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PROJECTIVE BIGRAPHS WITH RECURSIVE OPERATIONS

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1 *Introduction* All graphs under consideration will be undirected, simple (i.e., without loops or multiple edges), countable and connected. In [3] two effectiveness conditions were discussed for graphs $\mathfrak{G} = \langle \nu, \eta \rangle$, where ν (the set of vertices) and η (the set of edges) are subsets of ε (the set of nonnegative integers). $\mathfrak{G}$ is called an *α-graph* if there is an effective procedure which enables us to decide, given any two distinct vertices p and and q of $\mathfrak{G}$, whether they are adjacent, i.e., joined by an edge. On the other hand, $\mathfrak{G}$ is an *ω-graph* if there is an effective procedure which enables us, given any two distinct vertices p and q of $\mathfrak{G}$, to find a minimal path joining p and q. Note that p and q are adjacent if and only if the (only) minimal path which joins them has length one. We see therefore that *every ω-graph is an α-graph.* It will be shown below that the converse is false.

The present paper deals with *bigraphs,* i.e., graphs of the type $\mathfrak{B} = \langle \nu, \eta \rangle$, where ν can be decomposed into two disjoint nonempty sets α and β such that every edge of $\mathfrak{B}$ joins a vertex in α and a vertex in β. We call $\mathfrak{B}$ an *α-bigraph* (*ω-bigraph*) if it is both an α-graph (ω-graph) and a bigraph. In [2] the notion of a projective ω-plane was introduced. Projective planes correspond in a natural manner to certain bigraphs, the so-called projective bigraphs. Our main result is that *under this correspondence ω-planes correspond to ω-bigraphs.*

2 *Preliminaries* We consider nonnegative integers (*numbers*), collections of numbers (*sets*) and collections of sets (*classes*). The empty set of numbers is denoted by σ, the set of all numbers by ε, and the class of all finite sets by Q. We write $\subset$ for inclusion (proper or not), and $\mathfrak{c}$ for the cardinality of the continuum. We need an effective enumeration without repetitions of the class Q and we choose the following:

$$(1) \qquad \begin{cases} \rho_0 = \sigma, \\ \rho_{n+1} = \begin{cases} (a_1, \ldots, a_k), \text{ where } a_1, \ldots, a_k \text{ are distinct} \\ \text{and } n + 1 = 2^{a_1} + \ldots + 2^{a_k}. \end{cases} \end{cases}$$

The sequence $\langle \rho_0, \rho_1, \ldots \rangle$ is called the *canonical enumeration* of Q.

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For every finite set σ there is exactly one number i such that $\sigma = \rho_i$; this number i is called the *canonical index* of σ and denoted by $\text{can}(\sigma)$. Let $r_n = \text{card}(\rho_n)$; then r_n is a recursive function of n. We define for $\alpha \subset \varepsilon$, $i \in \varepsilon$,

(2) $$[\alpha; i] = \{n \mid \rho_n \subset \alpha \text{ and } r_n = i\}.$$

If e is the edge of a graph, we can characterize e by the two-element set of its endpoints. Every graph is therefore isomorphic to a graph of the form $\mathfrak{G} = \langle \nu, \eta \rangle$, where

(3) $$\nu \subset \varepsilon \text{ and } \eta \subset [\nu; 2].$$

Henceforth all graphs under consideration will satisfy (3). Thus, if the vertices a and b of $\mathfrak{G}$ are adjacent, the edge e which joins them can be computed from a and b, since $e = \text{can}(a, b) = 2^a + 2^b$. Conversely, given any edge e of $\mathfrak{G}$, we can compute its endpoints. The sets α and β are *separable* (written: $\alpha \mid \beta$) if they can be separated by disjoint r.e. sets. An *α-graph* can now be defined as a graph $\mathfrak{G} = \langle \nu, \eta \rangle$ such that $\eta \mid [\nu; 2] - \eta$.

If σ is any set, we write $(\sigma \times \sigma)'$ for $\{\langle x, y \rangle \in \sigma \times \sigma \mid x < y\}$ and $(\sigma \times \sigma)^-$ for $\{\langle x, y \rangle \in \sigma \times \sigma \mid x \neq y\}$. The domain of a function f is denoted by δf. Let us write $\langle q_0, q_1, \ldots \rangle$ for the sequence of all odd primes, arranged according to size. For any finite sequence $\pi = \langle p_0, \ldots, p_n \rangle$ of numbers with $p_0 < p_n$ we define the *Gödel number* $\mathfrak{G}(\pi)$ by

(4) $$\mathfrak{G}(\pi) = 2^{n+1} \prod_{i=0}^{n} q_i^{p_i},$$

and the *length* $l(\pi)$ as n. If p and q are vertices of a graph $\mathfrak{G}$, we denote their distance by $\text{d}(p, q)$; thus, if $p < q$ and π is a minimal path from p to q, then $\text{d}(p, q) = l(\pi)$.

