gentzen's rules for the appropriate fragment of LJ. The present formulation has been chosen in order to simplify certain calculations in the proofs that follow.

Terminology: For linguistic convenience, we introduce the following expressions: the explicitly mentioned instances of the term A in (R_0), of the term B in (R_1)-(R_2), and of the terms A and B in (R_3)-(R_7) will be called "active" and those of the remaining terms, "passive" terms of the inferences corresponding to the stated rules. We shall furthermore say that two consecutive inferences in a proof tree are "mutually passive" if the active terms of one inference are passive with respect to the other and conversely.

With result (ii) above in mind, we now introduce the following special class of multicategories:

Definition: A *Gentzen multicategory* is a multicategory $\mathfrak{M}$ closed under the "structural" operations schematically defined for L, subject to the following conditions: The multimaps of $\mathfrak{M}$ satisfy a number of equations, expressed conveniently as equations between "histories of construction" of such multimaps, i.e., mapping trees, as follows:

(i)
$$\frac{\Gamma \to A}{\Delta \to A}\,(a) = \frac{\Gamma \to A}{\Delta \to A}\,(b),$$

where (a) and (b) denote arbitrary, finitely many (possibly zero) antecedent thinnings, contractions, and interchanges, provided the same terms of Γ are active (R_2)-terms in the two trees and the trees admit the same generalization, i.e., the following steps transform both Γ's into identical sequences of numerals: (a) replace the terms of Δ by distinct numerals; (b) replace the "ancestors" of each term by the numeral which replaced that term in Δ; (c) choose the numerals which must be omitted in (R_1)'s so that the same numerals, with equal frequencies, occur in the sequences resulting from Γ;

(ii) cuts are permutable with arbitrary thinnings, contractions, and interchanges applied to either one of the two cut formulae, whenever such permutations are defined; and if any term occurs both as a thinning and subsequent cut term, then the thinning introducing the term and the cut eliminating it, are replaceable by suitable thinnings.

Remark: The category of sets and functions in several variables should be regarded as the paradigm of a Gentzen multicategory. The defining equations between multimaps are in fact an abstraction of the known coherence properties of such "monoidal" categories as proved by MacLane [8] and Kelly [1], under the obvious interpretation of thinning, contraction, and interchange as the adjunction, identification, and permutation of variables, respectively. These "structural" operations on the class of multimaps of a Gentzen multicategory are "combinatory" in the sense of Curry, and the precise connection between cartesian closed categories and combinatory logic is examined in detail in a paper by Lambek, to appear in the Proceedings of the Conference on the "Connection between Category Theory, Algebraic Geometry, and Intuitionistic Logic," held at Dalhousie University in January 1971.

2 *The Equivalence Relation* E. We refine the equivalence relation E_0 mentioned in the *Introduction* by requiring that E is the largest refinement of E_0 satisfying the following conditions:

(i) all equations between "mapping trees" in a Gentzen multicategory give rise to equivalent proofs;
(ii) the rules of inference of L preserve the equivalence of proofs;
(iii) the pairwise permutations of consecutive, mutually passive inferences in a proof tree preserve equivalence;
(iv) if fg is the result of substituting the multimap f in the multimap g, and $\mathsf{L}fg$, $\mathsf{L}f$, and $\mathsf{L}g$ are the induced axioms of L, then $\mathrm{Cut}(\mathsf{L}f, \mathsf{L}g)\,\mathsf{E}\,\mathsf{L}fg$;
(v) the following proofs act as "identities" with respect to the cut:

$$\text{(a)}\ \frac{\to \mathsf{I}}{\mathsf{I} \to \mathsf{I}} \qquad \text{(b)}\ \dfrac{\dfrac{A \xrightarrow{1_A} A \quad B \xrightarrow{1_B} B}{A, B \to A \wedge B}}{A \wedge B \to A \wedge B} \qquad \text{(c)}\ \dfrac{\dfrac{A \xrightarrow{1_A} A \quad B \xrightarrow{1_B} B}{A, A \supset B \to B}}{A \supset B \to A \supset B}\ ;$$

(vi) all proofs of the formula $\Gamma \to \mathsf{I}$ are equivalent;
(vii) the proof

$$\dfrac{\dfrac{A \xrightarrow{1_A} A \quad B \xrightarrow{1_B} B}{A, B \to A \wedge B} \quad \Gamma, A \wedge B, \Delta \to C}{\Gamma, A, B, \Delta \to C}$$

induces a bijection between $[\Gamma, A \wedge B, \Delta; C]$ and $[\Gamma, A, B, \Delta; C]$;
(viii) the proof

$$\dfrac{\Gamma \to A \supset B \quad \dfrac{A \xrightarrow{1_A} A \quad B \xrightarrow{1_B} B}{A, A \supset B \to B}}{A, \Gamma \to B}$$

induces a bijection between $[\Gamma; A \supset B]$ and $[A, \Gamma; B]$;
(ix) the proofs

