A CONTRIBUTION TO THE STUDY OF
EXTENDED MEREOLOGIES

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1 By *extended mereologies* I understand *atomistic mereology* and *atomless mereology*. Either is an extension of *general mereology*, which is a theory of part-whole relations, first established by Leśniewski about sixty years ago.[1] In presenting what follows, I will assume that the reader will be familiar with mereological vocabulary and also with a few elementary theses of general mereology.

Neither the term 'atm', short for 'mereological atom', nor 'at(A)', short for 'mereological atom of A', occurs in general mereology as developed by Leśniewski, but both are definable within the framework of his theory. In [10] Sobociński quotes the following definitions of these two notions:

(1) $[A]:A \, \varepsilon A \, . \, [B] \, . \sim (B \, \varepsilon \, \mathsf{pt}(A)) \, . \equiv . \, A \, \varepsilon \, \mathsf{atm}$
(2) $[A] \therefore A \, \varepsilon A : [B] : B \, \varepsilon \, \mathsf{el}(A) \, . \supset . \, B = A : \equiv . \, A \, \varepsilon \, \mathsf{atm}$
(3) $[AB]:A \, \varepsilon \, \mathsf{atm} \, . \, A \, \varepsilon \, \mathsf{el}(B) \, . \equiv . \, A \, \varepsilon \, \mathsf{at}(B)$

In terms of ordinary language (1) means that A is a mereological atom if and only if A is an object which has no proper parts; (2) says that A is a mereological atom if and only if A is an object identical with whatever is its proper or improper part; and according to (3) A is an atom of B if and only if both A is a mereological atom and a proper or improper part of B. Sobociński tells us that (1) is due to Clay, and he attributes (2) and (3) to V. F. Rickey.

It may be of interest to note that the notions defined with the aid of (1), (2), and (3) have been occasionally discussed by earlier authors. Thus, for instance, in connection with the notion of mereological class Leśniewski considers the notion of *mereological unit class*.[2] He defines the latter as a mereological class which is identical with its only element. A point-moment, that is to say an object which was indivisible as regards its volume and duration, would be described as a mereological unit class. However, Leśniewski feels that he should not prejudge the issue as to whether or not there are any point-moments in this world. In this respect

he prefers his mereology to remain neutral. If we put '$A\,\varepsilon\,$unit-Kl' to stand for 'A is a mereological unit class' then the relevant definition, informally suggested by Leśniewski, could be expressed as follows:

(4) $[A] \therefore A\,\varepsilon A : [a] : A\,\varepsilon\,\mathsf{Kl}(a) . \supset . a = A : \equiv . A\,\varepsilon\,\mathsf{unit\text{-}Kl}$

The Calculus of Individuals outlined by Goodman in [1] is also interpretable in terms of part-whole relations. In connection with the problem of applying the calculus to a given universe of discourse Goodman discusses the notion of *minimal basic unit*.[3] The discussion is informal but if we were to formulate its results in terms of Leśniewski's language, and in application to the universe of discourse consisting of everything that exists, we would achieve our aim by setting up the following definition:

(5) $[A] \therefore A\,\varepsilon A : [B] : B\,\varepsilon\,\mathsf{el}(A) . \supset . A = B : \equiv . A\,\varepsilon\,\mathsf{mbu},$

where '$A\,\varepsilon\,\mathsf{mbu}$' means the same as '$A$ is a minimal basic unit'.

A more systematic version of Goodman's calculus is presented by H. Hiż in [2]. Like Goodman, Hiż makes use of the notion of overlapping as the primitive notion of the calculus. This notion was never studied by Leśniewski, but it can easily be defined within the framework of general mereology. The following equivalence:

(6) $[AB] : A\,\varepsilon A . [\exists C] . C\,\varepsilon\,\mathsf{el}(A) . C\,\varepsilon\,\mathsf{el}(B) . \equiv . A\,\varepsilon\,\mathsf{ov}(B),$

in which '$A\,\varepsilon\,\mathsf{ov}(B)$' means the same as '$A$ overlaps B' or, more precisely, as 'A is an overlapper of B', can serve as the definition.[4] It is in terms of this notion that Hiż explicitly defines the notion of mereological atom. Translated into Leśniewski's language, Hiż's definition takes the form of the following expression:

(7) $[A] \therefore A\,\varepsilon A : [BC] : B\,\varepsilon\,\mathsf{ov}(A) . C\,\varepsilon\,\mathsf{ov}(A) . \supset . B\,\varepsilon\,\mathsf{ov}(C) : \equiv . A\,\varepsilon\,\mathsf{atm}$[5]

It is not difficult to prove, on the basis of general mereology, that the constant terms 'unit-Kl', 'mbu', and 'atm' are synonymous. Interestingly enough Hiż also considers the notion of being an atom of an object. He defines it with the aid of an equivalence which in the language adopted for the purpose of the present paper can be expressed as follows:

(8) $[AB] \therefore A\,\varepsilon\,\mathsf{ov}(B) : [CD] : C\,\varepsilon\,\mathsf{ov}(A) . D\,\varepsilon\,\mathsf{ov}(A) . \supset . C\,\varepsilon\,\mathsf{ov}(D) : \equiv . A\,\varepsilon\,\mathsf{at}(B)$[6]

And among the theses derived by Hiż from the presuppositions of his system we find one which says that

(9) $[AB] : A\,\varepsilon\,\mathsf{at}(B) . \equiv . A\,\varepsilon\,\mathsf{atm} . A\,\varepsilon\,\mathsf{el}(B),$

and thus anticipates Rickey's (3).[7]

However, by adding definitions of 'atm' or 'at' to a system of general mereology we do not achieve an extension of the latter. A system of extended mereology can only result from supplementing axiomatic foundations of general mereology with new axioms. Thus, for instance, a system of atomistic mereology, first proposed by Rickey, is based on an axiom

system of general mereology to which a new axiom has been added to the effect that

(10) $[A] \therefore A \, \varepsilon \, A \, . \supset : [\exists a] : A \, \varepsilon \, \mathsf{Kl}(a) : [BC] : B \, \varepsilon \, a \, . \, C \, \varepsilon \, \mathsf{el}(B) \, . \supset \, . \, C = B$[8]

In the light of (2) the new axiom means that all objects are in fact mereological classes of mereological atoms.

Owing to Rickey's researches we now know that in constructing a system of atomistic mereology we can use 'at' as the only primitive mereological term. And it is this idea that has been taken up by Sobociński who has proved that a system of atomistic mereology based on the following two axioms,

(11) $[AB] \vdots \because A \, \varepsilon \, \mathsf{el}(B) \, . \equiv \because B \, \varepsilon \, B \vdots [Ca] \vdots [D] \vdots D \, \varepsilon \, C \, . \equiv \therefore [E] : E \, \varepsilon \, a \, . \supset .$
      $E \, \varepsilon \, \mathsf{el}(D) \therefore [E] : E \, \varepsilon \, \mathsf{el}(D) \, . \supset . [\exists FG] . F \, \varepsilon \, a \, . \, G \, \varepsilon \, \mathsf{el}(E) \, . \, G \, \varepsilon \, \mathsf{el}(F) \because$
      $B \, \varepsilon \, \mathsf{el}(B) \, . \, B \, \varepsilon \, a \because \supset . A \, \varepsilon \, \mathsf{el}(C)$

(12) $[A] \therefore A \, \varepsilon \, A \, . \supset : [\exists B] : B \, \varepsilon \, \mathsf{el}(A) : [C] : C \, \varepsilon \, \mathsf{el}(B) \, . \supset . \, C = B,$

is inferentially equivalent to a system whose axiomatic foundations consist of the following theses:

SA 1 $[AB] : A \, \varepsilon \, \mathsf{at}(B) \, . \supset . B \, \varepsilon \, B$

SA 2 $[ABC] : A \, \varepsilon \, \mathsf{at}(B) \, . \, C \, \varepsilon \, \mathsf{at}(A) \, . \supset . \, C = A$

SA 3 $[AB] \therefore A \, \varepsilon \, A \, . \, B \, \varepsilon \, B : [C] : C \, \varepsilon \, \mathsf{at}(A) \, . \equiv . \, C \, \varepsilon \, \mathsf{at}(B) : \supset . A = B$

SA 4 $[Aa] \therefore A \, \varepsilon \, a \, . \supset : [\exists B] : [\exists C] . \, C \, \varepsilon \, \mathsf{at}(B) : [C] : C \, \varepsilon \, \mathsf{at}(B) \, . \equiv .$
        $[\exists D] . \, C \, \varepsilon \, \mathsf{at}(D) \, . \, D \, \varepsilon \, a$[9]

It is to be noted that Sobociński's supplementary axiom (12), which in a system of atomistic mereology is inferentially equivalent to Rickey's (10), means that if there are any objects then every one of them has a mereological atom as its proper or improper part.[10]

