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SEMANTICS FOR S4.3.2

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In [5], semantics for S4.4 was presented. The present paper will do the same for the system S4.3.2 of [3]. The author had conjectured some time ago that the semantics to be discussed was characteristic for S4.3.2; since then, correspondence has reached him indicating that K. Fine has confirmed the conjecture. The approach of this aritcle differs from that of Fine, and is like that of [5] in developing a Kripke-style semantics for the system in question (familiarity with [1] and [2] of Kripke as well as with [3] and [5] is assumed).

In models belonging to the model structure (m.s.) of [5] for S4.4 there is a distinction drawn between kinds of worlds, the distinction depending upon the properties of the accessibility relation for the respective worlds. One kind of world will have one and only one representative in any S4.4 model (this is the "real" world); this world has access to all worlds in the model including itself; the other kind of world may have any number of representatives in an S4.4 model; this kind of world has access to all worlds in the model *except* for the real world. If we think of the S4.4 semantics as a temporal structure, the "real" world corresponds to the moment in time called the "last instant of time" (before eternity "begins") and the other worlds correspond to the "instants of eternity" (thus the reference in [5] to S4.4 as a "logic of the end of the world"). The semantics for S4.3.2 will be like that for S4.4 in distinguishing between two kinds of world based on the accessibility relation appropriate to the respective worlds; here, however, there will be no limit on the number of worlds of either class that may belong to a model. We shall call one of the classes of worlds the "t-worlds" (analogous to the instant of time in S4.4 models) and the other the "e-worlds" (from the eternity part of S4.4 models). The accessibility relation for the S4.3.2 m.s. is as follows:

Every t-world has access to every world (t- or e-) in the model in which it occurs; every e-world has access to all e-worlds.

$L\alpha$ in a t-world means, then, that α is true in all worlds; $L\alpha$ in an e-world means that α holds in all e-worlds. $M\alpha$ in a t-world means that α holds in some world; in an e-world it means that α holds in some e-world.

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Let us look at $A\mathfrak{C}LpqCMLqp$, the proper axiom of S4.3.2, in this m.s. This formula will be falsified only if MLq, Np, and $MKLpNq$ (the negation of $\mathfrak{C}$Lpq) are true simultaneously. We examine these requirements:

MLq—means that q must be true in at least all the e-worlds.

Np—means that p must be false. But since $MKLpNq$ must also hold, p cannot be false in an e-world, else Lp can never hold (since all e-worlds have access to each other); p must then be false in a t-world, and so Lp must fail in all t-worlds (since they all have access to each other).

$MKLpNq$—means that in some world p must be necessary while q in the same world is false. But by the above, p becomes necessary precisely at the point at which q—by MLq—must be necessary. Falsification of S4.3.2 is then impossible here, and the suggested m.s. verifies this system.

It is easy, on the other hand, to find a falsifying instance for $\mathfrak{C}pCMLpLp$, the proper axiom of S4.4: for falsification, p, MLp, and NKp must hold simultaneously; let p hold in all e-worlds (then MLp holds) and in the real world (so p holds) and fail in a t-world other than the real one (so NLp holds in the real world). The proper axiom of S4.4 is then falsified.

We shall give rules for a system of semantic tableaux corresponding to this m.s.; as with S4.4, the rules for the **PC** connectives will be as usual and there shall be two sets of L rules, depending on whether the formula in question occurs in a t- or in an e-tableau (which correspond respectively to t- and e-worlds). We give first of all the rules for formulas beginning with L in e-tableaux (these are the same as those for formulas beginning with L in auxiliary tableaux for S4.4 (see [5])).

L-left$_e$: *If $L\alpha$ occurs on the left of an* e*-tableau, write α on the left of each* e*-tableau in the alternative set in question.*

L-rght$_e$: *If $L\alpha$ occurs on the right of an* e*-tableau, begin a new* e*-tableau starting with α on the right.*

The rule for L on the left of a t-tableau will reflect the auxiliarity of all tableaux in the alternative set in question to that tableau:

L-left$_t$: *If $L\alpha$ occurs on the left of a* t*-tableau, write α on the left of each tableau* (e- *and* t-) *in the alternative set in question.*

The statement of L-rght$_t$ will be somewhat more complex, as is that of the corresponding rule for S4.4. We note that given the accessibility relation for e-worlds, a formula $L\alpha$ will be true in all e-worlds, or else α will be false in at least one e-world; this is reflected in the tableau split that will be required for this rule; note that if Lα holds in the e-worlds but not in the t-worlds (as it must not if it is written right in a t-tableau) as then α must fail in some t-worlds.