$$\dfrac{\Gamma \to A \wedge B \quad \dfrac{\dfrac{A \xrightarrow{1_A} A}{A, B \to A}}{A \wedge B \to A}}{\Gamma \to A} \quad \text{and} \quad \dfrac{\Gamma \to A \wedge B \quad \dfrac{\dfrac{B \xrightarrow{1_B} B}{A, B \to B}}{A \wedge B \to B}}{\Gamma \to B}$$

induce a bijection between $[\Gamma; A \wedge B]$ and $[\Gamma; A] \times [\Gamma; B]$;
(x) the following proofs are equivalent:

$$\text{(a)}\ \dfrac{\dfrac{\Gamma \to A \quad \Delta \to B}{\Gamma, \Delta \to A \wedge B} \quad \dfrac{\Theta, A, B, \Phi \to C}{\Theta, A \wedge B, \Phi \to C}}{\Theta, \Gamma, \Delta, \Phi \to C} \qquad \text{(b)}\ \dfrac{\Gamma \to A \quad \dfrac{\Delta \to B \quad \Theta, A, B, \Phi \to C}{\Theta, A, \Delta, \Phi \to C}}{\Theta, \Gamma, \Delta, \Phi \to C}$$

"1_A" and "1_B" denote the corresponding instances of axiom schema (iii).
"Cut($\mathbf{L}f$, $\mathbf{L}g$)" denotes the cut $\dfrac{\Gamma \xrightarrow{\mathbf{L}g} A \quad \Delta, A, \Theta \xrightarrow{\mathbf{L}f} B}{\Delta, \Gamma, \Theta \to B}$.
"$[\Gamma; A]$" denotes the E-class of proof of the formula $\Gamma \to A$ and "$[\Gamma; A] \times [\Gamma; B]$" the cartesian product of the corresponding classes.
For the sake of simplicity of notation, we shall also write "$P \equiv Q$" for $P \in [P]$ and $Q \in [P]$.

Remark: Conditions (i)-(x) have been stated in their most perspicuous form and as such are highly redundant. This is immaterial for the purposes of this paper.

3 *Cut Elimination in* **L** *Relative to* E. We can now state and prove our first result:

Cut elimination theorem. *Every* E*-class of proofs in* **L** *has a cut-free representative.*

Proof: We assume familiarity with Gentzen's proof of the *Hauptsatz* for **LJ** and merely indicate the modifications required in Gentzen's argument in view of the differences between **L** and **LJ** and the presence of E. We first note from a cursory study of the conditions on the multimaps in a Gentzen multicategory that the following two proofs are not equivalent:

$$\text{(a)}\ \dfrac{\dfrac{\Gamma, A, A, \Delta \to B}{\Gamma, A, \Delta \to B}\ (R_2)}{\Gamma, A, A, \Delta \to B}\ (R_1) \qquad \text{(b)}\ \Gamma, A, A, \Delta \to B.$$

Hence Gentzen's "mix rule" is unsuitable for our purposes and we must establish the theorem directly for cuts. Since **L** contains the term I and is purely intuitionistic, we are also forced to specialize Gentzen's notions of the "degree" and "rank" of a cut Cut(Q, P) where Q and P are representatives of $[\Delta, A, \Theta; B]$ and $[\Gamma; A]$, respectively, as follows:

The *degree* of $\mathrm{Cut}(Q, P)$ is the total number of I's, $\wedge$'s, and $\supset$'s occurring in the cut term A. We agree that a cut-free proof has degree 0.

The *rank* of $\mathrm{Cut}(Q, P)$ is the sum of the left rank and right rank of $\mathrm{Cut}(Q, P)$. We agree that a cut-free proof has rank 0.

The *left rank* of $\mathrm{Cut}(Q, P)$ is the largest number of consecutive formulae in a path of P, up to and including $\Gamma \to A$, which contain the cut term A as their succedent.

The *right rank* of $\mathrm{Cut}(Q, P)$ is the largest number of consecutive formulae in a path of Q, up to and including $\Delta, A, \Theta \to B$, which contain the cut term A as an antecedent term.

We now carry out a double induction on the degree and rank of $\mathrm{Cut}(Q, P)$. In each case to be considered, the difficulty lies not in finding a proof of lower degree or rank (since the *Hauptsatz* already provides such proofs), but lies rather in finding *equivalent* proofs of lower degree or rank. We shall examine four cases in detail.