In the present paper I propose to do two things. In Section 2, I will show that Sobociński's system of atomistic mereology, to be referred to as System $\mathfrak{S}_1$, is inferentially equivalent to System $\mathfrak{S}_2$, which is based on the following single axiom:

AA 1 $[AB] \vdots A \, \varepsilon \, \mathsf{at}(B) \, . \equiv \because B \, \varepsilon \, B \because [CDa] : : [E] \therefore E \, \varepsilon \, C \, . \equiv : [F] : F \, \varepsilon \, \mathsf{at}(E) \, .$
        $\equiv . \, [\exists G] . \, F \, \varepsilon \, \mathsf{at}(G) \, . \, G \, \varepsilon \, a :: D \, \varepsilon \, \mathsf{at}(B) \, . \, B \, \varepsilon \, a :: \supset . \, \mathsf{at}(A) \, \varepsilon \, A \, . \, A \, \varepsilon \, \mathsf{at}(C)$

In Section 3, I will consider a system of atomless mereology as determined by the following two axioms:

BA 1 $[AB] \vdots A \, \varepsilon \, \mathsf{pt}(B) \, . \equiv \because B \, \varepsilon \, B \, . \sim (B \, \varepsilon \, \mathsf{pt}(A)) \vdots [af] \vdots [Cb] \because$
        $C \, \varepsilon \, f(b) \, . \equiv :: [\exists D] . \, D \, \varepsilon \, b \therefore [D] \therefore D \, \varepsilon \, b \, . \supset : C \, \varepsilon \, D \, . \, \mathsf{v} \, . \, D \, \varepsilon \, \mathsf{pt}(C) ::$
        $[D] \therefore D \, \varepsilon \, \mathsf{pt}(C) \, . \supset : [\exists E] : E \, \varepsilon \, D \, . \, \mathsf{v} \, . \, E \, \varepsilon \, \mathsf{pt}(D) : E \, \varepsilon \, b \, . \, \mathsf{v} \, . \, [\exists F] . \, F \, \varepsilon \, b \, .$
        $E \, \varepsilon \, \mathsf{pt}(F) \vdots B \, \varepsilon \, a \vdots \supset : [\exists c] : A \, \varepsilon \, c : A \, \varepsilon \, \mathsf{pt}(f(a)) \, . \, \mathsf{v} \, . \sim (f(c) \, \varepsilon \, f(c))$

BA 2 $[A] : A \, \varepsilon \, A \, . \supset . [\exists B] . \, B \, \varepsilon \, \mathsf{pt}(A)$

In particular I will prove that this system, to which I will refer as System $\mathfrak{S}_3$, is inferentially equivalent to System $\mathfrak{S}_4$, whose single axiom has the form of the following thesis:

$CA1$ $[AB] \ddots A \, \varepsilon \, \mathsf{pt}(B) \, . \equiv \, \vdots \, B \, \varepsilon \, B \, . \sim (B \, \varepsilon \, \mathsf{pt}(A)) \, \vdots \, [CDa] \, \vdots \, [E] \, . \vdots \, E \, \varepsilon \, C \, . \equiv \, :$
        $[F] \therefore F \, \varepsilon \, a \, . \supset : E \, \varepsilon \, F \, . \vee . F \, \varepsilon \, \mathsf{pt}(E) :: [F] : F \, \varepsilon \, \mathsf{pt}(E) \, . \supset . [\exists GH] \, . G \, \varepsilon \, a \, .$
        $H \, \varepsilon \, \mathsf{pt}(F) \, . H \, \varepsilon \, \mathsf{pt}(G) \, \vdots \, D \, \varepsilon \, \mathsf{pt}(B) \, . B \, \varepsilon \, a \, \ddots \supset . A \, \varepsilon \, \mathsf{pt}(C)$

BA1 is a single axiom of general mereology, with '**pt**' as the primitive
term, and *BA2*, which extends general mereology into atomless mereology,
says that if there are any objects then every one of them has a proper
part.[11]

      Incidentally, while the term 'atomistic mereology' brings to one's mind
the name of Democritus although his atoms are quite different from
mereological atoms, atomless mereology can perhaps be associated with
Anaxagoras, who appears to be the first to have claimed that there were no
smallest objects and that divisibility could go on *ad infinitum*.

2 The proof that $\mathfrak{S}_1$ is inferentially equivalent to $\mathfrak{S}_2$, involves the deduction
of the following theses within the framework of $\mathfrak{S}_1$:

$ST1$	$[Aa] : A \, \varepsilon \, a \, . \supset . [\exists B] \, . B \, \varepsilon \, \mathsf{at}(A)$	
PR	$[Aa] : \mathsf{Hp}(1) \, . \supset .$	
(2)	$A \, \varepsilon \, A \, .$	[1]
	$[\exists BC] \, .$	
(3)	$B \, \varepsilon \, \mathsf{at}(C) \, .$	$[SA4, 2]$
(4)	$C \, \varepsilon \, A \, .$	
(5)	$A = C \, .$	$[2, 4]$
	$[\exists B] \, . B \, \varepsilon \, \mathsf{at}(A)$	$[3, 5]$
$ST2$	$[ABC] : A \, \varepsilon \, \mathsf{at}(B) \, . C \, \varepsilon \, \mathsf{at}(A) \, . D \, \varepsilon \, \mathsf{at}(A) \, . \supset . C \, \varepsilon \, D$	
PR	$[ABC] : \mathsf{Hp}(3) \, . \supset .$	
(4)	$C = A \, .$	$[SA2, 1, 2]$
(5)	$D = A \, .$	$[SA2, 1, 3]$
	$C \, \varepsilon \, D$	$[4, 5]$
$ST3$	$[AB] : A \, \varepsilon \, \mathsf{at}(B) \, . \supset . \mathsf{at}(A) \, \varepsilon \, A$	
PR	$[AB] : \therefore \mathsf{Hp}(1) \, . \supset ::$	
	$[\exists C] \therefore$	
(2)	$C \, \varepsilon \, \mathsf{at}(A) :$	$[ST1, 1]$
(3)	$[DE] : D \, \varepsilon \, \mathsf{at}(A) \, . E \, \varepsilon \, \mathsf{at}(A) \, . \supset . D \, \varepsilon \, E \therefore$	$[ST2, 1]$
(4)	$\mathsf{at}(A) \, \varepsilon \, C \, .$	$[2, 3]$
(5)	$C = A ::$	$[SA2, 1, 2]$
	$\mathsf{at}(A) \, \varepsilon \, A$	$[4, 5]$
$ST4$	$[ABa] :: A \, \varepsilon \, a : [F] : F \, \varepsilon \, \mathsf{at}(A) \, . \equiv . [\exists G] \, . F \, \varepsilon \, \mathsf{at}(G) \, . G \, \varepsilon \, a \therefore$	
	$[F] : F \, \varepsilon \, \mathsf{at}(B) \, . \equiv . [\exists G] \, . F \, \varepsilon \, \mathsf{at}(G) \, . G \, \varepsilon \, a \therefore \supset . A = B$	$[ST1, SA1, SA3]$
$ST5$	$[ABCa] : \therefore A \, \varepsilon \, \mathsf{at}(B) \therefore [E] \therefore E \, \varepsilon \, C \, . \equiv : [F] : F \, \varepsilon \, \mathsf{at}(E) \, . \equiv . [\exists G] \, . F \, \varepsilon \, \mathsf{at}(G) \, .$	
	$G \, \varepsilon \, a :: B \, \varepsilon \, a :: \supset . A \, \varepsilon \, \mathsf{at}(C)$	
PR	$[ABCa] \vdots \mathsf{Hp}(3) :: \supset : \therefore$	
	$[\exists D] : \therefore$	
(4)	$[F] : F \, \varepsilon \, \mathsf{at}(D) \, . \equiv . [\exists G] \, . F \, \varepsilon \, \mathsf{at}(G) \, . G \, \varepsilon \, a \therefore$	$[SA4, 3]$
(5)	$A \, \varepsilon \, \mathsf{at}(D) \, .$	$[4, 1, 3]$
(6)	$D \, \varepsilon \, D \therefore$	$[SA1, 5]$
(7)	$[E] \therefore E \, \varepsilon \, D \, . \supset : [F] : F \, \varepsilon \, \mathsf{at}(E) \, . \equiv . [\exists G] \, . F \, \varepsilon \, \mathsf{at}(G) \, . G \, \varepsilon \, a ::$	$[6, 4]$
(8)	$[E] \therefore [F] : F \, \varepsilon \, \mathsf{at}(E) \, . \equiv . [\exists G] \, . F \, \varepsilon \, \mathsf{at}(G) \, . G \, \varepsilon \, a : \supset . E \, \varepsilon \, D ::$	
		$[ST4, 6, 4]$

(9) $\quad [E] \therefore E \varepsilon D . \equiv : [F] : F \varepsilon \text{at}(E) . \equiv . [\exists G] . F \varepsilon \text{at}(G) . G \varepsilon a$ $\qquad$ [7, 8]

