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MODALITY AND PREFERENCE RELATION

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Halldén's system $\mathcal{A}$ on logic of preference contains the following formula as one axiom.[1]

(1) $$p\mathbf{P}q \equiv (p \cdot \sim q)\, \mathbf{P}(\sim p \cdot q)$$

The formula means that a state p is preferred to a state q if and only if p and not q is preferred to not p and q. Hansson pointed out the absurdity of the formula as follows. If $\sim q$ and $\sim p$ are substituted respectively for p and q in (1), we will have the formula:[2]

$$\sim q\, \mathbf{P} \sim p \equiv (\sim q \cdot \sim\sim p)\, \mathbf{P}(\sim\sim q \cdot \sim p),$$

that is

$$\sim q\mathbf{P}\sim p \equiv (p \cdot \sim q)\, \mathbf{P}(\sim p \cdot q).$$

From this formula and (1) we can derive the formula:

(2) $$p\mathbf{P}q \equiv \sim q\mathbf{P}\sim p.$$

He illustrates the invalidity of the formula by the following example. "Suppose that a person A has bought some ticket in a lottery with two prizes of unequal worth . . . . Let p stand for 'A wins the first prize' and q for 'A wins some prize.' It is reasonable to think that $p\mathbf{P}q$ is true for A. If A accepts (2), then he will also claim that $\sim q\mathbf{P}\sim p$ is true, i.e., he will prefer not winning any prize to not winning the first prize."

In this example, one of α and β in $\alpha\mathbf{P}\beta$ logically implies the other. In case where we can think that $\alpha\mathbf{P}\beta$ is equivalent to $\alpha \cdot \sim \beta\mathbf{P}\sim\alpha \cdot \beta$, can we think that $\alpha(\beta)$ logically implies $\beta(\alpha)$? If $\alpha(\beta)$ logically implies $\beta(\alpha)$, $\alpha \cdot \sim\beta(\sim\alpha \cdot \beta)$ is self-contradictory, i.e., it cannot express any logically

1. S. Halldén, *On the Logic of 'Better,'* Lund (1957), p. 28.

2. B. Hansson, "Fundamental axioms for preference relations," *Synthese*, vol. 18 (1968), pp. 428-429.

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possible state. In such a case we may not be able to understand what it means that $\alpha \cdot \sim\beta$ is preferred to $\sim\alpha\cdot\beta$.

Instead, we are justified to think that (1) holds only when both $p\cdot\sim q$ and $\sim p\cdot q$ are logically possible, i.e., p and q do not imply each other. Insofar as we think so, we can avoid at least the difficulty pointed out by Hansson. If $\Box(p\supset q)$ stands for 'p logically implies q,' the formula expressing that (1) holds if p and q do not imply each other will be symbolized as follows:

$$\Box(p\supset q)\vee\Box(q\supset p)\vee(p\mathbf{P}q\equiv(p\cdot\sim q)\mathbf{P}(\sim p\cdot q)). \tag{3}$$

If $\sim q$ and $\sim p$ is substituted respectively for p and q in the formula, we will have the formula:

$$\Box(\sim q\supset\sim p)\vee\Box(\sim p\supset\sim q)\vee(\sim q\mathbf{P}\sim p\equiv(\sim q\cdot\sim\sim p)\mathbf{P}(\sim\sim q\cdot\sim p)),$$

that is,

$$\Box(p\supset q)\vee\Box(q\supset p)\vee(\sim q\mathbf{P}\sim p\equiv(p\cdot\sim q)\mathbf{P}(\sim p\cdot q)).$$

From this formula and (3), the following formula derives:

$$\Box(p\supset q)\vee\Box(q\supset p)\vee(p\mathbf{P}q\equiv\sim q\mathbf{P}\sim p).$$

The formula means that (2) holds if p and q do not imply each other.