(i) The following two proofs are equivalent by condition (iv) of E and are of ranks 2 and 0, respectively:

$$\text{(a)}\ \frac{\Gamma \xrightarrow{\mathsf{L}f} X_1 \quad \Delta, X_1, \Theta \xrightarrow{\mathsf{L}g} X_2}{\Delta, \Gamma, \Theta \to X_2} \qquad \text{(b)}\ \Delta, \Gamma, \Theta \xrightarrow{\mathsf{L}gf} X_2.$$

(ii) The following two proofs are equivalent by condition (iii) of E and are of ranks $n + 1$ and n respectively:

$$\text{(a)}\ \frac{\Gamma \to C \quad \dfrac{\Delta, A, B, \Theta, C, \Phi \to D}{\Delta, A \wedge B, \Theta, C, \Phi \to D}}{\Delta, A \wedge B, \Theta, \Gamma, \Phi \to D} \qquad \text{(b)}\ \frac{\dfrac{\Gamma \to C \quad \Delta, A, B, \Theta, C, \Phi \to D}{\Delta, A, B, \Theta, \Gamma, \Phi \to D}}{\Delta, A \wedge B, \Theta, \Gamma, \Phi \to D}$$

(iii) The following two proofs are equivalent by conditions (i) and (x) of E and are of degrees 1 and 0, respectively:

$$\text{(a)}\ \frac{\dfrac{\Gamma \xrightarrow{\mathsf{L}f} X_1 \quad \Gamma \xrightarrow{\mathsf{L}f} X_1}{\Gamma, \Gamma \to X_1 \wedge X_1} \quad \dfrac{\dfrac{\dfrac{\Delta, X_1, X_1, X_1, X_1, \Theta \xrightarrow{\mathsf{L}g} X_2}{\Delta, X_1, X_1, X_1 \wedge X_1, \Theta \to X_2}}{\Delta, X_1 \wedge X_1, X_1 \wedge X_1, \Theta \to X_2}}{\Delta, X_1 \wedge X_1, \Theta \to X_2}}{\Delta, \Gamma, \Gamma, \Theta \to X_2}$$

$$\text{(b)}\ \frac{\dfrac{\Gamma \xrightarrow{\mathsf{L}f} X_1 \quad \dfrac{\Gamma \xrightarrow{\mathsf{L}f} X_1 \quad \dfrac{\Gamma \xrightarrow{\mathsf{L}f} X_1 \quad \dfrac{\Gamma \xrightarrow{\mathsf{L}f} X_1 \quad \Delta, X_1, X_1, X_1, X_1, \Theta \xrightarrow{\mathsf{L}g} X_2}{\Delta, X_1, X_1, X_1, \Gamma, \Theta \to X_2}}{\Delta, X_1, X_1, \Gamma, \Gamma, \Theta \to X_2}}{\Delta, X_1, \Gamma, \Gamma, \Gamma, \Theta \to X_2}}{\Delta, \Gamma, \Gamma, \Gamma, \Gamma, \Theta \to X_2}}{\Delta, \Gamma, \Gamma, \Theta \to X_2}\ \text{(possibly } (\mathrm{R}_3)\text{'s and } (\mathrm{R}_2)\text{'s)}$$

By condition (iv) of E, (b) is equivalent to the cut-free proof $\Delta, \Gamma, \Gamma, \Theta \xrightarrow{\mathsf{L}h} X_2$, where h stands for the multimaps obtained from f and g by the indicated structural operations.

(iv) The following proofs are equivalent by conditions (i), (iii), and (viii) of E and (a) and (d) are of degrees $n + m + 1$, and m and n, respectively:

$$\text{(a)}\quad \dfrac{\dfrac{A,\Gamma \to B}{\Gamma \to A \supset B} \quad \dfrac{\Delta \to A \quad \Theta, B, \Phi \to C}{\Theta, \Delta, A \supset B, \Phi \to C}}{\Theta, \Delta, \Gamma, \Phi \to C}$$

$$\text{(b)}\quad \dfrac{\dfrac{A,\Gamma \to B}{\Gamma \to A \supset B} \quad \dfrac{\dfrac{\Delta \to A \quad \dfrac{A \xrightarrow{1_A} A \quad B \xrightarrow{1_B} B}{A, A \supset B \to B}}{\Delta, A \supset B \to B} \quad \Theta, B, \Phi \to C}{\Theta, \Delta, A \supset B, \Phi \to C}}{\Theta, \Delta, \Gamma, \Phi \to C}$$

$$\text{(c)}\quad \dfrac{\Delta \to A \quad \dfrac{\dfrac{\dfrac{A,\Gamma \to B}{\Gamma \to A \supset B} \quad \dfrac{A \xrightarrow{1_A} A \quad B \xrightarrow{1_B} B}{A, A \supset B \to B}}{A, \Gamma \to B} \quad \Theta, B, \Phi \to C}{\Theta, A, \Gamma, \Phi \to C}}{\Theta, \Delta, \Gamma, \Phi \to C}$$

$$\text{(d)}\quad \dfrac{\Delta \to A \quad \dfrac{A, \Gamma \to B \quad \Theta, B, \Phi \to C}{\Theta, A, \Gamma, \Phi \to C}}{\Theta, \Delta, \Gamma, \Phi \to C} \quad .$$

The remaining cases are proved similarly and establish the theorem.

4 *Canonical Proofs in* L *Relative to* E. Each E-class $[P]$ is sufficiently rich in cut-free representatives to enable us to reduce any given cut-free proof $P \epsilon [P]$ to a definite canonical form within $[P]$. We shall describe the canonically equivalent proof P' in several steps. All proofs occurring in the reduction procedure will be cut-free.