Definition 1 The graph $\mathfrak{G} = \langle \nu, \eta \rangle$ is an *ω-graph* if some function m such that

(i) $\delta m = (\nu \times \nu)'$,
(ii) $m(p, q) = \mathfrak{G}(\pi)$, for some minimal path π from p to q, has a partial recursive extension.

If $\mathfrak{G} = \langle \nu, \eta \rangle$ is an ω-graph and m a function related to $\mathfrak{G}$ as described in Definition 1, we have for $\langle p, q \rangle \in (\nu \times \nu)'$,

(5) $$\text{can}(p, q) \in \eta \Longleftrightarrow [4 \text{ divides } m(p, q) \text{ and } 8 \text{ does not divide } m(p, q)].$$

Thus *every ω-graph is an α-graph*. In Section 1 we defined α-bigraphs and ω-bigraphs. Thus every *ω-bigraph is an α-bigraph*. A graph $\mathfrak{G} = \langle \nu, \eta \rangle$ is *isolic* if the sets ν and η are isolated, i.e., have no infinite r.e. subsets; $\mathfrak{G}$ is *immune* if the sets ν and η are immune, i.e., both infinite and isolated.

3 *Two propositions* For a nonempty set σ we write $\mathfrak{K}_\sigma$ for the complete graph $\langle \sigma, \eta \rangle$ with $\eta = [\sigma; 2]$. We immediately see that $\mathfrak{K}_\sigma$ is both an α-graph and an ω-graph. If α and β are disjoint nonempty sets, we write $\mathfrak{K}_{\alpha,\beta}$ for the complete bigraph on α and β, i.e., for the graph $\mathfrak{B} = \langle \nu, \eta \rangle$ such that $\nu = \alpha \cup \beta$ and for $\langle x, y \rangle \in (\nu \times \nu)^-$,

(6) $$\text{can}(x, y) \in \eta \Longleftrightarrow (x \in \alpha \text{ and } y \in \beta) \text{ or } (x \in \beta \text{ and } y \in \alpha).$$

Proposition 1 *For disjoint nonempty sets α and β the following conditions are mutually equivalent:*

(i) $\alpha \mid \beta$,
(ii) $\mathfrak{K}_{\alpha\beta}$ *is an ω-bigraph,*
(iii) $\mathfrak{K}_{\alpha\beta}$ *is an α-bigraph.*

Proof: Note that $\mathfrak{K}_{\alpha\beta}$ is connected, since α and β are nonempty. Let $a \epsilon \alpha$, $b \epsilon \beta$; throughout this proof a and b remain fixed. We already know that (ii) $\Longrightarrow$ (iii). To show that (i) $\Longrightarrow$ (ii), assume $\alpha \mid \beta$. Let us write $\mathcal{M}(x, y)$ for the family of all minimal paths between vertices x and y. We have for $x, y \epsilon \alpha \cup \beta$, $x < y$,

$$\begin{aligned} &x, y \epsilon \alpha \Longrightarrow \langle x, b, y\rangle \epsilon \mathcal{M}(x, y),\\ &x, y \epsilon \beta \Longrightarrow \langle x, a, y\rangle \epsilon \mathcal{M}(x, y),\\ &x \epsilon \alpha \text{ and } y \epsilon \beta \Longrightarrow \langle x, y\rangle \epsilon \mathcal{M}(x, y),\\ &x \epsilon \beta \text{ and } y \epsilon \alpha \Longrightarrow \langle x, y\rangle \epsilon \mathcal{M}(x, y). \end{aligned}$$

Since $\alpha \mid \beta$ we can effectively decide which of the four premisses holds; thus we can effectively find (the Gödel number of) a minimal path from x to y. Hence $\mathfrak{K}_{\alpha\beta}$ is an ω-bigraph. To establish (iii) $\Longrightarrow$ (i), assume that $\mathfrak{K}_{\alpha\beta}$ is an α-bigraph. We have for $x \epsilon \alpha \cup \beta$,

$$\begin{aligned} &x \epsilon \alpha \Longleftrightarrow x = a \text{ or } [x \neq b \text{ and } \mathrm{can}(x, b) \epsilon \eta],\\ &x \epsilon \beta \Longleftrightarrow x = b \text{ or } [x \neq a \text{ and } \mathrm{can}(x, a) \epsilon \eta]. \end{aligned}$$

It follows that $\alpha \mid \beta$ because we can effectively decide whether $x \epsilon \alpha$ or $x \epsilon \beta$.