L-rght$_t$: *If $L\alpha$ occurs on the right of a* t*-tableau* (*call it* w), *split the alternative set to which* w *belongs, and*

1. *Auxiliary to the occurrence of* w *in one of the resulting sets, begin an* e*-tableau with* α *on its* right, and
2. Auxiliary to the occurrence of w in the other set, begin an e-tableau with $L\alpha$ *left*, and also begin a t-tableau with α right.

The reasons for the complexity of L-rght$_t$ will come out in the proofs to follow. We note that the relation of "auxiliarity" in tableau constructions using the **PC** rules and the above four rules will correspond to the accessibility relation for S4.3.2 worlds, with e- and t-tableaux corresponding to e- and t-worlds respectively; lemmas 1 and 2 of Kripke [2] will then hold, and a formula will be valid in the m.s. for S4.3.2 iff the tableau construction for it using the **PC** rules and the above four L rules closes.

As is usual, we shall think of the first stage in the construction of tableaux as being the initial main tableau with the formula to be checked written on its right. The $(i + 1)$th stage in the construction is derived from the ith by the application of one of the rules. Corresponding to the ith stage of construction is a formula, χ_i, which is the "characteristic wff" of that stage. χ_i as a whole is defined in terms of the characteristic wffs of the various parts of the ith stage of construction. The characteristic wff of any tableau at a given stage of construction is $KK\alpha_1 \ldots \alpha_r KN\beta_1 \ldots N\beta_s$ where the α's are the formulas on the left and the β's those on the right of the tableau. The characteristic wff of an alternative set of S4.3.2 tableaux is

(1) $KK\mu_1 KM\mu_2 \ldots M\mu_m LMKM\sigma_1 \ldots M\sigma_n$

where μ_1 is the characteristic wff of the main tableau, the other μ's are the characteristic wffs of the t-tableaux other than the main, and the σ's are the characteristic wffs of the e-tableaux; $m \geq 1$, $n \geq 0$. Note that in (1) all characteristic wffs of tableaux other than the main are preceded by M's. Finally, the characteristic wff χ_i of the whole ith stage of construction is the disjunction

(2) $A\delta_1 \ldots \delta_t$

where the δ's are the characteristic wffs of each of the alternative sets of the construction at that stage. The above formulas, especially (1), may be referred to in what follows. $\chi_{(i+1)}$ will differ from χ_i in some part as determined by the ith rule application in the construction. We wish to show, as usual, that the implication $C\chi_i\chi_{(i+1)}$ holds in the system in question, here S4.3.2. We first of all will note that if the ith rule application is by a **PC** rule, in any tableau, then $C\chi_i\chi_{(i+1)}$ holds in S4.3.2, since each **PC** rule corresponds to a strict implication holding in S4.3.2 and the semisubstitutivity of strict implication (see [3]) holds in S4.3.2.

Secondly, let us consider the situation in which the rule applied is one of those for L in an e-tableau. Here there will be involved the transformation of a part $LM\gamma_i$ if χ_i to a part $LM\gamma_{(i+1)}$ of $\chi_{(i+1)}$. The situation here is exactly like that for the rules for L in auxiliary tableaux in S4.4 tableaux in the S4.4 semantics; in [5, pp. 345-6] it is shown that the necessary principles hold in S4.2; so for the L rules in e-tableaux, $C\chi_i\chi_{(i+1)}$ holds as well in S4.3.2.

We now consider the cases in which the ith stage is converted to the $(i + 1)$th by a rule for L in a ŧ-tableau. We shall find it convenient here to use the notation and some of the notions of the Gentzen sequent logics (see, for example, [6] for terminology and general details); we shall do this informally, making the mechanisms of these logics part of our metalanguage. By doing this we can, I think, avoid too lengthy and hard-to-read formulas in our proofs; we shall summarize certain deductions by the form $\Gamma \to \Theta$, where Γ is a sequence of formulas (possibly empty) interpreted conjunctively, and Θ a sequence interpreted disjunctively (thought of as "meaning" the disjunction of the formulas of Θ).