(10) $\quad [E] : E \varepsilon C . \equiv . E \varepsilon D \therefore$ $\qquad$ [2, 9]

$\quad A \varepsilon \text{at}(C)$ $\qquad$ [Extensionality, 10, 5]

$ST6 \quad [ABCDa] : \cdots : A \varepsilon \text{at}(B) \therefore [E] \therefore E \varepsilon C . \equiv : [F] : F \varepsilon \text{at}(E) . \equiv .$

$\qquad [\exists G] . F \varepsilon \text{at}(G) . G \varepsilon a :: B \varepsilon a :: \supset . \text{at}(A) \varepsilon A . A \varepsilon \text{at}(C)$ $\qquad$ [ST3, ST5]

$ST7 \quad [AB] \vdots B \varepsilon B : \cdots : [CDa] : \cdots : [E] \therefore E \varepsilon C . \equiv : [F] : F \varepsilon \text{at}(E) . \equiv . [\exists G] .$

$\qquad F \varepsilon \text{at}(G) . G \varepsilon a :: D \varepsilon \text{at}(B) . B \varepsilon a :: \supset . A \varepsilon \text{at}(C) : \cdots : \supset . A \varepsilon \text{at}(B)$

$PR \quad [AB] \vdots \text{Hp}(2) : \cdots : \supset ::$

(3) $\quad [F] : F \varepsilon \text{at}(B) . \equiv . [\exists G] . F \varepsilon \text{at}(G) . G \varepsilon B \therefore$ $\qquad$ [1]

(4) $\quad [E] \therefore E \varepsilon B . \supset : [F] : F \varepsilon \text{at}(E) . \equiv : [\exists G] . F \varepsilon \text{at}(G) . G \varepsilon B ::$ $\qquad$ [1, 3]

(5) $\quad [E] \therefore [F] : F \varepsilon \text{at}(E) . \equiv . [\exists G] . F \varepsilon \text{at}(G) . G \varepsilon B : \supset . E \varepsilon B ::$ $\qquad$ [ST4, 1, 3]

(6) $\quad [E] \therefore E \varepsilon B . \equiv : [F] : F \varepsilon \text{at}(E) . \equiv . [\exists G] . F \varepsilon \text{at}(G) . G \varepsilon B ::$ $\qquad$ [4, 5]

(7) $\quad [\exists D] . D \varepsilon \text{at}(B) :$ $\qquad$ [ST1, 1]

$\quad A \varepsilon \text{at}(B)$ $\qquad$ [2, 6, 7, 1]

$ST8 (= AA1) \quad [AB] \vdots A \varepsilon \text{at}(B) . \equiv : \cdots : B \varepsilon B : \cdots : [CDa] : \cdots : [E] \therefore E \varepsilon C . \equiv :$

$\qquad [F] : F \varepsilon \text{at}(E) . \equiv . [\exists G] . F \varepsilon \text{at}(G) . G \varepsilon a ::$

$\qquad D \varepsilon \text{at}(B) . B \varepsilon a :: \supset . \text{at}(A) \varepsilon A . A \varepsilon \text{at}(C)$ $\qquad$ [SA1, ST6, ST7]

By deducing $ST8$ from $SA1$-$SA4$ we have shown that any thesis obtainable in $\mathfrak{S}_2$ is also obtainable in $\mathfrak{S}_1$. Now the converse can be established by the following deductions within the framework of $\mathfrak{S}_2$:

$AD1 \quad [Aa] \therefore A \varepsilon A : [B] : B \varepsilon \text{at}(A) . \equiv . [\exists C] . B \varepsilon \text{at}(C) . C \varepsilon a : \equiv . A \varepsilon \text{Kl}(a)$

$\qquad$ [Definition]

$AT1 (= SA1) \quad [AB] : A \varepsilon \text{at}(B) . \supset . B \varepsilon B .$ $\qquad$ [AA1]

$AT2 \quad [Aa] : A \varepsilon a . \supset . [\exists B] . B \varepsilon \text{at}(A)$

$PR \quad [Aa] \therefore \text{Hp}(1) . \supset :$

(2) $\quad A \varepsilon A :$ $\qquad$ [1]

(3) $\quad A \varepsilon \text{at}(A) . \text{v} . [\exists D] . D \varepsilon \text{at}(A) :$ $\qquad$ [AA1, 2]

$\quad [\exists B] . B \varepsilon \text{at}(A)$ $\qquad$ [3]

$AT3 \quad [ADa] \therefore D \varepsilon a : [B] : B \varepsilon \text{at}(A) . \equiv . [\exists C] . B \varepsilon \text{at}(C) . C \varepsilon a : \supset . A \varepsilon \text{Kl}(a)$

$PR \quad [ADa] \therefore \text{Hp}(2) : \supset :$

$\quad [\exists E] .$

(3) $\quad E \varepsilon \text{at}(D) .$ $\qquad$ [AT2, 1]

(4) $\quad E \varepsilon \text{at}(A) :$ $\qquad$ [2, 3, 1]

(5) $\quad A \varepsilon A .$ $\qquad$ [AT1, 4]

$\quad A \varepsilon \text{Kl}(a)$ $\qquad$ [AD1, 5, 2]

$AT4 \quad [ADa] :: D \varepsilon a . \supset \therefore A \varepsilon \text{Kl}(a) . \equiv : [B] : B \varepsilon \text{at}(A) . \equiv . [\exists C] . B \varepsilon \text{at}(C) . C \varepsilon a$

$\qquad$ [AD1, AT3]

$AT5 \quad [ABa] : A \varepsilon \text{at}(B) . B \varepsilon a . \supset . \text{at}(A) \varepsilon A . A \varepsilon \text{at}(\text{Kl}(a))$

$PR \quad [ABa] : \cdots : \text{Hp}(2) . \supset ::$

(3) $\quad [C] \therefore C \varepsilon \text{Kl}(a) . \equiv : [F] : F \varepsilon \text{at}(C) . \equiv . [\exists G] . F \varepsilon \text{at}(G) . G \varepsilon a ::$ $\qquad$ [AT4, 2]

$\quad \text{at}(A) \varepsilon A . A \varepsilon \text{at}(\text{Kl}(a))$ $\qquad$ [AA1, 1, 3, 2]

$AT6 (= SA2) \quad [ABC] : A \varepsilon \text{at}(B) . C \varepsilon \text{at}(A) . \supset . C = A$

$PR \quad [ABC] : \text{Hp}(2) . \supset .$

(3) $\quad B \varepsilon B .$ $\qquad$ [AT1, 1]

(4) $\quad \text{at}(A) \varepsilon A .$ $\qquad$ [AT5, 1, 3]

(5) $\quad C \varepsilon A .$ $\qquad$ [2, 4]

(6) $A \, \varepsilon \, \mathsf{at}(A)$.   [4, 1]

(7) $A \, \varepsilon \, C$.   [5, 6]

       $C = A$ [5, 7]

$AT7$ $[AB] : B \, \varepsilon \, \mathsf{at}(A) . \supset . [\exists C] . B \, \varepsilon \, \mathsf{at}(C) . C \, \varepsilon \, \mathsf{at}(A)$

PR $[AB] \therefore \mathrm{Hp}(1) . \supset :$

      $[\exists C] .$

(2) $C \, \varepsilon \, \mathsf{at}(B)$. $[AT2, 1]$

(3) $C = B :$ $[AT6, 1, 2]$

(4) $B \, \varepsilon \, \mathsf{at}(B)$. $[2, 3]$

      $[\exists C] . B \, \varepsilon \, \mathsf{at}(C) . C \, \varepsilon \, \mathsf{at}(A)$ $[4, 1]$

$AT8$ $[ABC] : B \, \varepsilon \, \mathsf{at}(C) . C \, \varepsilon \, \mathsf{at}(A) . \supset . B \, \varepsilon \, \mathsf{at}(A)$

PR $[ABC] : \mathrm{Hp}(2) . \supset .$

(3) $B = C$. $[AT6, 2, 1]$

      $B \, \varepsilon \, \mathsf{at}(A)$ $[2, 3]$

$AT9$ $[AB] : B \, \varepsilon \, \mathsf{at}(A) . \equiv . [\exists C] . B \, \varepsilon \, \mathsf{at}(C) . C \, \varepsilon \, \mathsf{at}(A)$ $[AT7, AT8]$

$AT10$ $[Aa] : A \, \varepsilon \, a . \supset . A \, \varepsilon \, \mathsf{Kl}(\mathsf{at}(A))$ $[AD1, AT9]$

$AT11$ $[Aa] : A \, \varepsilon \, a . \supset . \mathsf{Kl}(a) \, \varepsilon \, \mathsf{Kl}(a)$