Proposition 2 *Every ω-bigraph is an α-bigraph, but not conversely.*

Proof: We only need to exhibit an α-bigraph $\mathfrak{B} = \langle \nu, \eta\rangle$ which is not an ω-bigraph. In our example $\mathfrak{B}$ will be immune. For the definitions of regressive functions and regressive isols, see ([1], Section **3**). It follows from ([1], p. 25) that there exist regressive functions s_n and t_n from ϵ into ϵ with ranges σ and τ respectively such that σ and τ are separable and immune, while the set $\sigma \cup \tau$ is immune, but not regressive. Define three classes of two-element sets by:

$$E_1 = \{(s_n, t_n)\}_{n \epsilon \varepsilon}, \; E_2 = \{(s_{2n+1}, s_{2n+2})\}_{n \epsilon \varepsilon}, \; E_3 = \{(t_{2n}, t_{2n+1})\}_{n \epsilon \varepsilon}.$$

Let $\mathfrak{B} = \langle \nu, \eta\rangle$ be the bigraph such that

$$\begin{aligned} &\alpha = (s_0, t_1, s_2, t_3, \ldots), \; \beta = (t_0, s_1, t_2, s_3, \ldots),\\ &\nu = \alpha \cup \beta, \; \eta = \{\mathrm{can}(x, y) \mid (x, y) \epsilon E_1 \cup E_2 \cup E_3\}. \end{aligned}$$

Note that every edge of $\mathfrak{B}$ joins a vertex in α and a vertex in β. Since the functions s_n and t_n are regressive, the functions r_s and r_t defined by

$$\delta r_s = \sigma, \; r_s(x) = s^{-1}(x), \; \delta r_t = \tau, \; r_t(x) = t^{-1}(x),$$

have partial recursive extensions. We claim that (a) $\mathfrak{B}$ is an α-bigraph and (b) $\mathfrak{B}$ is not an ω-bigraph.

Re(a). Let $x, y \epsilon \nu$, $x < y$. We distinguish four cases.

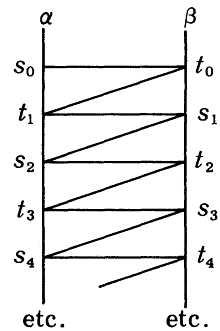

Figure 1.

(I) $x, y \in \sigma$, (II) $x, y \in \tau$, (III) $x \in \sigma, y \in \tau$, (IV) $x \in \tau, y \in \sigma$.

In view of $\sigma | \tau$ we can decide which of these four cases holds. Moreover,

if (I), $\text{can}(x, y) \in \eta \Leftrightarrow \min[r_s(x), r_s(y)]$ odd and $|r_s(x) - r_s(y)| = 1$,
if (II), $\text{can}(x, y) \in \eta \Leftrightarrow \min[r_t(x), r_t(y)]$ even and $|r_t(x) - r_t(y)| = 1$,
if (III), $\text{can}(x, y) \in \eta \Leftrightarrow r_s(x) = r_t(y)$,
if (IV), $\text{can}(x, y) \in \eta \Leftrightarrow r_t(x) = r_s(y)$.

The numbers $r_s(x)$, $r_s(y)$, $r_t(x)$, $r_t(y)$ can be computed from x and y; hence η is separable from $[\nu; 2] - \eta$, i.e., $\mathfrak{B}$ is an α-bigraph. Moreover, $\text{Req}\ \eta \leq \text{Req}\ [\nu; 2]$; thus, since ν is immune and η infinite, the set η is also immune. We conclude that $\mathfrak{B}$ is an immune α-bigraph.

Re (b). If $\mathfrak{B}$ were an ω-bigraph, we could, given any vertex x of $\mathfrak{B}$ different from s_0, compute the unique (hence minimal) path which joins x and s_0. Then the enumeration $s_0, t_0, t_1, s_1, s_2, \ldots$ of the set $\alpha \cup \beta = \sigma \cup \tau$ would be regressive. However, the set $\sigma \cup \tau$ is not regressive; hence $\mathfrak{B}$ is not an ω-bigraph.