Halldén's system $\mathscr{A}$ contains also the following formula as an axiom:

$$p\mathbf{S}q\equiv(p\cdot\sim q)\mathbf{S}(\sim p\cdot q),$$

where $\mathbf{S}$ means 'is equal in value to.' On the similar ground, it is desirable to replace by the following formula:

$$\Box(p\supset q)\vee\Box(q\supset p)\vee(p\mathbf{S}q\equiv p\cdot\sim q\mathbf{S}\sim p\cdot q).$$

Although we cannot think that (1) holds in case where one of p and q implies the other, we can think as follows. In case where p implies q that p is preferred to q is equivalent to that p is preferred to not p and q. In the example mentioned above, that winning the first prize is preferred to winning some prize is equivalent to that winning the first prize is preferred to not winning first prize and winning some prize, i.e., winning the second prize. Similarly in case where q implies p, we can think, that p is preferred to q is equivalent to that p and not q is preferred to q. In these cases, we can formulate as follows:

$$\Box(p\supset q)\supset(p\mathbf{P}q\equiv p\mathbf{P}(\sim p\cdot q)), \tag{4}$$
$$\Box(q\supset p)\supset(p\mathbf{P}q\equiv(p\cdot\sim q)\mathbf{P}q). \tag{5}$$

If p is substituted for q in (4), we will have

$$\Box(p\supset p)\supset(p\mathbf{P}p\equiv p\mathbf{P}(\sim p\cdot p)).$$

Since $p\supset p$ is a tautology, we have

$$p\mathbf{P}p\equiv p\mathbf{P}c,$$

where c is a self-contradictory formula. Since p is irreflexive, we have $\sim(p\mathbf{P}p)$. Therefore, we have

$$(6) \qquad \sim(p\mathbf{P}c).$$

Similarly by substituting p for q in (5), we can have

$$(7) \qquad \sim(c\mathbf{P}p).$$

It is shown by (6) and (7) that it is false that a state is preferred to a logically impossible state or a logically impossible state is preferred to a state. In the systems where some states may not be comparable with each other, the above result is not absurd. This use of 'prefer' may not be remote from the ordinary use of 'prefer.' In the ordinary sense of 'prefer,' we may not be able to understand what a comparison between a logically impossible state and a state means. In the system, however, where all states are comparable with each other, the above result leads to an absurd conclusion. In such a system,

$$\sim(p\mathbf{P}c)\cdot\sim(c\mathbf{P}p)$$

is equivalent to $p\mathbf{S}c$. From symmetricity and transitivity of $\mathbf{S}$, it follows that all states are equal in value. In order to avoid this absurdity, it is necessary to modify the system. That all states are comparable with each other should be replaced by the following. All logically possible states are comparable with each other. For example, if the formula

$$p\mathbf{P}q \vee p\mathbf{S}q \vee q\mathbf{P}p$$

is an axiom of a system, it should be replaced by the formula

$$\Box\sim p \vee \Box\sim q \vee p\mathbf{P}q \vee p\mathbf{S}q \vee q\mathbf{P}p.$$

If $\mathbf{t}$ (which stands for a tautology) is substituted for q in (4), we will have

$$\Box(p \supset \mathbf{t}) \supset (p\mathbf{P}t \equiv p\mathbf{P}(\sim p\cdot\mathbf{t})).$$

Since $p \supset \mathbf{t}$ is a tautology, we have

$$(8) \qquad p\mathbf{P}\mathbf{t} \equiv p\mathbf{P}\sim p.$$

If $\mathbf{t}$ is substituted for p in (5), we can derive

$$\Box(q \supset \mathbf{t}) \supset (\mathbf{t}\mathbf{P}q \equiv (\mathbf{t}\cdot\sim q)\mathbf{P}q).$$

Therefore, we have

$$(9) \qquad \mathbf{t}\mathbf{P}q \equiv \sim q\mathbf{P}q.$$

The above equivalences may not be remote from the ordinary use of 'prefer.' For example, let p stand for 'She will come tomorrow' and let us suppose that 'She will come tomorrow' is preferred to 'She will not come tomorrow,' i.e., that $p\mathbf{P}\sim p$ is true. Then, accepting the information of 'She will come tomorrow' is preferred to accepting no information. (Accepting a tautological information may be equal to accepting no information.) On the contrary, suppose that accepting information that she will come tomorrow is preferred to accepting no information. Then the information that she will come tomorrow may be preferred to the information that she will not come tomorrow. The formulas as to $\mathbf{S}$ corresponding to (4) and (5) are as follows:

$$\Box(p \supset q) \supset (p\mathbf{S}q \equiv p\mathbf{S}(\sim p \cdot q)), \tag{10}$$
$$\Box(q \supset p) \supset (p\mathbf{S}q \equiv (p . \sim q)\mathbf{S}q). \tag{11}$$