PR $[Aa] \therefore \mathrm{Hp}(1) . \supset :$

      $[\exists B] .$

(2) $B \, \varepsilon \, \mathsf{at}(A)$. $[AT2, 1]$

(3) $B \, \varepsilon \, \mathsf{at}(\mathsf{Kl}(a)) :$ $[AT5, 2, 1]$

      $\mathsf{Kl}(a) \, \varepsilon \, \mathsf{Kl}(a)$ $[AT1, 3]$

$AT12 \ (= SA3)$ $[AB] \therefore A \, \varepsilon \, A . B \, \varepsilon \, B : [C] : C \, \varepsilon \, \mathsf{at}(A) . \equiv . C \, \varepsilon \, \mathsf{at}(B) : \supset . A = B$

PR $[AB] :: \mathrm{Hp}(3) : \supset \therefore$

(4) $[\Phi] : \Phi(\mathsf{at}(A)) . \equiv . \Phi(\mathsf{at}(B)) \therefore$ [Extensionality, 3]

(5) $A \, \varepsilon \, \mathsf{Kl}(\mathsf{at}(A))$. $[AT10, 1]$

(6) $A \, \varepsilon \, \mathsf{Kl}(\mathsf{at}(B))$. $[4, 5]$

(7) $B \, \varepsilon \, \mathsf{Kl}(\mathsf{at}(B))$. $[AT10, 2]$

(8) $[\exists C] . C \, \varepsilon \, \mathsf{at}(B) :$ $[AT2, 2]$

(9) $\mathsf{Kl}(\mathsf{at}(B)) \, \varepsilon \, \mathsf{Kl}(\mathsf{at}(B))$. $[AT11, 8]$

      $A = B$ $[9, 6, 7]$

$AT13 \ (= SA4)$ $[Aa] \therefore A \, \varepsilon \, a . \supset : [\exists B] : [\exists C] . C \, \varepsilon \, \mathsf{at}(B) :$

      $[C] : C \, \varepsilon \, \mathsf{at}(B) . \equiv . [\exists D] . C \, \varepsilon \, \mathsf{at}(D) . D . D \, \varepsilon \, a$

PR $[Aa] :: \mathrm{Hp}(1) . \supset \therefore$

(2) $\mathsf{Kl}(a) \, \varepsilon \, \mathsf{Kl}(a)$. $[AT11, 1]$

(3) $[\exists C] . C \, \varepsilon \, \mathsf{at}(\mathsf{Kl}(a)) :$ $[AT2, 2]$

(4) $[C] : C \, \varepsilon \, \mathsf{at}(\mathsf{Kl}(a)) . \equiv . [\exists D] . C \, \varepsilon \, \mathsf{at}(D) . D \, \varepsilon \, a \therefore$ $[AD1, 2]$

      $[\exists B] \therefore [\exists C] . C \, \varepsilon \, \mathsf{at}(B) \therefore [C] : C \, \varepsilon \, \mathsf{at}(B) . \equiv . [\exists D] . C \, \varepsilon \, \mathsf{at}(D) . D \, \varepsilon \, a$ $[3, 4]$

     By deducing $AT1$, $AT6$, $AT12$, and $AT13$ from $AA1$ and $AD1$ we have proved that any thesis obtainable in $\mathfrak{S}_1$ is also obtainable in $\mathfrak{S}_2$, which completes the proof that $\mathfrak{S}_1$ and $\mathfrak{S}_2$ are inferentially equivalent.

3   The proof that $\mathfrak{S}_3$ is inferentially equivalent to $\mathfrak{S}_4$ involves the deduction of the following theses within the framework of $\mathfrak{S}_3$:

$BD1$ $[Aa] ::: A \, \varepsilon \, A :: [B] \therefore B \, \varepsilon \, a . \supset : A \, \varepsilon \, B . \vee . B \, \varepsilon \, \mathsf{pt}(A) :: [B] \therefore B \, \varepsilon \, \mathsf{pt}(A) . \supset :$

      $[\exists C] : C \, \varepsilon \, B . \vee . C \, \varepsilon \, \mathsf{pt}(B) : C \, \varepsilon \, a . \vee . [\exists D] . D \, \varepsilon \, a . C \, \varepsilon \, \mathsf{pt}(D) :: \equiv . A \, \varepsilon \, \mathsf{Kl}(a)$

                                                               [Definition]

$BT1$ $[AB]: A \, \varepsilon \, \mathsf{pt}(B) . \supset . B \, \varepsilon \, B$ $[BA1]$

$BT2$ $[AB]: A \, \varepsilon \, \mathsf{pt}(B) . \supset . \sim (B \, \varepsilon \, \mathsf{pt}(A))$ $[BA1]$

$BT3$ $[A] . \sim (A \, \varepsilon \, \mathsf{pt}(A))$ $[BT2]$

$BT4$ $[Aa]: A \, \varepsilon \, \mathsf{Kl}(a) . \supset . [\exists B] . B \, \varepsilon \, a$

PR $[Aa] \therefore \mathrm{Hp}(1) . \supset :$

(2) $[\exists B] . B \, \varepsilon \, \mathsf{pt}(A) :$ $[BA2, 1]$

     $[\exists B] . B \, \varepsilon \, a$ $[BD1, 1, 2]$

$BT5$ $[AaE] \therefore E \, \varepsilon \, a :: [B] \therefore B \, \varepsilon \, a . \supset : A \, \varepsilon \, B . \mathsf{v} . B \, \varepsilon \, \mathsf{pt}(A) :: [B] \therefore B \, \varepsilon \, \mathsf{pt}(A) . \supset :$
       $[\exists C] : C \, \varepsilon \, B . \mathsf{v} . C \, \varepsilon \, \mathsf{pt}(B) : C \, \varepsilon \, a . \mathsf{v} . [\exists D] . D \, \varepsilon \, a . C \, \varepsilon \, \mathsf{pt}(D) :: \supset . A \, \varepsilon \, \mathsf{Kl}(a)$

PR $[AaE] \therefore \mathrm{Hp}(3) :: \supset :$

(4) $A \, \varepsilon \, E . \mathsf{v} . E \, \varepsilon \, \mathsf{pt}(A) :$ $[2, 1]$

(5) $A \, \varepsilon \, A .$ $[4, BT1]$

     $A \, \varepsilon \, \mathsf{Kl}(a)$ $[BD1, 5, 2, 3]$

$BT6$ $[Aa] A \, \varepsilon \, \mathsf{Kl}(a) . \equiv :: [\exists B] . B \, \varepsilon \, a :: [B] \therefore B \, \varepsilon \, a . \supset : A \, \varepsilon \, B . \mathsf{v} . B \, \varepsilon \, \mathsf{pt}(A) ::$
       $[B] \therefore B \, \varepsilon \, \mathsf{pt}(A) . \supset : [\exists C] : C \, \varepsilon \, B . \mathsf{v} . C \, \varepsilon \, \mathsf{pt}(B) : C \, \varepsilon \, a . \mathsf{v} . [\exists D] .$
       $D \, \varepsilon \, a . C \, \varepsilon \, \mathsf{pt}(D)$ $[BT4, BD1, BT5]$

$BT7$ $[Aa]: A \, \varepsilon \, a . \supset . \mathsf{Kl}(a) \, \varepsilon \, \mathsf{Kl}(a)$

PR $[Aa] \vdots \mathrm{Hp}(1) . \supset \vdots$
     $[\exists f] \vdots$

(2) $[Cb] \therefore C \, \varepsilon \, f(b) . \equiv :: [\exists D] . D \, \varepsilon \, b :: [D] \therefore D \, \varepsilon \, b . \supset : C \, \varepsilon \, D . \mathsf{v} .$
         $D \, \varepsilon \, \mathsf{pt}(C) :: [D] \therefore D \, \varepsilon \, \mathsf{pt}(C) . \supset : [\exists E] : E \, \varepsilon \, D . \mathsf{v} . E \, \varepsilon \, \mathsf{pt}(D) : E \, \varepsilon \, b . \mathsf{v} .$
         $[\exists F] . F \, \varepsilon \, b . E \, \varepsilon \, \mathsf{pt}(E) \vdots$

(3) $[c]: A \, \varepsilon \, c . \supset . f(c) \, \varepsilon \, f(c) \therefore$ $[BA1, 1, BT3]$

(4) $f(a) \, \varepsilon \, f(a) \therefore$ $[3, 1]$

(5) $[C]: C \, \varepsilon \, f(a) . \equiv . C \, \varepsilon \, \mathsf{Kl}(a) \therefore$ $[2, BT6]$

     $\mathsf{Kl}(a) \, \varepsilon \, \mathsf{Kl}(a)$ $[\text{Extensionality}, 5, 4]$

$BT8$ $[ABa]: A \, \varepsilon \, \mathsf{pt}(B) . B \, \varepsilon \, a . \supset . A \, \varepsilon \, \mathsf{pt}(\mathsf{Kl}(a))$