We consider first of all the rule L-rght$_ŧ$; this consideration will be in two stages: first, when the rule is applied for the first time in a given alternative set, and secondly, when the rule is applied in an alternative set after having been applied at least once.

In the first case of L-rght$_ŧ$, only the main tableau is in the alternative set at the ith stage, and so the characteristic wff of the set at that stage is simply μ_1. This means that, where Lα is the formula for which the rule is applied, NLα is a conjunct of the characteristic wff of that set. A split is called for by this rule; by its first clause, an e-tableau with α right is begun for one of the resulting alternative sets; the characteristic wff for that set at the next stage will then include $LMMN\alpha$ (= $LMN\alpha$ in S4) as a conjunct; the other set has added an e-tableau with $L\alpha$ left and also a ŧ-tableau with α right. Its characteristic wff at the next stage will then contain as a conjunct (equivalently) $KMN\alpha LMML\alpha$ (= $KMN\alpha LML\alpha$ in S4). In moving from χ_i to $\chi_{(i+1)}$, then, μ_1 transforms to $AK\mu_1LMN\alpha K\mu_1KMN\alpha LML\alpha$. This holds in S4.3.2 if the deduction corresponding to

(3) $NL\alpha \to LMN\alpha, KMN\alpha LML\alpha$

holds there. But (3) does hold; $NL\alpha \to MN\alpha$ is an S1° principle, and $\to LMN\alpha, LMN\alpha$ is characteristic of S4.2. By **PC** procedures form there, (3) holds in S4.2, and so in S4.3.2.

Now consider the case of L-rght$_ŧ$ in which that rule has been applied before in a given set. Here $L\alpha$, the formula for which the rule is applied, may be in the main tableau or in one of the other ŧ-tableaux. Assume the latter case. We have then as a conjunct of the characteristic wff of the set in question the formula $MK\beta NL\alpha$ (if $L\alpha$ is in the main tableau, we have as a conjunct simply $K\beta NL\alpha$ (= μ_1); whatever follows from $M\mu_j$ follows here from μ_j as well, so it is sufficient to consider explicitly the case in which $L\alpha$ occurs in a ŧ-tableau other than the main one). Also, since L-rght$_ŧ$ has by hypothesis been applied before in this set, we have at least one e-tableau already in it, and so as a conjunct of its characteristic wff we have $LM\gamma$. As before, a split is called for. In one of the resulting sets an e-tableau beginning with α right is added; the characteristic wff for that set at the next stage will then have $LMK\gamma MN\alpha$ as a conjunct. In the other of the resulting alternative sets, an e-tableau beginning with $L\alpha$ left is added; the characteristic wff for that set then has $LMK\gamma ML\alpha$ as a conjunct. That set also has a ŧ-tableau added beginning with α right; $MN\alpha$ will also then be a conjunct of the characteristic wff of that set at the next stage of construction. We must then show that the principle

(4) $MK\beta NL\alpha, LM\gamma \rightarrow LMK\gamma MN\alpha, KMN\alpha LMK\gamma L\alpha$

holds in S4.3.2. We note that the deduction corresponding to $MK\beta NL\alpha \rightarrow MN\alpha$ holds in S4°. And in S2° we have $LM\gamma \rightarrow LMAK\gamma MN\alpha K\gamma L\alpha$. LM distributes over A in S4.2, so this last sequent yields $LM\gamma \rightarrow LMK\gamma MN\alpha, LMK\gamma L\alpha$. We are then able to derive (4) in S4.3.2. This covers all cases of L-rght$_{\mathfrak{t}}$; when this rule is the ith in a construction, then, $C\chi_i\chi_{(i+1)}$ holds in S4.3.2.

We now turn to rule L-left$_{\mathfrak{t}}$. Consider first of all the case in which $L\alpha$ occurs left in the main tableau. Here $L\alpha$ itself is a conjunct of the characteristic formula of the set of tableaux involved. In any case the insertion of α left in the same tableau in which $L\alpha$ occurs left follows by $\mathfrak{C}Lpp$. If α is inserted into a $\mathfrak{t}$-tableau other than the main, then there is a formula $M\gamma$ as a conjunct of the characteristic wff of the set corresponding to that tableau; the characteristic wff of the set at the next stage will then include as a conjunct $MK\gamma\alpha$; the relevant deductive principle is then:

(5) $L\alpha, M\gamma \rightarrow MK\alpha\gamma$

which holds in S1°. If α is inserted into an $\mathfrak{e}$-tableau, the relevant principle is

(6) $L\alpha, LMK\delta M\gamma \rightarrow LMK\delta MK\gamma\alpha$

which holds in S3°. If L-left$_{\mathfrak{t}}$ is the ith rule with $L\alpha$ left in the main tableau, then, $C\chi_i\chi_{(i+1)}$ holds in S4.3.2.