As (11) is derived from (10) we need not consider (11) hereafter. If p is substituted for q in (10), we will have

$$\Box(p \supset p) \supset (p\mathbf{S}p \equiv p\mathbf{S}(\sim p \cdot p)).$$

Since $p \supset p$ is a tautology, we have

$$p\mathbf{S}p \equiv p\mathbf{S}c.$$

Since $\mathbf{S}$ is reflexive, we have

$$p\mathbf{S}c. \tag{12}$$

Since $\mathbf{S}$ is symmetrical and transitive, it follows from (12) that all states are equal in value. In order to avoid this absurdity, (10) should be modified as follows:

$$\sim\Box(p \supset q) \vee \Box(q \supset p) \vee (p\mathbf{S}q \equiv p\mathbf{S}(\sim p \cdot q)).$$

From the above consideration we shall construct the system $\mathcal{A}'$, a modification of Halldén's system $\mathcal{A}$,[3] as follows.

$\mathcal{A}'$ is constructed by adding the following axioms and the rule of replacement to the modal system S5.

A1 $p\mathbf{P}q \supset \sim(q\mathbf{P}p)$
A2 $(p\mathbf{P}q \cdot q\mathbf{P}r) \supset p\mathbf{P}r$
A3 $p\mathbf{S}p$
A4 $p\mathbf{S}q \supset q\mathbf{S}p$
A5 $(p\mathbf{S}q \cdot q\mathbf{S}r) \supset p\mathbf{S}r$
A6 $(p\mathbf{P}q \cdot q\mathbf{S}r) \supset p\mathbf{P}r$
A7 $\Box(p \supset q) \vee \Box(q \supset p) \vee (p\mathbf{P}q \equiv (p \cdot \sim q)\mathbf{P}(\sim p \cdot q))$
A8 $\Box(p \supset q) \vee \Box(q \supset p) \vee (p\mathbf{S}q \equiv (p \cdot \sim q)\mathbf{S}(\sim p \cdot q))$
A9 $\Box(p \supset q) \supset (p\mathbf{P}q \equiv p\mathbf{P}(\sim p \cdot q))$
A10 $\Box(q \supset p) \supset (p\mathbf{P}q \equiv (p \cdot \sim q)\mathbf{P}q)$
A11 $\sim\Box(p \supset q) \vee \Box(q \supset p) \vee (p\mathbf{S}q \equiv p\mathbf{S}(\sim p \cdot q))$

The consistency of $\mathcal{A}'$ can be proved as follows.[4] Let us assign 1 or 0 to all formulas in $\mathcal{A}'$ in the following way. To truth functions, either 1 or 0

3. Halldén's system $\mathcal{A}$ is constructed by adding the following axioms and the rule of replacement to ordinary propositional logic.

A1 - - - A6 : same as those of $\mathcal{A}'$
A7 $p\mathbf{P}q \equiv (p \cdot \sim q)\mathbf{P}(q \cdot \sim p)$
A8 $p\mathbf{S}q \equiv (p \cdot \sim q)\mathbf{P}(q \cdot \sim p)$

4. The proof of consistency is due to Prof. S. Maehara.

is assigned by the method of the ordinary truth table. The value assignments to $\Box\alpha$, $\alpha\mathbf{P}\beta$ and $\alpha\mathbf{S}\beta$ are respectively as follows.

$$
\begin{array}{ll}
V(\Box\alpha) = 1 & \text{if } V(\alpha) = 1 \\
V(\Box\alpha) = 0 & \text{if } V(\alpha) = 0 \\
V(\alpha\mathbf{P}\beta) = 0 & \text{for all values assigned to } \alpha, \beta \\
V(\alpha\mathbf{S}\beta) = 1 & \text{for all values assigned to } \alpha, \beta
\end{array}
$$

where $V(\alpha)$ stands for a value assigned to α. Then values of the axioms are 1 for any value assignments to all propositional variables. Also a value for a formula derived by the rules of $\mathcal{A}'$ from any formula always having the value 1 is 1 in any case. Therefore a formula whose value is 0 in some cases (e.g., $p\mathbf{P}q$) cannot be derived from $\mathcal{A}'$. In consequence, $\mathcal{A}'$ is consistent.[5]

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5. A preliminary report of this work was published in Japanese in *Philosophy of Science*, Vol. III (1970), (RISOSHA).