Remark: Note that $\mathfrak{B}$ is a tree, in fact, a one-way infinite path. Thus there exists an immune α-tree which is not an ω-tree.

4 *Projective bigraphs*

Definition 2 A *projective plane* is an ordered triple $\Pi = \langle \alpha, \lambda, \text{inc.} \rangle$ consisting of two disjoint sets α and λ and an incidence relation inc. so that the three classical axioms hold.

The elements of α are called the *points*, those of λ the *lines* of Π. With every projective plane $\Pi = \langle \alpha, \lambda, \text{inc.} \rangle$ we associate the functions L and P:

$$\delta L = (\alpha \times \alpha)^{-},\ \delta P = (\lambda \times \lambda)^{-},$$
$$L(a, b) = \text{the line through } a \text{ and } b,$$
$$P(l, m) = \text{the point in which } l \text{ and } m \text{ intersect}.$$

We also write $a \cdot b$ for $L(a, b)$ and $l \cap m$ for $P(l, m)$.

Definition 3 A *projective ω-plane* is a projective plane $\Pi = \langle \alpha, \lambda, \text{inc.} \rangle$, where $\alpha|\lambda$ and the functions L and P have partial recursive extensions. Π is called *isolic* (*immune*), if the sets α and λ are isolated (immune).

Definition 4 A *projective bigraph* is a graph $\mathfrak{B} = \langle \nu, \eta \rangle$ for which there exist sets α and λ such that

(a) $\nu = \alpha \cup \lambda$, where α and λ are disjoint,
(b) $\mathrm{can}(x, y) \in \eta \Rightarrow [x \in \alpha$ and $y \in \lambda]$ or $[x \in \lambda$ and $y \in \alpha]$,
(c) the relation inc. defined by:

$$x \text{ inc. } y \Leftrightarrow \mathrm{can}(x, y) \in \eta, \text{ for } x, y \in \nu,$$

is such that $\Pi = \langle \alpha, \lambda, \text{inc.} \rangle$ is a projective plane.

If the projective bigraph $\mathfrak{B}$ and the projective plane Π are related in this manner and $\min(\alpha \cup \lambda) \in \alpha$, we say that $\mathfrak{B}$ and Π are *associated.* The condition $\min(\alpha \cup \lambda) \in \alpha$ guarantees that α and λ are uniquely determined by the bigraph $\mathfrak{B}$ and cannot be interchanged. Note that every projective bigraph is connected.

Proposition 3 *Let the projective plane* $\Pi = \langle \alpha, \lambda, \text{inc.} \rangle$ *and the projective bigraph* $\mathfrak{B} = \langle \nu, \eta \rangle$ *be associated. Then* Π *is a projective* ω*-plane if and only if* $\mathfrak{B}$ *is an* ω*-bigraph.*

Proof: Assume the hypothesis.

(a) Suppose that Π is a projective ω-plane. Choose distinct elements $a, b \in \alpha$; from now on we keep a and b fixed. Let p and q be distinct vertices of $\mathfrak{B}$, say, $p < q$. We distinguish four cases. Since $\alpha|\lambda$ we can effectively decide which of these four cases holds.

Case 1 $p, q \in \alpha$. Then $\mathrm{d}(p, q) = 2$, for if $p \cdot q = r$, the path $\langle p, r, q \rangle$ is a minimal path from p to q. Since $r = L(p, q)$ can be computed from p and q, so can the path $\langle p, r, q \rangle$.

Case 2 $p, q \in \lambda$. If $s = p \cap q$, the minimal path $\langle p, s, q \rangle$ from p to q can be computed from p and q.

Case 3 $p \in \alpha, q \in \lambda$. Then

$$\mathrm{d}(p, q) = \begin{cases} 1, & \text{if } p \text{ inc. } q, \\ 3, & \text{if not}[p \text{ inc. } q]. \end{cases}$$

If p inc. q, $\langle p, q \rangle$ is the only minimal path from p to q. Suppose not$[p$ inc. $q]$ and assume $p \neq a$. Define $s = (p \cdot a) \cap q$. Then $\langle p, p \cdot a, s, q \rangle$ is a minimal path from p to q; it can be computed from p and q, because P and L have partial recursive extensions. If $p = a$, put $s = (p \cdot b) \cap q$; then $\langle p, p \cdot b, s, q \rangle$ is a minimal path. Given p and q we can by [2, §3, (e)] effectively decide whether p inc. q. Hence we can compute a minimal path from p to q.