PR $[ABa] :: \mathrm{Hp}(2) . \supset \therefore$
     $[\exists c] :$

(3) $A \, \varepsilon \, c :$ $[BA1, 1, BT6, 2]$

(4) $A \, \varepsilon \, \mathsf{pt}(\mathsf{Kl}(a)) . \mathsf{v} . \sim (\mathsf{Kl}(c) \, \varepsilon \, \mathsf{Kl}(c)) :$

(5) $\mathsf{Kl}(c) \, \varepsilon \, \mathsf{Kl}(c) \therefore$ $[BT7, 3]$

     $A \, \varepsilon \, \mathsf{pt}(\mathsf{Kl}(a))$ $[4, 5]$

$BT9$ $[ABC]: A \, \varepsilon \, \mathsf{pt}(B) . B \, \varepsilon \, \mathsf{pt}(C) . \supset . A \, \varepsilon \, \mathsf{pt}(C)$

PR $[ABC]: \mathrm{Hp}(2) . \supset .$

(3) $C \, \varepsilon \, \mathsf{Kl}(\mathsf{pt}(C)) .$ $[BT5, 2]$

(4) $\mathsf{Kl}(\mathsf{pt}(C)) \, \varepsilon \, \mathsf{Kl}(\mathsf{pt}(C)) .$ $[BT7, 2]$

(5) $C = \mathsf{Kl}(\mathsf{pt}(C)) .$ $[3, 4]$

(6) $A \, \varepsilon \, \mathsf{pt}(\mathsf{Kl}(\mathsf{pt}(C))) .$ $[BT8, 1, 2]$

   $A \, \varepsilon \, \mathsf{pt}(C)$ $[\text{Extensionality}, 5, 6]$

$BT10$ $[AaB] \therefore [G] \therefore G \, \varepsilon \, \mathsf{pt}(A) . \supset : [\exists E] : E \, \varepsilon \, G . \mathsf{v} . E \, \varepsilon \, \mathsf{pt}(G) : E \, \varepsilon \, a . \mathsf{v} .$
       $[\exists F] . F \, \varepsilon \, a . E \, \varepsilon \, \mathsf{pt}(F) :: B \, \varepsilon \, \mathsf{pt}(A) :: \supset . [\exists CD] : C \, \varepsilon \, a . D \, \varepsilon \, \mathsf{pt}(B) . D \, \varepsilon \, \mathsf{pt}(C)$

PR $[AaB] \therefore \mathrm{Hp}(2) :: \supset ::$
     $[\exists E] \therefore$

(3) $E \, \varepsilon \, B . \mathsf{v} . E \, \varepsilon \, \mathsf{pt}(B) :$

(4) $E \, \varepsilon \, a . \mathsf{v} . [\exists F] . F \, \varepsilon \, a . E \, \varepsilon \, \mathsf{pt}(F) \therefore$ $[1, 2]$

     $[\exists D] :$

(5) $D \, \varepsilon \, \text{pt}(E)$. $[3, BA2]$

(6) $D \, \varepsilon \, \text{pt}(B)$: $[3, \text{Extensionality, } 2, 5, BT9]$

       $[\exists C]$.

(7) $C \, \varepsilon \, a$. $\Big\}$

(8) $D \, \varepsilon \, \text{pt}(C)$ :: $\Big\}$ $[4, 5, BT9]$

    $[\exists CD]$. $C \, \varepsilon \, a$. $D \, \varepsilon \, \text{pt}(B)$. $D \, \varepsilon \, \text{pt}(C)$ $[7, 6, 8]$

$BT11$ $[AaB] : A \, \varepsilon \, \text{Kl}(a)$. $B \, \varepsilon \, \text{pt}(A)$. $\supset$. $[\exists CD]$. $C \, \varepsilon \, a$. $D \, \varepsilon \, \text{pt}(B)$. $D \, \varepsilon \, \text{pt}(C)$

PR $[AaB] :\cdot: \text{Hp}(2)$. $\supset ::$

(3) $[G] \therefore G \, \varepsilon \, \text{pt}(A)$. $\supset : [\exists E] : E \, \varepsilon \, G$. $\vee$. $E \, \varepsilon \, \text{pt}(G)$:

   $E \, \varepsilon \, a$. $\vee$. $[\exists F]$. $F \, \varepsilon \, a$. $E \, \varepsilon \, \text{pt}(F)$ :: $[BD1, 1]$

   $[\exists CD]$. $C \, \varepsilon \, a$. $D \, \varepsilon \, \text{pt}(B)$. $D \, \varepsilon \, \text{pt}(C)$ $[BT10, 3, 2]$

$BT12$ $[AaB] :: [G] : G \, \varepsilon \, \overline{\text{pt}}(A)$. $\supset$. $[\exists EF]$. $E \, \varepsilon \, a$. $F \, \varepsilon \, \text{pt}(G)$. $F \, \varepsilon \, \text{pt}(E)$ $\therefore$

  $B \, \varepsilon \, \text{pt}(A) \therefore \supset : [\exists C] : C \, \varepsilon \, B$. $\vee$. $C \, \varepsilon \, \text{pt}(B) : C \, \varepsilon \, a$. $\vee$. $[\exists D]$. $D \, \varepsilon \, a$. $C \, \varepsilon \, \text{pt}(D)$

PR $[AaB] :: \text{Hp}(2) \therefore \supset :$

  $[\exists EF]$.

(3) $E \, \varepsilon \, a$.

(4) $F \, \varepsilon \, \text{pt}(B)$.

(5) $F \, \varepsilon \, \text{pt}(E)$:

(6) $F \, \varepsilon \, B$. $\vee$. $F \, \varepsilon \, \text{pt}(B)$: $[4]$

(7) $F \, \varepsilon \, a$. $\vee$. $[\exists D]$. $D \, \varepsilon \, a$. $F \, \varepsilon \, \text{pt}(D)$: $[3, 5]$

  $[\exists C] : C \, \varepsilon \, B$. $\vee$. $C \, \varepsilon \, \text{pt}(B) : C \, \varepsilon \, a$. $\vee$. $[\exists D]$. $D \, \varepsilon \, a$. $C \, \varepsilon \, \text{pt}(D)$ $[6, 7]$

$BT13$ $[AaE] :\cdot: E \, \varepsilon \, a :: [B] \therefore B \, \varepsilon \, a$. $\supset : A \, \varepsilon \, B$. $\vee$. $B \, \varepsilon \, \text{pt}(A) :: [B] : B \, \varepsilon \, \text{pt}(A)$. $\supset$.

  $[\exists CD]$. $C \, \varepsilon \, a$. $D \, \varepsilon \, \text{pt}(B)$. $D \, \varepsilon \, \text{pt}(C) :: \supset$. $A \, \varepsilon \, \text{Kl}(a)$

PR $[AaE] :\cdot: \text{Hp}(3) :: \supset \therefore$

(4) $A \, \varepsilon \, A \therefore$ $[BT1, 2, 1]$

(5) $[B] : B \, \varepsilon \, \text{pt}(A)$. $\supset : [\exists C] : C \, \varepsilon \, B$. $\vee$. $C \, \varepsilon \, \text{pt}(B)$:

  $C \, \varepsilon \, a$. $\vee$. $[\exists D]$. $D \, \varepsilon \, a$. $C \, \varepsilon \, \text{pt}(D) \therefore$ $[BT12, 3]$

  $A \, \varepsilon \, \text{Kl}(a)$ $[BD1, 4, 2, 5]$

$BT14$ $[Aa] :\cdot: A \, \varepsilon \, \text{Kl}(a)$. $\equiv :: [\exists B]$. $B \, \varepsilon \, a :: [B] \therefore B \, \varepsilon \, a$. $\supset : A \, \varepsilon \, B$. $\vee$. $B \, \varepsilon \, \text{pt}(A) ::$

  $[B] : B \, \varepsilon \, \text{pt}(A)$. $\supset$. $[\exists CD]$. $C \, \varepsilon \, a$. $D \, \varepsilon \, \text{pt}(B)$. $D \, \varepsilon \, \text{pt}(C)$

            $[BT4, BD1, BT11, BT13]$

$BT15$ $[ABCa] \overset{.\,.}{\cdot} A \, \varepsilon \, \text{pt}(B) \mathbin{\vdots} [E] :\cdot: E \, \varepsilon \, C$. $\equiv :: [F] \therefore F \, \varepsilon \, a$. $\supset : E \, \varepsilon \, F$. $\vee$. $F \, \varepsilon \, \text{pt}(E) ::$

  $[F] : F \, \varepsilon \, \text{pt}(E)$. $\supset : [\exists GH]$. $G \, \varepsilon \, a$. $H \, \varepsilon \, \text{pt}(F)$. $H \, \varepsilon \, \text{pt}(G) \mathbin{\vdots} B \, \varepsilon \, a \mathbin{\vdots} \supset$. $A \, \varepsilon \, \text{pt}(C)$

PR $[ABCa] \overset{.\,.}{\cdot} \text{Hp}(3) \mathbin{\vdots} \supset \therefore$

(4) $A \, \varepsilon \, \text{pt}(\text{Kl}(a)) \therefore$ $[BT8, 1, 3]$

(5) $[E] : E \, \varepsilon \, C$. $\equiv$. $E \, \varepsilon \, \text{Kl}(a) \therefore$ $[2, BT14, 3]$

  $A \, \varepsilon \, \text{pt}(C)$ $[\text{Extensionality}, 5, 4]$

$BT16$ $[BC] : C \, \varepsilon \, \text{pt}(B)$. $\supset$. $[\exists DE]$. $D \, \varepsilon \, \text{pt}(B)$. $E \, \varepsilon \, \text{pt}(C)$. $E \, \varepsilon \, \text{pt}(D)$

PR $[BC] : \text{Hp}(1)$. $\supset$.