(i) Inductions on the number of $\wedge$'s and $\supset$'s, together with condition (v) of E, show that P is equivalent to a proof Q_1 which does not involve axiom schema (iii).

(ii) The following sequence of equivalent proofs shows that Q_1 is equivalent to a proof Q_2 which does not contain thinning terms of the form $B \wedge C$:

$$\text{(a)}\quad \dfrac{\dfrac{\dfrac{\Gamma, D, \Delta \to A}{\Gamma, C, D, \Delta \to A}}{\Gamma, B, C, D, \Delta \to A}}{\Gamma, B \wedge C, D, \Delta \to A}$$

$$\text{(b)}\quad \dfrac{\dfrac{\dfrac{C \xrightarrow{1_C} C}{B, C \to C} \quad \dfrac{\dfrac{D \xrightarrow{1_D} D}{C, D \to D} \quad \Gamma, D, \Delta \to A}{\Gamma, C, D, \Delta \to A}}{\Gamma, B, C, D, \Delta \to A}}{\Gamma, B \wedge C, D, \Delta \to A}$$

$$\text{(c)}\quad \dfrac{\dfrac{\dfrac{C \xrightarrow{1_C} C}{B, C \to C}}{B \wedge C \to C} \quad \dfrac{\dfrac{D \xrightarrow{1_D} D}{C, D \to D} \quad \Gamma, D, \Delta \to A}{\Gamma, C, D, \Delta \to A}}{\Gamma, B \wedge C, D, \Delta \to A}$$

$$\text{(d)}\quad \dfrac{\dfrac{\dfrac{C \xrightarrow{1_C} C}{B, C \to C}}{B \wedge C \to C} \quad \dfrac{\Gamma, D, \Delta \to A}{\Gamma, C, D, \Delta \to A}}{\Gamma, B \wedge C, D, \Delta \to A}$$

$$\text{(e)}\quad \dfrac{\Gamma, D, \Delta \to A}{\Gamma, B \wedge C, D, \Delta \to A}\ .$$

The equivalences of proofs (a)-(e) follow from conditions (i), (iii), (i), and (i) of E, respectively.

(iii) The following two proofs are equivalent by conditions (i) and (iii) of E and show that Q_2 is equivalent to a proof Q_3 which contains at most an instance of (b):

$$\text{(a)}\quad \dfrac{\dfrac{\dfrac{\Gamma, A, B, A, B, \Delta \to C}{\Gamma, A, B, A \wedge B, \Delta \to C}}{\Gamma, A \wedge B, A \wedge B, \Delta \to C}}{\Gamma, A \wedge B, \Delta \to C}$$

$$\text{(b)}\quad \dfrac{\dfrac{\dfrac{\dfrac{\Gamma, A, B, A, B, \Delta \to C}{\Gamma, A, A, B, B, \Delta \to C}}{\Gamma, A, A, B, \Delta \to C}}{\Gamma, A, B, \Delta \to C}}{\Gamma, A \wedge B, \Delta \to C}\ .$$

(iv) The following pairs (a)-(b) and (c)-(d) of respectively equivalent proofs show that Q_3 is equivalent to a proof Q_4 which does not contain contraction terms of the form $A \supset B$, if B was introduced by a thinning:

$$\text{(a)}\quad \dfrac{\dfrac{\dfrac{B \to A \quad \dfrac{\Gamma, B, A \supset B, \Delta \to C}{\Gamma, B, B, A \supset B, \Delta \to C}}{\Gamma, B, B, A \supset B, A \supset B, \Delta \to C}}{\Gamma, B, B, A \supset B, \Delta \to C}}{\Gamma, B, A \supset B, \Delta \to C}$$

$$\text{(b)}\quad \Gamma, B, A \supset B, \Delta \to C$$

$$\text{(c)}\quad \dfrac{\dfrac{\dfrac{A \to A \quad \dfrac{\Gamma, A, A \supset B, \Delta \to C}{\Gamma, A, B, A \supset B, \Delta \to C}}{\Gamma, A, A, A \supset B, A \supset B, \Delta \to C}}{\Gamma, A, A, A \supset B, \Delta \to C}}{\Gamma, A, A \supset B, \Delta \to C}$$

$$\text{(d)}\quad \Gamma, A, A \supset B, \Delta \to C.$$

The equivalence of (a) and (b), for example, is established by the following sequence of equivalent proofs: (a) as above

$$\text{(b)}\quad \dfrac{\dfrac{\dfrac{B \to A \quad \dfrac{\dfrac{A \xrightarrow{1_A} A \quad B \xrightarrow{1_B} B}{A, A \supset B \to B} \quad \dfrac{\dfrac{B \xrightarrow{1_B} B}{B, B \to B} \quad \Gamma, B, A \supset B, \Delta \to C}{\Gamma, B, B, A \supset B, \Delta \to C}}{\Gamma, B, A, A \supset B, A \supset B, \Delta \to C}}{\Gamma, B, B, A \supset B, A \supset B, \Delta \to C}}{\Gamma, B, B, A \supset B, \Delta \to C}}{\Gamma, B, A \supset B, \Delta \to C}$$