Case 4 $p \in \lambda, q \in \alpha$. This is the dual of Case 3.

(b) Suppose that $\mathfrak{B} = \langle \nu, \eta \rangle$ is a projective ω-bigraph. Let α and λ be related to ν as described in Definition 4 and let $\min(\alpha \cup \lambda) \in \alpha$. We wish to prove:

(i) $\alpha | \lambda$, (ii) the function P has a partial recursive extension, (iii) the function L has a partial recursive extension. Note that (ii) is the dual of (iii). Let $x = \min(\alpha \cup \lambda)$. Then $x \in \alpha$ and for $y \in (\alpha \cup \lambda) - (x)$ we have: $y \in \alpha \Leftrightarrow \mathrm{d}(x, y) = 2$. Since $\mathfrak{B}$ is an ω-graph we can compute $\mathrm{d}(x, y)$; thus $\alpha | \lambda$. We now prove (iii). Let p and q be distinct points of Π with $p < q$. Then there is exactly one vertex x of $\mathfrak{B}$ so that $\mathrm{con}(p, x)$ and $\mathrm{con}(q, x)$ belong to η, say, $x = r$; moreover, $r = L(p, q)$. The only minimal path from p to q is $\langle p, r, q\rangle$. Since $\langle p, r, q\rangle$ can be computed from p and q, so can $r = L(p, q)$. Thus L has a partial recursive extension.

Let the projective ω-bigraph $\mathfrak{B} = \langle \nu, \eta\rangle$ and the projective ω-plane $\Pi = \langle \alpha, \lambda, \text{inc.}\rangle$ be associated. For $p \in \alpha$, $l \in \lambda$ we define α_l as the set of all points on l and λ_p as the set of all lines through p. By ([2], p. 2) there is a unique recursive equivalence type [see 1, Section 1] $\mathcal{M}$ such that

$$\text{Req}\,\alpha_l = \text{Req}\,\lambda_p = \mathcal{M} + 1, \text{ for all } p \in \alpha,\ l \in \lambda,$$
$$\text{Req}\,\alpha = \text{Req}\,\lambda = \mathcal{M}^2 + \mathcal{M} + 1,$$

the so-called *order* of Π. It follows that

$$\text{Req}\,\nu = \text{Req}\,(\alpha \cup \lambda) = 2\,\text{Req}\,(\alpha) = 2(\mathcal{M}^2 + \mathcal{M} + 1),$$
$$\text{Req}\,\eta = (\mathcal{M} + 1)(\mathcal{M}^2 + \mathcal{M} + 1).$$

Clearly, $\mathcal{M} \leqslant \mathcal{M}^2 + \mathcal{M} + 1$. Thus, if we write Λ for the collection of all isols, we have

$$\Pi \text{ isolic} \Rightarrow \mathcal{M} \in \Lambda \Rightarrow \text{Req}\,\nu,\ \text{Req}\,\eta \in \Lambda \Rightarrow \mathfrak{B} \text{ isolic},$$
$$\mathfrak{B} \text{ isolic} \Rightarrow 2(\mathcal{M}^2 + \mathcal{M} + 1) \in \Lambda \Rightarrow \mathcal{M} \in \Lambda \Rightarrow \Pi \text{isolic}.$$

Also, Π is infinite if and only if $\mathfrak{B}$ is infinite, so that

(7) $$\Pi \text{ immune} \Leftrightarrow \mathfrak{B} \text{ immune}.$$

Proposition 4 *There are exactly* $\mathfrak{c}$ ω-*bigraphs. Among these exactly* $\mathfrak{c}$ *are immune.*

Proof: Every graph is of the form $\mathfrak{G} = \langle \nu, \eta\rangle$, where $\nu, \eta \subset \varepsilon$; hence there are at most $\mathfrak{c}$ graphs and at most $\mathfrak{c}$ ω-bigraphs. Thus we only need to show that there are at least $\mathfrak{c}$ immune ω-bigraphs. It follows from ([2], p. 7) that there exists a family of $\mathfrak{c}$ immune ω-planes which are mutually nonisomorphic. We may assume that all these planes satisfy $\min(\alpha \cup \lambda) \in \alpha$, since every immune ω-plane is isomorphic to an immune ω-plane in which $o \in \alpha$. Using (7) we conclude that there are at least $\mathfrak{c}$ immune ω-bigraphs.

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