  $[\exists D]$.

(2) $D \, \varepsilon \, \text{pt}(C)$. $[BA2, 1]$

  $[\exists DE]$. $D \, \varepsilon \, \text{pt}(B)$. $E \, \varepsilon \, \text{pt}(C)$. $E \, \varepsilon \, \text{pt}(D)$ $[1, 2]$

$BT17$ $[AB] \mathbin{\vdots} B \, \varepsilon \, B \overset{.\,.}{\cdot} [CDa] \mathbin{\vdots} [E] :\cdot: E \, \varepsilon \, C$. $\equiv :: [F] \therefore F \, \varepsilon \, a$. $\supset : E \, \varepsilon \, F$. $\vee$.

  $F \, \varepsilon \, \text{pt}(E) \therefore [F] : F \, \varepsilon \, \text{pt}(E)$. $\supset$. $[\exists GH]$. $G \, \varepsilon \, a$. $H \, \varepsilon \, \text{pt}(F)$.

  $H \, \varepsilon \, \text{pt}(G) \mathbin{\vdots} D \, \varepsilon \, \text{pt}(B)$. $B \, \varepsilon \, a \mathbin{\vdots} \supset$. $A \, \varepsilon \, \text{pt}(C) \overset{.\,.}{\cdot} \supset$. $A \, \varepsilon \, \text{pt}(B)$

PR $\quad [AB] \stackrel{..}{..} \text{Hp(2)} \stackrel{.}{..} \supset :$

(3) $\quad [\exists D] . D \varepsilon \, \text{pt}(B) :$ $\hfill [BA2, 1]$

(4) $\quad A \varepsilon \, \text{pt}(\mathsf{Kl}(B))$ $\hfill [2, BT14, 3, 1]$

(5) $\quad B \varepsilon \, \mathsf{Kl}(B) .$ $\hfill [BT14, 1, BT16]$

(6) $\quad \mathsf{Kl}(B) \, \varepsilon \, \mathsf{Kl}(B) .$ $\hfill [BT7, 1]$

(7) $\quad B = \mathsf{Kl}(B) .$ $\hfill [5, 6]$

$\quad A \varepsilon \, \text{pt}(B)$ $\hfill [\text{Extensionality}, 7, 4]$

$BT18 \; (= CA1) \quad [AB] \stackrel{..}{..} A \varepsilon \, \text{pt}(B) . \equiv \stackrel{.}{..} B \varepsilon B . \sim (B \varepsilon \, \text{pt}(A)) \stackrel{.}{..} [CDa] \stackrel{..}{..}$

$\qquad\qquad [E] : \cdot : E \varepsilon C . \equiv :: [F] \therefore F \varepsilon a . \supset : E \varepsilon F . \vee . F \varepsilon \, \text{pt}(E) ::$

$\qquad\qquad [F] : F \varepsilon \, \text{pt}(E) . \supset . [\exists GH] . G \varepsilon a . H \varepsilon \, \text{pt}(F) . H \varepsilon \, \text{pt}(G) \stackrel{..}{..}$

$\qquad\qquad D \varepsilon \, \text{pt}(B) . B \varepsilon a \stackrel{..}{..} \supset . A \varepsilon \, \text{pt}(C) \hfill [BT1, BT2, BT15, PT17]$

By deducing *BT18* from *BA1*, *BA2*, and *BD1* we have shown that any thesis of $\mathfrak{S}_4$ is derivable in $\mathfrak{S}_3$. The purpose of the deductions presented below within the framework of $\mathfrak{S}_4$ is to establish that the converse holds too.

$CD1 \quad [Aa] : \cdot : A \varepsilon A :: [B] \therefore B \varepsilon a . \supset : A \varepsilon B . \vee . B \varepsilon \, \text{pt}(A) :: [B] : B \varepsilon \, \text{pt}(A) . \supset .$

$\qquad [\exists CD] . C \varepsilon a . D \varepsilon \, \text{pt}(B) . D \varepsilon \, \text{pt}(C) :: \equiv . A \varepsilon \, \mathsf{Kl}_1(a)^{12} \hfill [\text{Definition}]$

$CT1 \quad [AB] : A \varepsilon \, \text{pt}(B) . \supset . B \varepsilon B \hfill [CA1]$

$CT2 \quad [AB] : A \varepsilon \, \text{pt}(B) . \supset . \sim (B \varepsilon \, \text{pt}(A)) \hfill [CA1]$

$CT3 \quad [A] . \sim (A \varepsilon \, \text{pt}(A)) \hfill [CT2]$

$CT4 \; (= BA2) \; [Aa] : A \varepsilon a . \supset . [\exists B] . B \varepsilon \, \text{pt}(A) \hfill [CA1, CT3]$

$CT5 \quad [Aa] : A \varepsilon \, \mathsf{Kl}_1(a) . \supset . [\exists B] . B \varepsilon a$

PR $\quad [Aa] : \text{Hp(1)} . \supset .$

(2) $\quad [\exists B] . B \varepsilon \, \text{pt}(A) .$ $\hfill [CT4, 1]$

$\quad [\exists B] . B \varepsilon a$ $\hfill [CD1, 1, 2]$

$CT6 \quad [AaE] : \cdot : E \varepsilon a \therefore [B] \therefore B \varepsilon a . \supset : A \varepsilon B . \vee . B \varepsilon \, \text{pt}(A) :: [B] :$

$\qquad B \varepsilon \, \text{pt}(A) . \supset . [\exists CD] . C \varepsilon a . D \varepsilon \, \text{pt}(B) . D \varepsilon \, \text{pt}(C) :: \supset . A \varepsilon \, \mathsf{Kl}_1(a)$

PR $\quad [AaE] : \cdot : \text{Hp(3)} :: \supset :$

(4) $\quad A \varepsilon E . \vee . E \varepsilon \, \text{pt}(A) :$ $\hfill [2, 1]$

(5) $\quad A \varepsilon A .$ $\hfill [4, CT1]$

$\quad A \varepsilon \, \mathsf{Kl}_1(a)$ $\hfill [CD1, 5, 2, 3]$

$CT7 \quad [Ea] \stackrel{..}{..} E \varepsilon a . \supset : \cdot : [A] : \cdot : A \varepsilon \, \mathsf{Kl}_1(a) . \equiv :: [B] \therefore B \varepsilon a . \supset : A \varepsilon B . \vee .$

$\qquad B \varepsilon \, \text{pt}(A) :: [B] : B \varepsilon \, \text{pt}(A) . \supset . [\exists CD] . C \varepsilon a . D \varepsilon \, \text{pt}(B) . D \varepsilon \, \text{pt}(C)$

$\hfill [CD1, CT6]$

$CT8 \quad [ABa] : A \varepsilon \, \text{pt}(B) . B \varepsilon a . \supset . A \varepsilon \, \text{pt}(\mathsf{Kl}_1(a))$

PR $\quad [ABa] \stackrel{..}{..} \text{Hp(2)} . \supset : \cdot$

(3) $\quad [E] : \cdot : E \varepsilon \, \mathsf{Kl}_1(a) . \equiv :: [F] \therefore F \varepsilon a . \supset : E \varepsilon F . \vee . F \varepsilon \, \text{pt}(E) ::$

$\qquad [F] : F \varepsilon \, \text{pt}(E) . \supset . [\exists GH] . G \varepsilon a . H \varepsilon \, \text{pt}(F) . H \varepsilon \, \text{pt}(G) : \cdot :$ $\hfill [CT7, 2]$

$\quad A \varepsilon \, \text{pt}(\mathsf{Kl}_1(a))$ $\hfill [CA1, 1, 3, 2]$

$CT9 \quad [Aa] : A \varepsilon a . \supset . \mathsf{Kl}_1(a) \, \varepsilon \, \mathsf{Kl}_1(a)$

PR $\quad [Aa] \therefore \text{Hp(1)} . \supset :$

$\quad [\exists B] . \qquad\qquad\qquad \Big\}$

(2) $\qquad B \varepsilon \, \text{pt}(A) . \quad \Big\}$ $\hfill [CT4, 1]$

(3) $\qquad B \varepsilon \, \text{pt}(\mathsf{Kl}_1(a)) :$ $\hfill [CT8, 2, 1]$

$\quad \mathsf{Kl}_1(a) \, \varepsilon \, \mathsf{Kl}_1(a)$ $\hfill [CT1, 3]$

$CT10 \quad [AB] : B \varepsilon \, \text{pt}(A) . \supset . [\exists CD] . C \varepsilon \, \text{pt}(A) . D \varepsilon \, \text{pt}(B) . D \varepsilon \, \text{pt}(C)$

PR [AB]:Hp(1).⊃.
[∃D].