(c)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B\to A \quad B\xrightarrow{1_B}B}{B, A\supset B\to B}\quad \dfrac{B\xrightarrow{1_B}B}{B, B\to B}}{B, B, A\supset B\to B}\quad \Gamma, B, A\supset B, \Delta\to C}{\Gamma, B, B, A\supset B, A\supset B, \Delta\to C}}{\Gamma, B, B, A\supset B, \Delta\to C}}{\Gamma, B, A\supset B, \Delta\to C}$$

(d)
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B\xrightarrow{1_B}B}{B, A\supset B\to B}}{B, B, A\supset B\to B}\quad \Gamma, B, A\supset B, \Delta\to C}{\Gamma, B, B, A\supset B, A\supset B, \Delta\to C}}{\Gamma, B, B, A\supset B, \Delta\to C}}{\Gamma, B, A\supset B, \Delta\to C}$$

(e)
$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, B, A\supset B, \Delta\to C}{\Gamma, B, A\supset B, A\supset B, \Delta\to C}}{\Gamma, B, B, A\supset B, A\supset B, \Delta\to C}}{\Gamma, B, B, A\supset B, \Delta\to C}}{\Gamma, B, A\supset B, \Delta\to C} \quad .$$

Conditions (i) and (iii) of E yield the equivalence of proofs (a)-(e) and of (e) and $\Gamma, B, A\supset B, \Delta\to C$.

(v) Calculations similar to those in (iv) also show that Q_4 is equivalent to a proof Q_5 which contains at most the shorter proofs of the following pairs (a)-(b) and (c)-(d) of respectively equivalent proofs:

(a)
$$\dfrac{\dfrac{A\to A \quad \dfrac{\Gamma, A, \Delta\to B}{\Gamma, C, A, \Delta\to B}}{\Gamma, A, A\supset C, A, \Delta\to B}}{\dfrac{\Gamma, A\supset C, A, A, \Delta\to B}{\Gamma, A\supset C, A, \Delta\to B}}$$

(b)
$$\dfrac{\Gamma, A, \Delta\to B}{\Gamma, A\supset C, A, \Delta\to B}$$

(c)
$$\dfrac{\dfrac{A\to C \quad \dfrac{\Gamma, A, \Delta\to B}{\Gamma, A, A, \Delta\to B}}{\Gamma, A, C\supset A, A, \Delta\to B}}{\dfrac{\Gamma, C\supset A, A, A, \Delta\to B}{\Gamma, C\supset A, A, \Delta\to B}}$$

(d)
$$\dfrac{\Gamma, A, \Delta\to B}{\Gamma, C\supset A, A, \Delta\to B}\,.$$

(vi) By conditions (iii) and (ix) of E, Q_5 is equivalent to a proof Q_6 which contains at most a segment similar to (a) of the following two equivalent proofs (a) and (b):

(a)
$$\dfrac{\dfrac{A\xrightarrow{p}A \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A\to A \quad A\to B}{A, A\supset A\to B}\quad \dfrac{A\to A \quad A\to C}{A, A\supset A\to C}}{A, A\supset A, A, A\supset A\to B\wedge C}}{A, A, A\supset A, A\supset A\to B\wedge C}}{A, A, A\supset A\to B\wedge C}}{A, A\supset A\to B\wedge C}}{A, A\supset A, A\supset A\to B\wedge C}}{A, A\supset A\to B\wedge C}$$

$$
\text{(b)}\quad
\cfrac{\cfrac{\cfrac{\cfrac{A \xrightarrow{P} A \quad \cfrac{A \to A \quad A \to B}{A, A \supset A \to B}}{A, A \supset A, A \supset A \to B}}{A, A \supset A \to B} \qquad \cfrac{\cfrac{A \xrightarrow{P} A \quad \cfrac{A \to A \quad A \to C}{A, A \supset A \to C}}{A, A \supset A, A \supset A \to C}}{A, A \supset A \to C}}{\cfrac{A, A \supset A, A, A \supset A \to B \wedge C}{\cfrac{A, A, A \supset A, A \supset A \to B \wedge C}{A, A, A \supset A \to B \wedge C}}}}{A, A \supset A \to B \wedge C}
$$

(vii) By condition (i) of E, Q_6 is equivalent to a proof Q_7 which contains at most the shorter proofs of the following pairs (a)-(b) and (c)-(d) of respectively equivalent proofs:

$$
\text{(a)}\quad \cfrac{\cfrac{\Gamma, A, \Delta \to B}{\Gamma, A, A, \Delta \to B}}{\Gamma, A, \Delta \to B} \qquad \text{(b)}\quad \Gamma, A, \Delta \to B
$$

$$
\text{(c)}\quad \begin{array}{l} \Gamma, A, \Delta, B, \Theta \to C \\ \hline\hline \Gamma, B, \Delta, A, \Theta \to C \\ \hline\hline \Gamma, A, \Delta, B, \Theta \to C \end{array} \begin{array}{l} (\mathrm{R}_3)\text{'s} \\ (\mathrm{R}_3)\text{'s} \end{array} \qquad \text{(d)}\quad \Gamma, A, B, \Delta \to C\ ,
$$

where the steps in (a) and (c) may have been made consecutive by the permutation of several inferences in agreement with conditions (i) and (iii) of E.