Consider now L-left$_{\mathfrak{t}}$ with $L\alpha$ left in a $\mathfrak{t}$-tableau other than the main. Here $MKL\alpha\beta$ is a conjunct of the characteristic wff of the alternative set in question; the formula $M\gamma$ (corresponding to the tableau into which α is inserted) is also such a conjunct. There is by hypothesis a $\mathfrak{t}$-tableau other than the main; for this to be the case, clause 2. of L-rght$_{\mathfrak{t}}$ must have been applied in this set, inserting a formula $L\delta$ left in an $\mathfrak{e}$-tableau while the $\mathfrak{t}$-tableau eventually corresponding to $M\gamma$ is begun by the insertion of δ right in that new $\mathfrak{t}$-tableau. $N\delta$ is then a conjunct of γ, and since $\gamma = KL\alpha\beta$, also of β. We then have $\overline{\mathfrak{C}}\beta N\delta$ as an S1° thesis.

As a conjunct of the characteristic wff of the set in question at the $(i + 1)$th stage (because of that $\mathfrak{e}$-tableau which began with $L\delta$ left) we will have the formula $LMK\theta MK\eta L\delta$. This latter formula strictly implies $LMML\delta$ (in S2°) and so (in S4) strictly implies $ML\delta$. This fact and the fact that, as indicated above, $\overline{\mathfrak{C}}\beta N\delta$ holds we shall employ a bit later in the proof.

We note that the principle

(7) $MKL\alpha\beta, ML\delta \rightarrow ML\delta$

holds for **PC**. Now, as we have noted, the proper axiom of S4.3.2 is $A\overline{\mathfrak{C}}LpqCMLqp$. In S4 this easily gives $A\overline{\mathfrak{C}}LpqCMLqLq$, and so the deductive form

(8) $\rightarrow \mathfrak{C}Lpq, CMLqLp$

holds for S4.3.2. With p/α and q/δ and a simple **PC** operation, this gives

(9) $ML\delta \rightarrow \mathfrak{C}L\alpha\delta, L\alpha$

With (9), (7), and the rule "cut" ($ML\delta$ as cut formula) we get:

(10) $MKL\alpha\beta, ML\delta \rightarrow \mathfrak{C}L\alpha\delta, L\alpha$

which converts by **PC** methods to

(11) $N\mathfrak{C}L\alpha\delta, MKL\alpha\beta, ML\delta \rightarrow L\alpha$

Deductive form (11) is equivalent by S1° methods to

(12) $MKL\alpha N\delta, MKL\alpha\beta, ML\delta \rightarrow L\alpha$

Nδ is effectively in an "A-pos" (see [4]) in (12); by the semisubstitutivity of strict implication and the fact that $\mathfrak{C}\beta N\delta$ holds, we may replace $N\delta$ in (12) by β, converting $MKL\alpha N\delta$ to $MKL\alpha\beta$. "Contraction" then gives us

(13) $MKL\alpha\beta, ML\delta \rightarrow L\alpha$

We noted above that $LMK\theta MK\eta L\delta$ (the portion of the characteristic wff representing all the e-tableaux in the set in question) strictly implies $ML\delta$; it may then replace $ML\delta$ in (13):

(14) $MKL\alpha\beta, LMK\theta MK\eta L\delta \rightarrow L\alpha$

We then see that given the characteristic wff of the t-tableau in which Lα occurs left and the part of the characteristic wff representing the e-tableaux, we can derive $L\alpha$; by forms (5) and (6), then, analogs to the insertion of α on the left of any t- or e-tableau hold in S4.3.2, and so our L-left$_t$ as a whole holds there; when this is the ith rule application, then, $C\chi_i\chi_{(i+1)}$ is an S4.3.2 thesis; indeed, now that we have covered the cases involving all of the rules of this system of tableaux, this latter formula holds in general in S4.3.2.

The remainder of the completeness proof follows just as in [2]; we have then shown that the semantics proposed in this paper is characteristic for S4.3.2.

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