(2) D ε pt(B).   [CT4, 1]

(3) B ε pt(A).D ε pt(B).D ε pt(B)   [1, 2]

[∃CD].C ε pt(A).D ε pt(B).D ε pt(C)   [3]

CT11 [Aa]:A ε a.⊃.A ε Kl$_1$(pt(A))

PR [Aa]:Hp(1).⊃.

(2) [B]∴B ε pt(A).⊃.A ε B.∨.B ε pt(A):   [Prototetic]

A ε Kl$_1$(pt(A))   [CD1, 1, 2, CT10]

CT12 [ABC]:A ε pt(B).B ε pt(C).⊃.A ε pt(C)

PR [ABC]:Hp(2).⊃.

(3) C ε C.   [CT1, 2]

(4) C ε Kl$_1$(pt(C)).   [CT11, 3]

(5) Kl$_1$(pt(C)) ε Kl$_1$(pt(C)).   [CT9, 2]

(6) C = Kl$_1$(pt(C)).   [4, 5]

(7) A ε pt(Kl$_1$(pt(C))).   [CT8, 1, 2]

A ε pt(C)   [Extensionality, 6, 7]

CT13 [AB]:B ε pt(A).⊃.[∃CD].C ε A.D ε pt(B).D ε pt(C)

PR [AB]∴Hp(1).⊃:

(2) A ε A.   [CT1, 1]

[∃D].

(3) D ε pt(B).   [CT4, 1]

(4) D ε pt(A):   [CT12, 3, 1]

[∃CD].C ε A.D ε pt(B).D ε pt(C)   [2, 3, 4]

CT14 [Aa]:A ε a.⊃.A ε Kl$_1$(A)

PR [Aa]∴·Hp.⊃::

(2) A ε A::   [1]

(3) [B]∴B ε A.⊃:A ε B.∨.B ε pt(A)::   [1]

A ε Kl$_1$(A)   [CD1, 2, 3, CT13]

CT15 [AaB]∴A ε Kl$_1$(a).B ε pt(A).⊃:[∃C]:C ε B.∨.C ε pt(B):C ε a.∨.
[∃D].D ε a.C ε pt(D)

PR [Aab]::Hp(2):⊃∴

[∃CD]∴

(3) D ε a.   [CD1, 1, 2]

(4) C ε pt(B)

(5) C ε pt(D).

(6) C ε B.∨.C ε pt(B):   [4]

(7) [∃E].E ε a.C ε pt(E):   [3, 5]

(8) C ε a.∨.[∃E].E ε a.C ε pt(E)∴   [7]

[∃C]:C ε B.∨.C ε pt(B):C ε a.∨.[∃D].D ε a.C ε pt(D)   [6, 8]

CT16 [AaB]∴·[E]∴E ε pt(A).⊃:[∃F]:F ε E.∨.F ε pt(E):F ε a.∨.
[∃G].G ε a.F ε pt(G)::B ε pt(A)::⊃.[∃CD].C ε a.D ε pt(B).D ε pt(C)

PR [AaB]∴·Hp(2)::⊃::

[∃F]∴

(3) F ε B.∨.F ε pt(B):   [1, 2]

(4) F ε a.∨.[∃G].G ε a.F ε pt(G)∴

[∃D]:

(5) $\qquad D \,\varepsilon\, \mathsf{pt}(F)\,.$ $\qquad\qquad [3, CT4]$

(6) $\qquad D \,\varepsilon\, \mathsf{pt}(B):$ $\qquad$ [Extensionality, 3, 2, 5, $CT12$]
$\qquad [\exists C]\,.$

(7) $\qquad\quad C\,\varepsilon\,a\,.$ $\qquad\qquad\qquad\qquad [4, 5, CT12]$

(8) $\qquad\quad D\,\varepsilon\,\mathsf{pt}(C)::$
$\qquad [\exists CD]\,.\,C\,\varepsilon\,a\,.\,D\,\varepsilon\,\mathsf{pt}(B)\,.\,D\,\varepsilon\,\mathsf{pt}(C) \qquad\qquad [7, 6, 8]$

$CT17 \quad [AaE]:\!\cdot\!:E\,\varepsilon\,a::[B]\,\therefore\,B\,\varepsilon\,a\,.\,\supset:A\,\varepsilon\,B\,.\,\vee\,.\,B\,\varepsilon\,\mathsf{pt}(A)::[B]\,\therefore\,B\,\varepsilon\,\mathsf{pt}(A)\,.\,\supset:$
$\qquad [\exists C]:C\,\varepsilon\,B\,.\,\vee\,.\,C\,\varepsilon\,\mathsf{pt}(B):C\,\varepsilon\,a\,.\,\vee\,.\,[\exists D]\,.\,D\,\varepsilon\,a\,.\,C\,\varepsilon\,\mathsf{pt}(D)::\supset\,.\,A\,\varepsilon\,\mathsf{Kl}_1(a)$

PR $\quad [AaE]:\!\cdot\!:\mathrm{Hp}(3)::\supset\,\therefore$

(4) $\quad A\,\varepsilon\,A:$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad [2, 1, CT1]$

(5) $\quad [B]:B\,\varepsilon\,\mathsf{pt}(A)\,.\,\supset\,.\,[\exists CD]\,.\,C\,\varepsilon\,a\,.\,D\,\varepsilon\,\mathsf{pt}(B)\,.\,D\,\varepsilon\,\mathsf{pt}(C)\,\therefore \qquad [CT16, 3]$
$\quad A\,\varepsilon\,\mathsf{Kl}_1(a)$ $\qquad\qquad\qquad\qquad\qquad\qquad [CD1, 4, 2, 5]$

$CT18 \quad [Aa]:\!\cdot\!:A\,\varepsilon\,\mathsf{Kl}_1(a)\,.\,\equiv\,::[\exists B]\,.\,B\,\varepsilon\,a\,\therefore\,[B]\,\therefore\,B\,\varepsilon\,a\,.\,\supset:A\,\varepsilon\,B\,.\,\vee\,.\,B\,\varepsilon\,\mathsf{pt}(A)::$
$\qquad [B]\,\therefore\,B\,\varepsilon\,\mathsf{pt}(A)\,.\,\supset:[\exists C]:C\,\varepsilon\,B\,.\,\vee\,.\,C\,\varepsilon\,\mathsf{pt}(B):C\,\varepsilon\,a\,.\,\vee\,.\,[\exists D]\,.$
$\qquad D\,\varepsilon\,a\,.\,C\,\varepsilon\,\mathsf{pt}(D)$ $\qquad\qquad\qquad [CT5, CD1, CT15, CT17]$

$CT19 \quad [ABaf]\!\stackrel{..}{\cdot}\!A\,\varepsilon\,\mathsf{pt}(B):\!\cdot\!:[C]:\!\cdot\!:C\,\varepsilon\,f(a)\,.\,\equiv\,::[\exists D]\,.\,D\,\varepsilon\,a::[D]\,\therefore$
$\qquad D\,\varepsilon\,a\,.\,\supset:C\,\varepsilon\,D\,.\,\vee\,.\,D\,\varepsilon\,\mathsf{pt}(C)::[D]\,\therefore\,D\,\varepsilon\,\mathsf{pt}(C)\,.\,\supset:[\exists E]:E\,\varepsilon\,D\,.\,\vee\,.$
$\qquad E\,\varepsilon\,\mathsf{pt}(D):E\,\varepsilon\,a\,.\,\vee\,.\,[\exists F]\,.\,F\,\varepsilon\,a\,.\,E\,\varepsilon\,\mathsf{pt}(F)\!\stackrel{..}{::}\!B\,\varepsilon\,a\!\stackrel{..}{:}\!\supset:[\exists c]:$
$\qquad A\,\varepsilon\,c:A\,\varepsilon\,\mathsf{pt}(f(a))\,.\,\vee\,.\,\sim(f(c)\,\varepsilon\,f(c))$

PR $\quad [ABaf]\!\stackrel{..}{\cdot}\!\mathrm{Hp}(3)\!\stackrel{..}{::}\!\supset\,\therefore$

(4) $\quad A\,\varepsilon\,\mathsf{pt}(\mathsf{Kl}_1(a)):$ $\qquad\qquad\qquad\qquad [CT8, 1, 3]$