(viii) By conditions (i), (iii), and (iv) of E, Q_7 is equivalent to a proof Q_8 which contains no applications of (R_1)-(R_3) which can be relegated to the corresponding structural operations on $\mathfrak{M}$, and which can therefore be avoided altogether.

(ix) By conditions (i) and (iii) of E, Q_8 is equivalent to a proof Q_9 with the following properties: Q_9 contains a maximal contraction-free subproof R (with the obvious meaning of "subproof") such that Q_9 results from R by means of applications of (R_1)-(R_3) alone in such a way that

(a) all contractions precede all thinnings;
(b) the only interchanges which follow R are interchanges required in (a);
(c) all interchanges precede all thinnings;
(d) the only thinnings which follow contractions are thinnings by which contraction terms are re-introduced.

(x) We agree that "the" canonical proof of $\Gamma \to \mathsf{I}$ is the proof $\begin{array}{c} \to \mathsf{I} \\ \hline\hline \Gamma \to \mathsf{I} \end{array}\ (a)$,
where (a) consists of thinnings only. By condition (vi) of E, Q_9 is therefore equivalent to a proof Q_{10} with the property that any "subproof" of Q_{10} terminating in I is canonical.

Q_{10} is *irredundant* in an obvious way and we let $P' = Q_{10}$. This completes the description of a canonical representative of $[P]$.

Terminology: The maximal contraction-free subproof R of P' will be called the "principal part" and the part following the principal part, the "ending" of P'.

5 *The Categorical Generality of Canonical Proofs in* **L**. In order to define the "generality" of canonical proofs, we introduce the notion of an "arrow" multicategory (Lambek calls it a "testing multicategory").

Definition: An *arrow multicategory* is a finite multicategory $\mathfrak{m}$ whose objects are denoted by "0", "1", . . ., "m - 1", and "m", respectively, and which has precisely one non-identity map $a : 0, 1, \ldots, m - 1 \rightarrow m$.

Notation: "The" finite coproduct of the arrow multicategories $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ in the category of small multicategories (i.e., the "disjoint union" of $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$) will be denoted by $\sum_i^n \mathfrak{m}_i$.

The generality of a canonical proof $P : \Gamma \rightarrow A$ in **L** is now defined as follows:

Definition: The *generality* of $P : \Gamma \rightarrow A$ is the generality of the principal part $P' : \Gamma' \rightarrow A'$ of P. The generality of P' is the set of all pairs $\langle H, \sum_i^n \mathfrak{m}_i \rangle$, where $H: \sum_i^n \mathfrak{m}_i \rightarrow \mathfrak{M}$ is a functor for which there exists a proof $P^0 : \Gamma^0 \rightarrow A^0$ in $\mathsf{L}\left(\sum_i^n \mathfrak{m}_i\right)$ such that $H(P^0) = P'$ (with the obvious meaning of $H(P^0)$). "P^0" will be called a "generalization" of P'. We define $H(\mathsf{I}) = \mathsf{I}$ for all functors H.

Definition: Two canonical proofs P and Q of the same formula $\Gamma \rightarrow A$ are *equi-general* iff P and Q have the same generality, contain the same number of contractions, and P^0 and Q^0 are proofs of the same formula whenever $H(P^0)$ and $H(Q^0)$ are the principal parts of P and Q, respectively.

We can now prove the existence of an effective criterion for the equivalence of canonical proofs:

Generality theorem. *Two canonical proofs P and Q of a formula $\Gamma \rightarrow A$ in* **L** *are* E*-equivalent iff they are equi-general.*

Proof: Suppose that P and Q are equivalent. We may assume that they contain eigenaxioms, since the theorem holds trivially otherwise. By the definition of generality, it suffices to show that P and Q can differ only in the order of application of permutable mutually passive antecedent inferences. Since it is clear from the lack of choice in the construction of succedents that P and Q must contain the same number of applications of rules (R_5) and (R_7), we need merely show that they also contain the same number of applications of (R_4) and (R_6), that the relative order of application of rules (R_5) and (R_6) is the same in P and Q, and that the eigenaxioms of P, ordered from left to right, contain the same number of antecedent terms as the corresponding eigenaxioms of Q. These properties of P and Q are a direct consequence of the definition of canonical proofs and the following easy calculations:

$$\text{(i)}\quad \frac{\Gamma,\ X_1,\ X_1,\ \Delta \xrightarrow{\mathsf{L}f} X_2}{\Gamma,\ X_1 \wedge X_1,\ \Delta \to X_2} \not\equiv \frac{\Gamma,\ \Delta \xrightarrow{\mathsf{L}g} X_2}{\Gamma,\ X_1 \wedge X_1,\ \Delta \to X_2}\ ;$$

$$\text{(ii)}\quad \cfrac{\cfrac{\cfrac{\Gamma,\ X_1,\ X_2,\ X_1,\ X_2,\ \Delta \xrightarrow{\mathsf{L}f} X_3}{\Gamma,\ X_1,\ X_2,\ X_1 \wedge X_2,\ \Delta \to X_3}}{\Gamma,\ X_1 \wedge X_2,\ X_1 \wedge X_2,\ \Delta \to X_3}}{\Gamma,\ X_1 \wedge X_2,\ \Delta \to X_3} \equiv \frac{\Gamma,\ X_1,\ X_2,\ \Delta \xrightarrow{\mathsf{L}f'} X_3}{\Gamma,\ X_1 \wedge X_2,\ \Delta \to X_3}\ ,$$

where f' results from f in $\mathfrak{M}$ by the required interchange and contractions;

$$\text{(iii)}\quad \frac{A \to C \quad A,\ \Gamma \to B}{A,\ C \supset A,\ \ \Gamma \to B} \not\equiv \frac{A,\ \Gamma \to B}{A,\ C \supset A,\ \Gamma \to B}\ ;$$

$$\text{(iv)}\quad \cfrac{\cfrac{C \to D \quad \cfrac{C \to D \quad \cfrac{C \to D \quad A,\ \Gamma \to B}{C,\ D \supset A,\ \Gamma \to B}}{C,\ D \supset\ C,\ D \supset A,\ \Gamma \to B}}{C,\ D \supset\ C,\ D \supset C,\ D \supset A,\ \Gamma \to B}}{C,\ D \supset C,\ D \supset A,\ \Gamma \to B} \not\equiv \frac{C \to D \quad \cfrac{C \to D \quad A,\ \Gamma \to B}{C,\ D \supset A,\ \ \Gamma \to B}}{C,\ D \supset C,\ D \supset A,\ \Gamma \to B}\ ;$$

(v) step (vi) of the reduction to canonical form assures the identical order of application of rules (R_5) and (R_6) in P and Q;

(vi) if $P_1 \not\equiv P_2$, then

$$\frac{\Gamma \xrightarrow{P_1} A \quad \Delta \xrightarrow{P_2} A}{\Gamma,\ \Delta \to A \wedge A} \not\equiv \cfrac{\cfrac{\Delta \xrightarrow{P_2} A \quad \Gamma \xrightarrow{P_1} A}{\Delta,\ \Gamma \to A \wedge A}}{\Gamma,\ \Delta \to A \wedge A}\ (R_3)\ ;$$

(vii) if $P_1 \not\equiv P_2$, then

$$\frac{A,\ \Gamma \xrightarrow{P_1} B \quad A,\ \Delta \xrightarrow{P_2} B}{A,\ \Gamma,\ B \supset A,\ \Delta \to B} \not\equiv \cfrac{\cfrac{A,\ \Delta \xrightarrow{P_2} B \quad A,\ \Gamma \xrightarrow{P_1} B}{A,\ \Delta,\ B \supset A,\ \Gamma \to B}}{A,\ \ \Gamma,\ B \supset A,\ \Delta \to B}\ (R_3)\ ;$$

$$\text{(viii)}\quad \cfrac{\cfrac{\Gamma,\ A,\ A,\ \Delta \to B}{\Gamma,\ A,\ \Delta \to B}}{\Gamma,\ A,\ A,\ \Delta \to B} \not\equiv \Gamma,\ A,\ A,\ \Delta \to B.$$

Remark: Here and below, we are loosely labelling the following two equivalent proofs as "mutually passive":

$$\text{(a)}\quad \cfrac{\cfrac{\cfrac{\Gamma,\ X_1,\ X_2,\ \Delta \to X_3}{\Gamma,\ X_1,\ X_2,\ X_1,\ \Delta \to X_3}}{\Gamma,\ X_1,\ X_2,\ X_1,\ X_2,\ \Delta \to X_3}}{\Gamma,\ X_1,\ X_2,\ X_1 \wedge X_2,\ \Delta \to X_3} \qquad \text{(b)}\quad \cfrac{\cfrac{\cfrac{\Gamma,\ X_1,\ X_2,\ \Delta \to X_3}{\Gamma,\ X_1 \wedge X_2,\ \Delta \to X_3}}{\Gamma,\ X_2,\ X_1 \wedge X_2,\ \Delta \to X_3}}{\Gamma,\ X_1,\ X_2,\ X_1 \wedge X_2,\ \Delta \to X_3}\ .$$

This is reasonable since their equivalence follows from condition (iii) of E via a trivial calculation using conditions (i) and (ii) of E.