(5) $\quad [C]:C\,\varepsilon\,f(a)\,.\,\equiv\,.\,C\,\varepsilon\,\mathsf{Kl}_1(a)\,\therefore$ $\qquad\qquad [2, CT18]$

(6) $\quad A\,\varepsilon\,\mathsf{pt}(f(a))\,\therefore$ $\qquad\qquad$ [Extensionality, 5, 4]

(7) $\quad A\,\varepsilon\,A:A\,\varepsilon\,\mathsf{pt}(f(a))\,.\,\vee\,.\,\sim(f(A)\,\varepsilon\,f(A))\,\therefore$ $\qquad [1, 6]$
$\quad [\exists c]:A\,\varepsilon\,c:A\,\varepsilon\,\mathsf{pt}(f(a))\,.\,\vee\,.\,\sim(f(c)\,\varepsilon\,f(c))$ $\qquad\qquad [7]$

$CT20 \quad [AB]\!\stackrel{..}{::}\!B\,\varepsilon\,B\!\stackrel{..}{\cdot}\![af]\!\stackrel{..}{\cdot}\![Cb]:\!\cdot\!:C\,\varepsilon\,f(b)\,.\,\equiv\,::[\exists D]\,.\,D\,\varepsilon\,b\,\therefore\,[D]\,\therefore\,D\,\varepsilon\,b\,.\,\supset:$
$\qquad C\,\varepsilon\,D\,.\,\vee\,.\,D\,\varepsilon\,\mathsf{pt}(C)::[D]\,\therefore\,D\,\varepsilon\,\mathsf{pt}(C)\,.\,\supset:[\exists E]:E\,\varepsilon\,D\,.\,\vee\,.\,E\,\varepsilon\,\mathsf{pt}$
$\qquad (D):E\,\varepsilon\,b\,.\,\vee\,.\,[\exists F]\,.\,F\,\varepsilon\,b\,.\,E\,\varepsilon\,\mathsf{pt}(F)\!\stackrel{..}{::}\!B\,\varepsilon\,a\!\stackrel{..}{::}\!\supset:[\exists c]:A\,\varepsilon\,c:$
$\qquad A\,\varepsilon\,\mathsf{pt}(f(a))\,.\,\vee\,.\,\sim(f(c)\,\varepsilon\,f(c))\!\stackrel{..}{:}\!\supset\,.\,A\,\varepsilon\,\mathsf{pt}(B)$

PR $\quad [AB]\!\stackrel{..}{::}\!\mathrm{Hp}(2)\!\stackrel{..}{:}\!\supset\,\therefore$
$\quad [\exists c]$

(3) $\qquad A\,\varepsilon\,c:$
(4) $\qquad A\,\varepsilon\,\mathsf{pt}(\mathsf{Kl}_1(B))\,.\,\vee\,.\,\sim(\mathsf{Kl}_1(c)\,\varepsilon\,\mathsf{Kl}_1(c)):$ $\qquad [2, CT18, 1]$

(5) $\qquad\quad \mathsf{Kl}_1(c)\,\varepsilon\,\mathsf{Kl}_1(c)$ $\qquad\qquad\qquad\qquad\qquad [CT9, 3]$

(6) $\quad A\,\varepsilon\,\mathsf{pt}(\mathsf{Kl}_1(B))\,.$ $\qquad\qquad\qquad\qquad\qquad\qquad [4, 5]$

(7) $\quad B\,\varepsilon\,\overline{\mathsf{Kl}_1}(B)\,.$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad [CT14, 1]$

(8) $\quad \mathsf{Kl}_1(B)\,\varepsilon\,\mathsf{Kl}_1(B)\,.$ $\qquad\qquad\qquad\qquad\qquad\qquad [CT9, 1]$

(9) $\quad B = \mathsf{Kl}_1(B)\,.$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad [7, 8]$
$\quad A\,\varepsilon\,\mathsf{pt}(B)$ $\qquad\qquad\qquad\qquad$ [Extensionality, 9, 6]

$CT21\,(= BA1)\quad [AB]\!\stackrel{..}{\cdot}\!A\,\varepsilon\,\mathsf{pt}(B)\,.\,\equiv\!\stackrel{..}{\cdot}\!B\,\varepsilon\,B\,.\,\sim(B\,\varepsilon\,\mathsf{pt}(A))\!\stackrel{..}{\cdot}\![af]\!\stackrel{..}{\cdot}\![Cb]$
$\qquad C\,\varepsilon\,f(b)\,.\,\equiv\,::[\exists D]\,.\,D\,\varepsilon\,b::[D]\,\therefore\,D\,\varepsilon\,b\,.\,\supset:C\,\varepsilon\,D\,.\,\vee\,.\,D\,\varepsilon\,\mathsf{pt}(C)::$
$\qquad [D]\,\therefore\,D\,\varepsilon\,\mathsf{pt}(C)\,.\,\supset:[\exists E]\,.\,E\,\varepsilon\,D\,.\,\vee\,.\,E\,\varepsilon\,\mathsf{pt}(D):E\,\varepsilon\,b\,.\,\vee\,.\,[\exists F]\,.$
$\qquad F\,\varepsilon\,b\,.\,E\,\varepsilon\,\mathsf{pt}(F)\!\stackrel{..}{::}\!B\,\varepsilon\,a\!\stackrel{..}{:}\!\supset:[\exists c]:A\,\varepsilon\,c:A\,\varepsilon\,\mathsf{pt}((a))\,.\,\vee\,.$
$\qquad \sim(f(c)\,\varepsilon\,f(c))$ $\qquad\qquad\qquad [CT1, CT2, CT19, CT20]$

By proving $CT4$ and $CT21$ within the framework of $\mathfrak{S}_4$ we have established that any thesis of $\mathfrak{S}_3$ can be shown to be a thesis of $\mathfrak{S}_4$, which completes the proof that the two systems of atomless mereology are inferentially equivalent.

NOTES

1. See Leśniewski [6] and Leśniewski [7]; for a general introduction to mereology see Sobociński [9] and Luschei [8].

2. See Leśniewski [7], vol. 30 (1927), p. 197 f.

3. See Goodman [1], pp. 86 and 118.

4. See Lejewski [3], thesis $W2$; a system of general mereology with the functor '**ov**' as the primitive term can be based on the following single axiom:

$[AB] \vdots A \varepsilon \text{ ov}(B) . \equiv : : A \varepsilon A . B \varepsilon B \vdots [f] \vdots [Da] \vdots D \varepsilon f(a) . \equiv . D \varepsilon D \therefore$
$[E] : D \varepsilon \text{ ov} (E) . \equiv . [\exists F] . F \varepsilon a . F \varepsilon \text{ ov} (E) :\cdot: B \varepsilon \text{ ov} (B) :\cdot: \supset \vdots [\exists C] \vdots$
$[b] \therefore A \varepsilon b . \lor . B \varepsilon b : \supset . [\exists d] . C \varepsilon d . f(b) \varepsilon f(d)$

5. See Hiż [2], p. 22, $D4$.

6. See Hiż [2], p. 22, $D4$ (second occurrence), which is a misprint and should read: $D5$.

7. See Hiż [2], p. 23, $D14$.

8. See Sobociński [10], p. 89 f.

9. See Sobociński [11].

10. In [12] Sobociński has proved that (12) supplemented by

    (13) $[AB] \vdots A \varepsilon \text{ el} (B) . \equiv : : B \varepsilon B \vdots [Ca] \vdots [D] \vdots D \varepsilon C . \equiv \therefore [E] : E \varepsilon a.$
    $\supset . E \varepsilon \text{ el} (D) \therefore [E] : E \varepsilon \text{ el} (D) . \supset . [\exists FG] . F \varepsilon a . G \varepsilon \text{ el} (E).$
    $G \varepsilon \text{ el} (F) :\cdot: B \varepsilon a :\cdot: \supset . A \varepsilon \text{ el} (C)$

    yields an axiom system inferentially equivalent to the one consisting of (11) and (12).

11. In connection with BA1 see Lejewski [5]. The text of [5] contains a few misprints, which I would like to take this opportunity to correct:

    p. 280, line 7 from the top:
    instead of: $D \varepsilon b . \lor [\exists F] . F \varepsilon a$ read: $E \varepsilon b . \lor . [\exists F] . F \varepsilon b$

    p. 281, lines 6 and 1 from the bottom, and p. 282, lines 7 and 12 from the top:
    instead of: $E \varepsilon a . \lor . [\exists F] . F \varepsilon a$ read: $E \varepsilon b . \lor . [\exists F] . F \varepsilon b$

    p. 282, line 23 from the top:
    instead of: driving   read: deriving.

12. It can be proved that '**Kl**' in $\mathfrak{S}_3$ and '**Kl**$_1$' in $\mathfrak{S}_4$ are synonymous. Different symbols have been used in order not to presuppose the synonimity, which is not required for the purpose of proving that $\mathfrak{S}_3$ and $\mathfrak{S}_4$ are inferentially equivalent.

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