Suppose now that P and Q are equi-general. It suffices to show that (i) P and Q contain identical axioms in identical order from left to right, (ii) the endformulae of the principal parts of P and Q are identical, (iii) the principal parts of P and Q differ at most in the order of application of permutable mutually passive antecedent inferences, and (iv) the endings of P and Q differ at most in the order of application of permutable structural inferences. The equivalence of P and Q will then follow from conditions (iii) and (i) of E, respectively.

(i) is an easy consequence of the paucity of multimaps in arrow multicategories and the fact that P and Q are canonical proofs of the same formula, and that therefore each generalization of the principal part of P must be a proof of the same formula in $\mathsf{L}\left(\sum_i^n \mathfrak{m}_i\right)$ as the corresponding generalization of the principal part of Q.

(ii) and (iii) are seen as follows: By the definition of equi-general proofs, P and Q involve the same number of contractions and these contractions, if any, can occur only as the result of applications of rules (R_5) and (R_6). The subterm property of canonical proofs and the fact that eigenaxioms contain only atomic terms, imply moreover, that P and Q involve the same number of applications of rules (R_4), (R_5), and (R_7). Since the respective axioms and conclusions of P and Q contain the same terms in the same order, the canonical nature of the two proofs ensures, in addition, that the same $\supset$ is not introduced by a thinning in one proof and by an application of (R_6) in the other. Hence the number of applications of (R_6) and that of thinnings in the principal parts of P and Q with terms of the form $A \supset B$ coincide. By the definition of canonical proofs, the endformulae of the principal parts of P and Q are therefore identical.

(iv) follows at once from (ii) and (iii) since P and Q prove the same formula canonically. This establishes the theorem.

6 *Applications.*

1. The deductive system L can be considered as an extension by constants and function symbols of the deductive system L_0 whose terms are the objects of $\mathfrak{M}$, whose axioms are the eigenaxioms of L, and whose rules of inference are the structural rules of L. By the subterm property of cut-free proofs, condition (i) of E, and the *generality theorem* it follows that L is a conservative extension of L_0. This answers question (ii) raised at the beginning of this paper.

2. By taking the terms of L as objects, and the E-classes of proofs of L as multimaps, we obtain a cartesian closed multicategory $F(\mathfrak{M})$ "free" on $\mathfrak{M}$ (with a definition of "cartesian closed multicategory" based on the definition of a "biclosed monoidal multicategory" in Lambek [4]).

3. By virtue of conditions (i) and (ii) of E, any arbitrary category is isomorphic to a category generated by a deductive system whose terms are the objects of the given category and whose only rule of inference is the cut, specialized to formulae with non-empty single-term antecedents. Frequently categories "with additional structure" such as distinguished objects, endofunctors, etc., can be studied by considering extensions of the generating system of the "underlying" categories and by lifting well-known theorems of logic to the level of categories. The fact that E separates the

structural and operational rules of **L** conservatively, for example, means that any arbitrary (small) multicategory can be fully and faithfully embedded in a free *cartesian closed* multicategory.

4. Professor Bernays once pointed out to the author that one reason for the success of Gentzen's *Hauptsatz* is the fact that (R_6) comes almost trivially close to being a cut rule if we identify A and B in the schema above. The categorical interpretation of (R_0) as "generalized composition" and of (R_6) as "generalized evaluation" sheds new light on this remark since it shows that in the case of cartesian closed multicategories, for example, the cartesian closed structure is "closed under substitution" precisely because of the presence of "evaluation."

5. The non-equivalent canonical proofs of a formula $\Gamma \to A$ in **L** form a recursive set relative to the eigenaxioms of **L**. This fact follows from Gentzen's decision procedure for the propositional fragment of **LJ**, together with the *generality theorem*: If $\Gamma \to A$ does not contain a subterm of the form $B \supset C$, then $\Gamma \to A$ has only finitely many non-equivalent canonical proof schemata. If $\Gamma \to A$ does contain subterms of the form $B \supset C$ and admits contractions, then it follows from the *generality theorem* that $\Gamma \to A$ has countably many non-equivalent proof schemata. We can give a recursive description of these schemata by indexing them by means of recursively ordered k-tuples of natural numbers, where k denotes the number of places in a canonical proof of $\Gamma \to A$ which admit contractions. In view of the *generality theorem*, only finitely many non-equivalent schemata exist at each stage, and these can be calculated by Gentzen's decision procedure for the propositional fragment of **LJ**.

6. The *generality theorem* gives a recursive solution of the "coherence problem" for cartesian closed categories via cartesian closed multicategories by showing that although coherence in the sense of MacLane-Kelly, *cf.* [1], [2] and [8], does not hold in general for free cartesian closed categories, it is possible to associate with each canonical map in such categories a measure of complexity, which we shall call the "degree of coherence" of that map, and which is defined as the pair $\langle g, n\rangle$, where "g" denotes the generality of one of the canonical proofs P representing the given map in the appropriate deductive system **L**, and "n" denotes the number of contractions in P, such that a diagram of composite canonical maps commutes iff the composite maps of the two sides of the diagram have the same degree of coherence, i.e., iff their representing canonical proofs in **L** are equi-general.

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